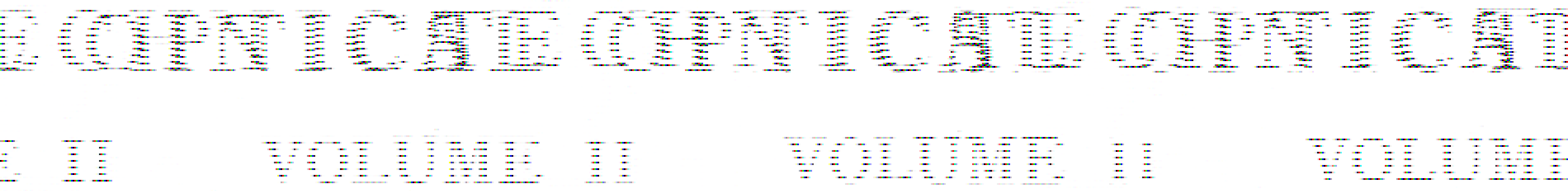
VOLUME II
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TECHNICAL OPTICS

A REVISED AND ENLARGED EDITION OF
"AN INTRODUCTION TO APPLIED OPTICS"

BY

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PREFACE TO SECOND EDITION

SINCE the last revision of this book, important advances have been made in optical theory and instrumentation—for example, in connection with the specification of optical performance in terms of contrast transmission or contrast transfer factors, and in the further development of interferometric testing methods as well as new optical systems and lenses. An attempt has been made to give a brief account of some of these topics, though a really adequate discussion would have entailed a prohibitive enlargement of the book; references to many original papers have therefore been added.

I am greatly indebted to my sometime colleagues Professor W. D. Wright and Dr. W. D. Welford for valuable help in revising parts of the text, and to numerous other friends, especially Dr. H. H. Hopkins, Dr. C. G. Wynne, and Mr. F. H. Smith, for information. Some of the firms mentioned in the original preface have given further help, and Messrs Wray Ltd. have also assisted. I have also had the privilege of correspondence with the Optical Institutes in Paris and Stockholm.

I hope, therefore, in spite of its limitations the book may serve as an introduction to this field of activity and a guide to further study. The time has not arrived when the relative value of the new advances can be fully appreciated, but the text should help in the recognition and exploitation of the present possibilities.

PREFACE TO FIRST EDITION

THE first volume of *Technical Optics* (a new and revised edition of the author's *Introduction to Applied Optics*) has been received with much kindness, and it is hoped that this second volume, which deals more specifically with instruments, may prove itself useful in elucidating their construction and performance. Considerable advances have taken place in recent years, and some account of the theory of phase-contrast microscopy, Schmidt cameras, the problems of aspheric surface optics, and other new topics has caused an enlargement of the scope of the previous work. Efforts have been made to supply the optical concepts on which a rigorous mathematical treatment of the newer problems may be based, rather than making the optics secondary to a great deal of analysis.

In addition to the thanks contained in the Preface to Volume II of the previous book, I must gratefully acknowledge help from other friends, especially including Mr. E. W. H. Selwyn of Messrs. Kodak, Ltd., and Dr. J. H. Jeffree of Combined Optical Industries, Ltd. I am also grateful to the Royal Photographic Society for permission to reproduce a number of star-image patterns, and to H.M. War Office (M.I. 10) for particulars of a sighting telescope.

A number of firms have furnished invaluable help by the loan of blocks and by giving particulars of their instruments and methods, amongst which are Messrs. C. Baker, Ltd.; Messrs. R. & J. Beck, Ltd.; Messrs. Ross, Ltd.; Messrs. Cooke, Troughton & Simms, Ltd.; Messrs. Hilger & Watts, Ltd., with Mr. G. Loeck; and the Polaroid Corporation of America.

Several colleagues and students have given assistance with the revision of the proofs, and my hearty thanks are due to them also.

L. C. MARTIN

IMPERIAL COLLEGE OF SCIENCE
AND TECHNOLOGY

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CHAPTER I

THE MAGNIFICATION PRODUCED BY LENSES—THE SIMPLE MICROSCOPE

IN the first volume of this work, the term *magnification* has been restricted to mean the ratio of a linear dimension of the image to the corresponding size of the object; thus the formulae

$$\frac{h'}{h} = -\frac{f}{x} = -\frac{x'}{f'} = m'$$

express the *lateral magnification* measured perpendicular to the axis, while the equation

$$\frac{dx'}{dx} = \left(\frac{h'}{h}\right)^2 \frac{n'}{n}$$

gives the longitudinal magnification, or the ratio of image to object size measured in the axial direction, close to a particular pair of points where the lateral magnification is $\frac{h'}{h}$; it is assumed, of course, that the image exists in three dimensions.

The Lagrange relation permits the derivation of an expression for magnification in terms of the distances of object and image from *any* pair of conjugate points for which the magnification is known. Thus in Fig. 1 we might take B and B′, P and P′ to be any two pairs of conjugate points. Let $PB = l$ and $P'B' = l'$. Let $BB_1 = h_1$ and $B'B_1' = h_1'$ where BB_1 and $B'B_1'$ are a small object and image both perpendicular to the axis. We will apply the Lagrange relation to the imagery at the points P and P′, considering the ray B_1P and its emergent path $P'B_1'$. The ordinary formula is

$$nh\omega = n'h'\omega'$$

h and h' are now the sizes of a small object and image at P and P′ respectively, while ω and ω' are the angles with the axis made by the rays B_1P and $P'B_1'$; but $\omega = -\frac{h_1}{l}$ and $\omega' = -\frac{h_1'}{l'}$. Hence

$$nh\left(-\frac{h_1}{l}\right) = n'h'\left(-\frac{h_1'}{l'}\right).$$

or

$$\frac{h_1'}{h_1} = \frac{nhl'}{n'h'l}$$

In a particular case of importance, P and P′ may represent the axial points of the entrance and exit pupils; the ray passing through them then becomes the "principal ray" of the bundle entering the instrument. In this case the ratio $\frac{h}{h'}$ above may be written $\frac{p}{p'}$, where p and p' are the radii of the pupils; and l and l' can be written q and q', using these symbols to denote the distances of object and image from entrance and exit pupils respectively. The formula then becomes

$$\frac{h_1'}{h_1} = \frac{npq'}{n'p'q} \quad . \quad . \quad . \quad (1.01)$$

The great majority of optical instruments are, however, used as

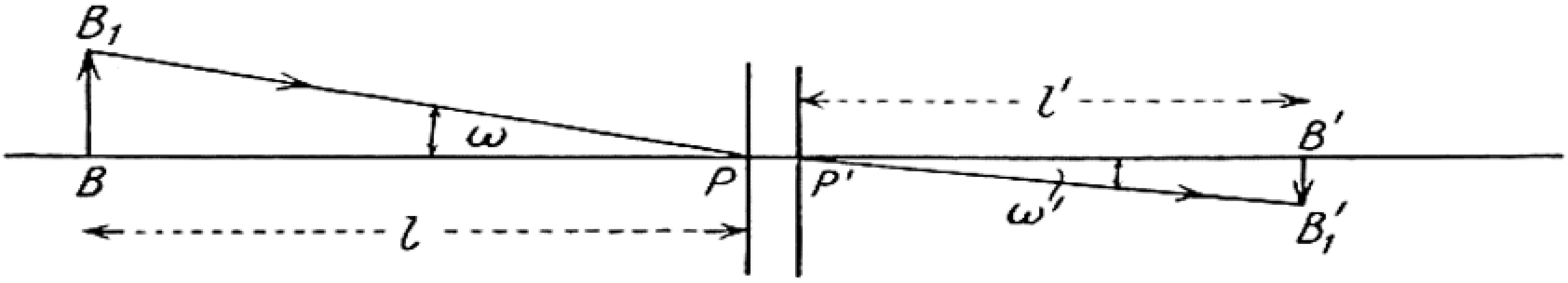


FIG. 1

direct aids to vision, and the conception of the "magnifying power" of an instrument is then—

Magnifying power

$$= \frac{\text{Size of retinal image obtained with the aid of the instrument}}{\text{Size of image obtained with the unaided eye}}$$

As explained in Vol. I, it has been found that the accommodation of the eye is largely effected by the variation in curvature of the lens. In consequence of this the distance of the principal and nodal points from the retina varies very little, in fact less than half a millimetre in changing the accommodation from distance vision to near vision. The "stop" which limits the bundle of rays arriving at the retina may sometimes be the pupil of the eye itself, and sometimes it may be the exit pupil of some instrument projected into the same approximate position. In most cases the action of the accommodation will secure the sharpness of the image, but, if not, the image position will be assumed to be defined by the intersection of the principal ray with the retina. Further, since the eye entrance pupil and first principal point are only separated by about one or two millimetres, the principal ray may be regarded as that one passing through the principal points.

Refer again to Fig. 1, and let P and P′ now represent the principal points of the eye; the object BB_1 subtends an angle ω at the first principal point P. The ray makes an angle ω' with the axis after refraction. The Lagrange relation applied to the principal points where the magnification is unity gives

$$n\omega = n'\omega'$$

if the discussion is limited to paraxial conditions.

As mentioned above, the size of the image perpendicular to the axis may be defined by the intersection of the principal ray with the image plane. If this plane is at a distance l' from the second principal point, we may then write for the reasons given above,

$$h_1' = -l'\omega' = -l'\frac{n}{n'}\omega$$

the size of the image depending mainly on the angle subtended by the object at the first principal point, even allowing for variations in accommodation. While the angles are small we can deal with their angular measure, though when the angles are large, and the images are measured on flat screens, we shall have to deal with their "tangents." Hence, the above equation for magnifying power can be written—for small angles, at least—

$$\text{Magnifying power} = \frac{\text{Angular subtense of image obtained with the instrument}}{\text{Angular subtense of object seen with the unaided eye}}$$

Magnifying Power of an Optical Instrument. It is always necessary to consider the state of accommodation or "refraction" of the eye when dealing with the magnification produced by an optical instrument. The condition is conveniently specified by giving the position of the accommodation point on which the eye is focused. Call this point M (conjugate to the macula M′), and let its distance from the first principal point of the eye be k. The condition of affairs might be as suggested in Fig. 2, in which "*a*" represents the optical system, and "*b*" is the system of the eye.

The object is situated at some point B, and the lens projects an image of height h' into the accommodation point of the eye.

The angle ω subtended by the image at the first principal point of the eye is

$$\omega = -\frac{h'}{k}$$

In order to investigate the dependence of ω on the size of the object, this equation can be written

$$\begin{aligned}\omega &= -h\left(\frac{h'}{h}\right)\frac{1}{k}\\ &= h\left(\frac{x'}{f'_a}\right)\frac{1}{k} \text{ (from equation (202), Vol. I)}\end{aligned}$$

where x' is the distance F'_aM.

Since from Fig. 2,

$$\begin{aligned} x' &= d + k - f'_a \\ \omega &= h\left(\frac{d + k - f'_a}{f'_a k}\right) \\ &= h\left(\frac{1}{f'_a} - \frac{1}{k} + \frac{d}{f'_a k}\right) \\ &= h\ (\mathbf{F}_a - \mathbf{K} + \mathbf{F}_a d\mathbf{K}) \\ &= h\ \{\mathbf{F}_a(1 + d\mathbf{K}) - \mathbf{K}\} \quad . \quad . \quad . \qquad (1.02) \end{aligned}$$

In order to see the object distinctly with the unaided eye in the same state of accommodation, it will be necessary to place it in the point of accommodation, M. This would, of course, be impossible for a real object with the case shown in Fig. 2, but it would be possible in all cases where k is numerically negative.

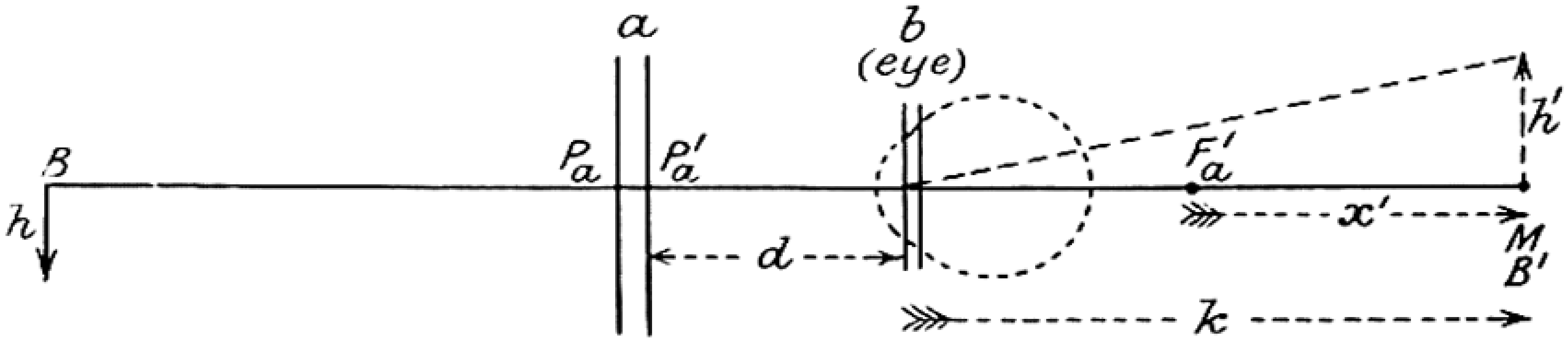


Fig. 2

When placed at M, the angle subtended at the first principal point of the eye by the object is given by

$$\omega_o = -\frac{h}{k} = -h\mathbf{K}$$

Hence magnification

$$\begin{aligned} &= \frac{\omega}{\omega_o} = \frac{h\ (\mathbf{F}_a - \mathbf{K} + d\mathbf{F}_a\mathbf{K})}{-h\mathbf{K}} \\ &= 1 - \frac{\mathbf{F}_a}{\mathbf{K}} - d\mathbf{F}_a \\ &= 1 - \mathbf{F}_a(k + d) \quad . \quad . \quad . \quad . \qquad (1.03) \end{aligned}$$

If l' is the distance of the accommodation point from the second principal point of the lens system, then $l' = k + d$, and the formula takes the form in which it is often written—

$$\text{Magnification} = 1 - \frac{l'}{f'} \quad . \quad . \qquad (1.04)$$

In the case of a small lens of high power, it will usually be held close to the eye, and the distance of the point of accommodation will usually be the "least distance of distinct vision." As has been seen from the table in Chapter IX, Vol. I, this distance varies. A mean value for adult vision may be taken as -250 mm. (or about -10 in.). The magnification then becomes (f' in millimetres)

$$1 + \frac{250}{f'}$$

or sufficiently nearly, if f' is small,

$$\text{Magnification} = 0{\cdot}25\mathbf{F} = \frac{\mathbf{F}}{4} \quad . \quad . \qquad (1.05)$$

where $\mathbf{F}$ is the power of the lens in diopters.

Spectacle Magnification. The user of a pair of spectacles will, however, judge "magnification" rather differently in that he will compare the apparent size of the more or less indistinct image seen with the unaided eye, and the size of the image seen with his spectacles, the object remaining fixed.

From above, the angular subtense of the image will be

$$\omega = h\ \{\mathbf{F}_a(1 + d\mathbf{K}) - \mathbf{K}\}$$

We now calculate the position of the real object corresponding to the image position at M, where it must be formed to be seen distinctly by the eye.

The usual formula,

$$\frac{1}{l'_a} - \frac{1}{l_a} = \frac{1}{f'_a}$$

gives $$l_a = \frac{f'_a l'_a}{f'_a - l'_a} = \frac{l'_a}{1 - l'_a\mathbf{F}},$$ omitting the suffix from $\mathbf{F}$.

Let the interval $P_aP'_a = t$, then the distance of the object from the *first* principal point of the eye is given by

$$l_b = l_a - (d + t)$$

$$= \frac{l'_a}{1 - l'_a\,\mathbf{F}} - (d + t)$$

If the thickness t be neglected, as it is usually small in spectacles, and remembering that $l'_a = d + k$, we have

$$l_b = \frac{l'_a - d + l'_a\mathbf{F}d}{1 - l'_a\mathbf{F}} = \frac{k + (d + k)\,d\mathbf{F}}{1 - (d + k)\mathbf{F}}$$

And the angular subtense of the object is therefore

$$-\frac{h}{l_b} = -\,h\left\{\frac{1 - (d + k)\,\mathbf{F}}{k + (d + k)d\mathbf{F}}\right\}$$

i.e. $$-\frac{h}{l_b} = -\,h\left\{\frac{\mathbf{K} - (d\mathbf{K} + 1)\mathbf{F}}{1 + (d\mathbf{K} + 1)\,d\mathbf{F}}\right\}$$

$$\text{Hence} \qquad \text{Magnification} = \frac{\text{Angular subtense of image}}{\text{Angular subtense of object}}$$

$$= \frac{h\{\mathbf{F}(1 + d\mathbf{K}) - \mathbf{K}\}}{-h\left\{\dfrac{\mathbf{K} - (d\mathbf{K} + 1)\,\mathbf{F}}{1 + (d\mathbf{K} + 1)\,d\mathbf{F}}\right\}}$$

$$= 1 + (1 + d\mathbf{K})\,d\,\mathbf{F} \qquad . \qquad . \qquad (1.06)$$

This expression is quite general for any thin lens and any state of accommodation, provided that a sharp image is obtained with the aid of the spectacle lens. The distance of the lens from the eye is not restricted.

Example I

Take $\mathbf{K} = -4\text{D}$, $\mathbf{F} = 6\text{D}$, and $d = 1{\cdot}0$ (metres)

Then $\qquad \text{Magnification} = 1 + (-3)\,6$

$$= -17$$

This shows that the lens is forming an inverted image between the lens and the eye; and the angular subtense of this inverted image is considerably greater than that of the object seen directly. The distance of the object from the lens proves to be $-21{\cdot}4$ cm., while the distance of the inverted image is 75 cm. from the lens, and -25 cm. from the eye. Thus $\dfrac{h'}{h}$ for the lens $= -3{\cdot}5$. From these figures the above value for the magnification is easily checked. The student should make a sketch of the arrangement.

Example II

Take $\mathbf{K} = -4\text{D}$, and $\mathbf{F} = 6\text{D}$ (as above), but let d now be equal to $0{\cdot}01$ metre. Then magnification $= 1{\cdot}06$, by the formula.

The image is now virtual and erect, and the lens is being used close to the eye as an ordinary magnifier. The low value for the "magnification" is due to the fact that we are comparing the angular subtense of the image seen sharply with the lens, and that of the diffuse image which is seen without it. These angles are not greatly different in magnitude.

Example III

Take $\mathbf{K} = +0{\cdot}5\text{D}$, $\mathbf{F} = +0{\cdot}2\text{D}$, $d = 3{\cdot}0$ (metres).

In this case the magnification works out to $+2{\cdot}50$. We therefore should obtain an erect image of moderate magnification.

The system is made use of in the so-called "window telescope," made by some opticians, which consists of a large single lens of low power made to hang on the window frame. Most persons of normal vision can manage to relax the accommodation far enough to bring the refraction of the eye to a small positive value, and the combination of lens and eye then produces a system having a focal length considerably longer than that of the eye alone. An erect, magnified image is the result.

Magnification given by Spectacles giving Distance Correction. The above expression for magnification takes a simple form in the case where the "glass" employed is suitable for distance correction. In that case we have the simple relations—

$$\boldsymbol{K} = \frac{\boldsymbol{F}}{1 - d\boldsymbol{F}}, \text{ and } \boldsymbol{F} = \frac{\boldsymbol{K}}{1 + d\boldsymbol{K}} \quad \text{(See Vol. I, eqn. 901)}$$

By a simple substitution the expression for the magnification can then be put into the simple forms—

$$\text{Magnification} = \frac{1}{1 - d\boldsymbol{F}} = 1 + d\boldsymbol{K} \quad . \qquad (1.07)$$

We see at once that a positive correcting lens produces a magnification greater than unity, while a negative lens as used in myopia produces a magnification less than unity—sometimes called a "minification."

Single Lenses as Magnifiers. The use of single lenses as magnifiers has been known since very early times. The Greeks were familiar with the effects of refraction of light at curved surfaces, such as might be studied with solid balls of glass or rock crystal, and the perfection of certain antique handwork seems to call for the explanation that it was produced with the aid of a magnifier.

Antony van Leeuwenhoek, born at Delft in 1632, made great advances in grinding and polishing small lenses for magnifiers, and obtained magnifications up to 160. He observed infusoria and bacteria for the first time. In 1702 J. Wilson produced a pocket microscope with which, it is said, magnifications up to 400 could be obtained. The convenient mechanical construction of this instrument ensured its popularity for quite 100 years.

We may distinguish two main functions of single-lens magnifiers. First a lens may be used, as in one side of a stereoscope, to obtain a general view of a large object or picture at a suitable angle. Secondly, a lens may be employed to obtain an enlarged image of a very small portion of an object. Naturally the optical arrangements differ in the two cases.

Lenses for Viewing Pictures. The first case above brings us to considerations very similar to those encountered in the discussion of spectacle lenses. The eye turns in its socket in order to view the different parts of the object, and the pencils of light reaching the retina under different angles have an eccentric passage through the lens. The conditions are optically similar to the case in which a small stop is situated in the centre of rotation of the eye (Fig. 3). The conditions have already been discussed in connection with high power spectacle lenses. It was shown in that connection that one of the chief troubles arising was the astigmatism of the oblique pencils. This was shown to be diminished by the choice of a lens of suitable figure, generally of meniscus type with the concavity towards the eye, but correction could not be given beyond a power of about 10D and higher without the employment of aspherical surfaces. The ellipse of Fig. 201, Vol. I, will provide approximate data for the radii required for various powers, but the spectacle lens was intended to form images of *distant* objects, and the results do not hold exactly for near objects. Calculations for near vision have been made by Whitwell.[1]

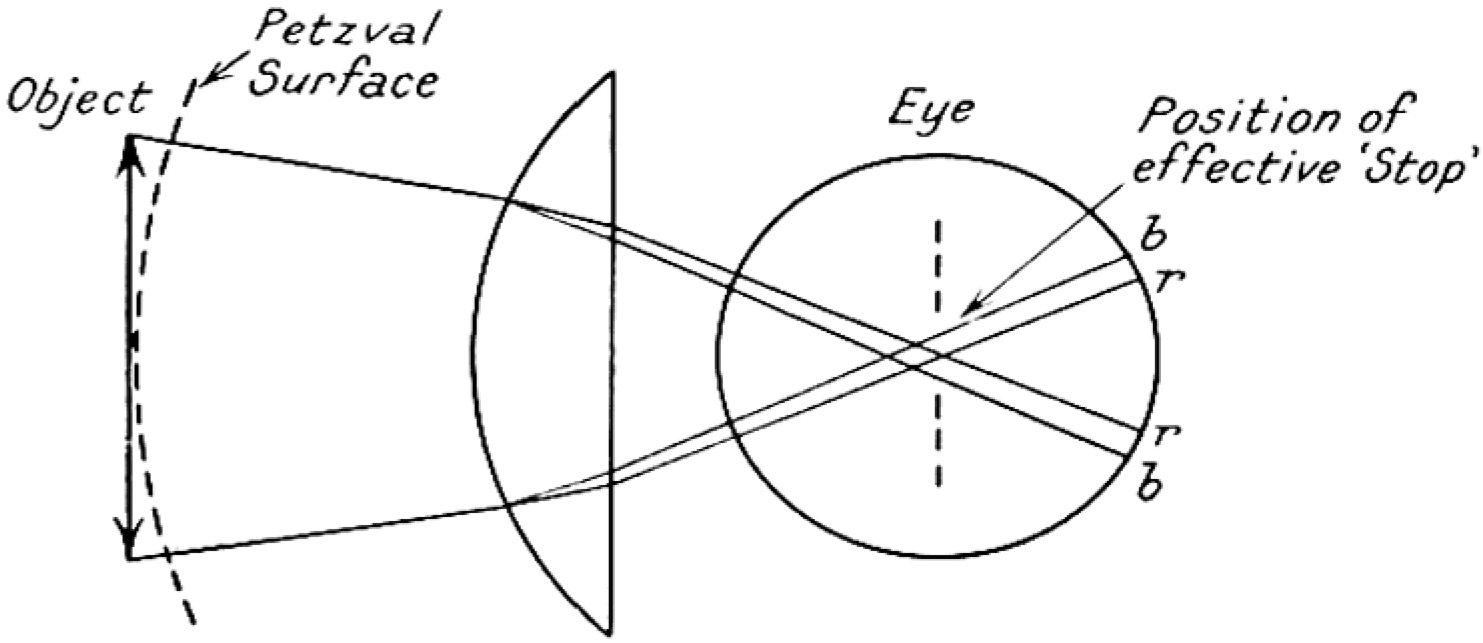


FIG. 3. USE OF A SIMPLE MAGNIFIER WITH ROTATING EYE

If it is sought to view a near flat object with such a lens, several additional defects in the image are at once noticed.

1. *Roundness of the Field.* The exterior parts of the object are farther away from the lens than the centre. Even if the lens has been freed from "oblique astigmatism," the object field will not be seen in focus as the eye is moved unless it has the curvature of the Petzval surface, of radius nf' for a single thin lens. With an ordinary double convex lens, the presence of astigmatism makes matters very much worse. If the object is flat we can first bring radial, then tangential, lines to a focus in the outer parts of the field by bringing the object closer to the lens.

2. *Chromatic Aberration.* On tracing a principal ray through the system of Fig. 3, it will at once be noticed that considerable lateral

chromatic aberration must arise through the passage of the pencils of light through the outer parts of the single lens. If the object consists of a number of small bright points on a dark ground (pinholes in a thin card illuminated from behind) each object point in the outer part of the field is imaged as a short radial "spectrum," the red being innermost.

In the case of lenses required to give a general view of objects or pictures, they are not often required to be of shorter focal length than about 5 cm., and, as will be shown below, the spherical aberration of the pencils which are transmitted by the pupil of the eye (which will probably not exceed 4 mm. in diameter during the day) does not produce any appreciable deterioration of the image.

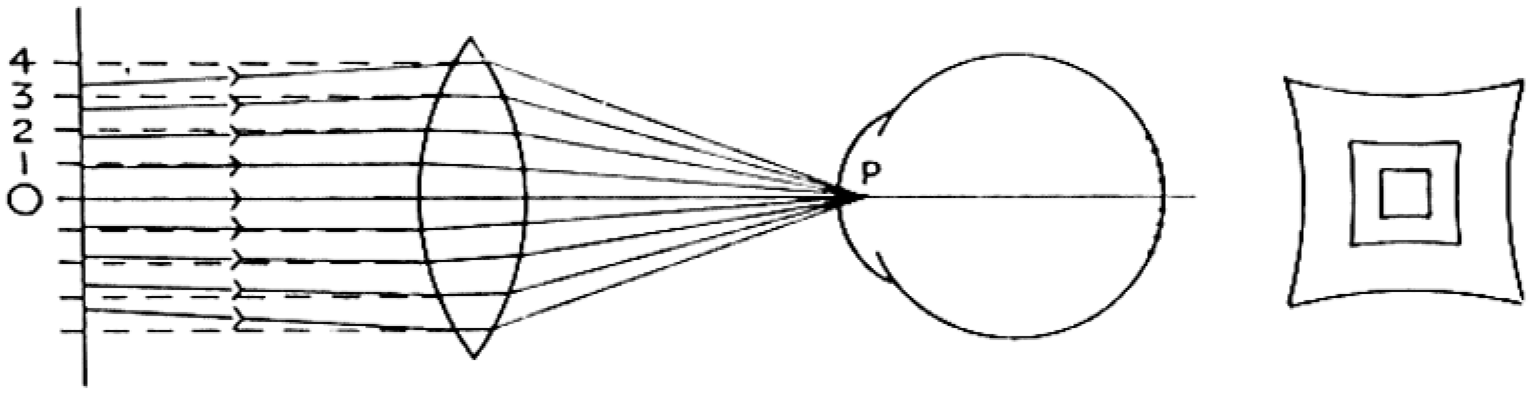


FIG. 4. PIN-CUSHION DISTORTION PRODUCED BY A SIMPLE MAGNIFIER

3. *Distortion,* of the pin-cushion variety, is noticeable with a single lens. The bending of the rays increases too rapidly in the outer parts of the field, and the magnification thus increases with the distance from the centre.

Fig. 4 illustrates the effect in the particular case when a double-convex lens forms the image of a scale placed in its first principal focus; the viewing eye is so placed that the entrance pupil, P, of the eye coincides with the paraxial principal focus of the lens. Imagine a set of rays from P traced from right to left through the lens such that *if the lens were free from spherical aberration* these rays would intersect the equidistant graduations 1, 2, 3, 4, etc., of the scale; they would be parallel to each other between the lens and the scale; they are suggested by the broken lines in the diagram.

In the case of such a lens of negligible thickness, the tangents of the angles of slope of the consecutive rays through P would have equal increments; the angles under which the graduations are seen vary in the same way as when a scale is viewed directly. Therefore, supposing these scale divisions to be the origins of bundles of rays, with the principal ray of each bundle passing through P, the image formation would be recognized as free from distortion.

In the actual case of a double-convex lens the spherical aberration and finite thickness of the lens have to be considered. The *same* set of rays traced "right to left" from P would intersect the scale at points closer to the axis than the corresponding graduations by amounts increasing with (at least) the cube of the zonal height of the ray owing to the undercorrected spherical aberration of the lens; consequently the graduations would be imaged on the retina at greater distances than the intersection points of the ideal case; and these distances would again increase with (at least) the cube of the zonal height, so that the discrepancy would grow in the same way with the image height. The distortion of the magnifier is seen to be directly bound up with the spherical aberration of the lens in regard to the pupil position.

If h is the height of an object point above the axis, and α' is the angle between the axis and a principal ray originating from the point and passing through the pupil, then the condition for freedom from distortion for the object plane (perpendicular to the axis) is

$$h \cot \alpha' = \text{const.}$$

The effects of various shapes of lenses, and various pupil positions are easily recognized in practical trials. Take, for example, a small plano-convex achromatic lens ($1\frac{1}{2}$ in. focus, 0·75 in. diameter, for example); it exhibits only small spherical aberration for the principal focus on the plane side. If it is used as a magnifier, with the eye pupil in this focus, there is no recognizable distortion, but distortion appears if the eye is drawn back while object and lens remain stationary; also if the lens is simply reversed after using it in the circumstances first mentioned. It will be noticed that, with this lens, the best central *definition* of the image would be found when the plane side of the lens is turned to the object; but this arrangement is not distortion-free. A better compromise for both definition and distortion freedom can be obtained by a triple-cemented lens (see below). The use of aspheric surfaces to obtain distortion freedom by removing the spherical aberration is another possibility.

True Perspective. In Vol. I, Chapter I, the question of the proper presentation of a perspective projection was discussed, and it was shown that a proper judgment of the space values of the picture can only be obtained if the latter is seen under the proper angle.

The majority of pictures obtained by small cameras are made with lenses of considerably shorter focal length than the least distance of distinct vision; hence, if the print is to be seen with the unaided eye, it must be held at too great a distance to attain the proper angle. Matters could be corrected, actually, by placing the projection (or print) back in the camera in which the photograph was made, illuminating it in some way, and then looking at it through the camera lens.

This, however, would be awkward because the camera lens may be provided with a stop through which the eye must look. It would be necessary to move the head about. This type of "keyhole observation" is not convenient as we could not get a good view of the picture as a whole. Hence, although the perspective conditions require that the viewing lens shall have the same focal length as the camera lens, it is desirable that the former shall be corrected for use with an (imaginary) stop situated in the centre of rotation of the eye when properly positioned with respect to the system.

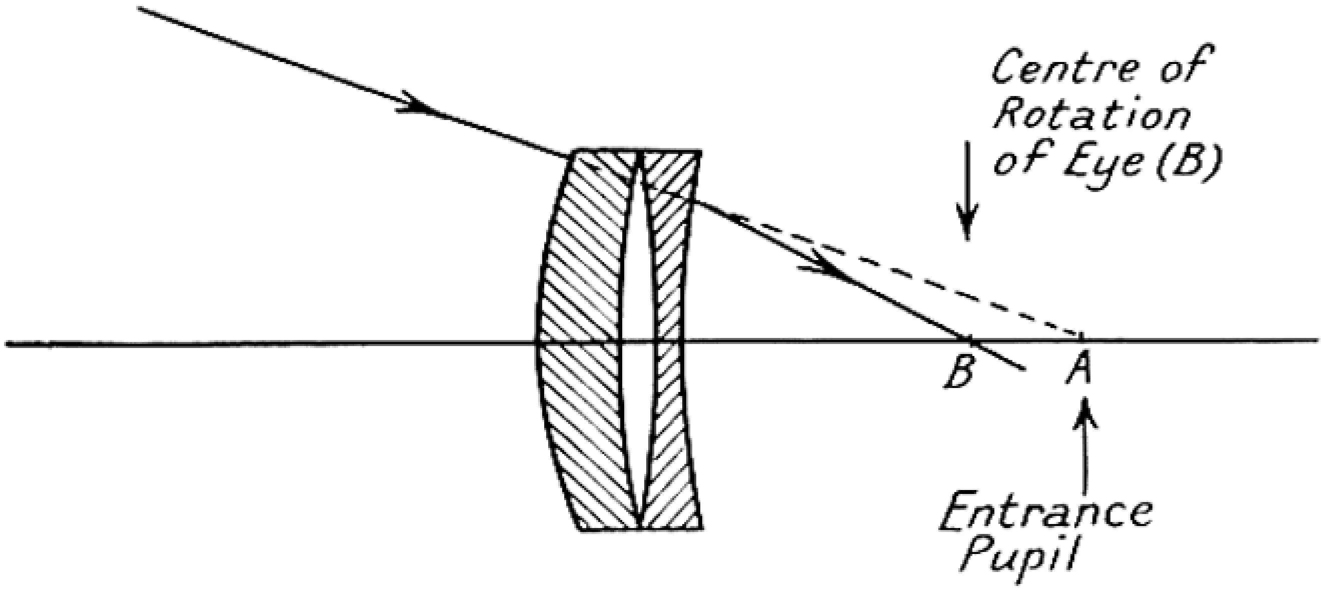


FIG. 5. THE VERANT LENS

The best-known lens of this type is the "Verant," designed by M. von Rohr and made by Zeiss.[2] It corrects chromatic aberration of the above type, and distortion on the lines laid down by Gullstrand, and it is illustrated in Fig. 5. There is some residual astigmatism at certain angles of obliquity, but it is not large. The curvature of field can be allowed for by variation of the accommodation of the eye.

A lens of somewhat similar character but different construction, intended for use with a stereoscope, has been designed by Albada.[3]

Nature of the Image to be Presented to the Eye. In spite of the fact that the human eye is admittedly subject to chromatic aberration in the sense of "undercorrection" characteristic of an ordinary lens, and also that it suffers from zonal spherical aberration, it has been shown in the chapter on the eye, Vol. I, that the acuity of normal human vision for the small pupilary diameters characteristic of day-time is little lower than the limits set by the wave-nature of light, even for a perfect optical system.

The vision of actual objects, either at a great distance, or at the near point, manifests no trace of coloured fringes or haziness due to spherical aberration. It is, indeed, just conceivable that the mental receiving apparatus has some means of automatic compensation, which causes the brain to interpret a particular distribution of light (in regard to colour) as characteristic of an elementary point. However that may be, it has always been judged best to design any optical instrument so that the *image* shall be as perfect as possible, in the physical sense, as it is presented to the eye. The defects of vision can then have no worse effect on the appearance of this image than on the appearance of a real object.

A few attempts have, indeed, been made to design systems which should compensate the chromatic aberration of the eye, and secure a greater concentration on the retina. It is possible that systematic research on such lines might yield results of interest, but so far nothing has been done which has led to any departure from the general rule of making the image as physically perfect as possible.

The Simple Microscope. In contra-distinction from the lens required for giving a general view of an object, it is frequently required to obtain a greatly enlarged view of a small area—small, perhaps, in comparison with the diameter of the lens itself.

It will be shown in the chapter on the microscope that the power of the system to yield very sharp images of small objects depends upon the angular divergence of the cone of rays derived from the object, and brought without appreciable aberration to the retinal focus. In Fig. 6, the cone has a small angular diameter ω, and this could be enlarged by increasing the size of the central stop. On the other hand, this will rapidly increase the spherical aberration arising through refraction at each surface, and the effects of chromatic aberration will also become serious. The chromatic aberration here referred to is the difference in the focusing position of different colours measured along the axis of symmetry of a bundle; thus if the green is focused on the retina, the other spectral components will be represented by blur patches of lessened concentration. It is to be noted that this chromatic aberration will be in the same sense as that of the eye itself, the shorter wave-lengths being focused nearer the lens.

In order to obtain a perfect image, as projected by the lens, it may be required that rays diverging from the object point shall all be rendered parallel after leaving the lens, i.e. that the wavefronts shall be plane on entering the eye. The case is then just the reverse of that when a lens is to form a sharp image of a very distant object. The numerical amounts of the optical path differences are therefore the same.

When we considered the case of primary spherical aberration, the maximum residual optical path differences arising at the best focus position (midway between marginal and paraxial foci) were shown to be one-quarter of the optical path difference between marginal and paraxial rays arising at the paraxial focus. The formula (Vol. I, p. 110) for this latter was

$$p = A_1 y^4$$

If we use a plano-convex lens with the plane surface towards the object, calculations with this formula show that the minimum focal length consistent with the maintenance of good definition will be about 16 mm., giving a magnifying power of about 15.

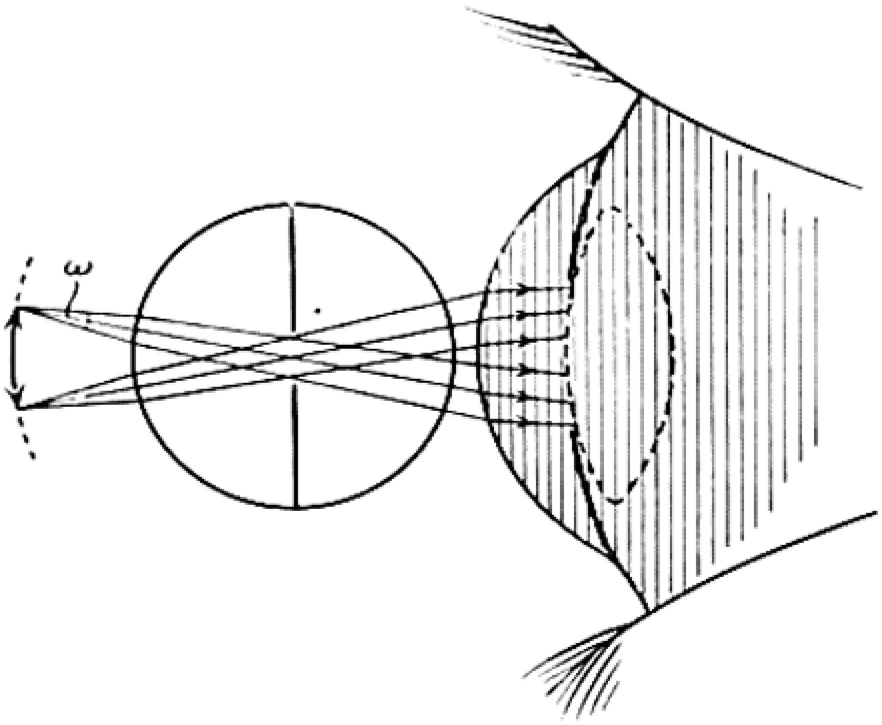


FIG. 6

Theoretical or practical tests soon show that a lens of 15 mm. focal length will have fairly pronounced chromatic aberration on the axis; this results in the presence of coloured haloes in the image of a very small source of light, such as a pinhole, and the colour is independent as to whether the lens is held with the curved or the flat side towards the object.

The "Steinheil" Magnifier. Spherical and chromatic aberration can be eliminated by using a biconvex crown-glass lens between two menisci of flint. The lens is designed after Steinheil's "aplanatic" magnifier or "lupe."

The symmetrical shape of the lens would tend to give it freedom from the aberrations of oblique pencils with regard to principal rays passing through the centre, somewhat as in the Coddington lens. When, however, the aberrations are reckoned with respect to a stop outside the lens represented by the pupil of the eye, or an imaginary stop at the centre of rotation of the eye-ball, they may not appear in such a good light. The lens is shown in Fig. 7. The magnification is about × 6.

Symmetry is a very real advantage in a small pocket magnifier; in cases where it is necessary to obtain a compromise between the opposing claims of freedom from spherical aberration, and freedom from undue astigmatism of the oblique pencils, the double convex form of the lens may be frequently selected.

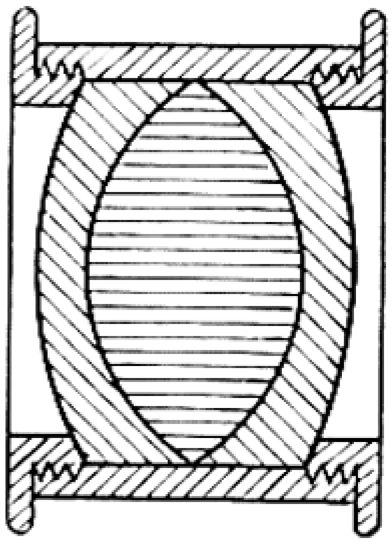


FIG. 7. STEINHEIL TYPE APLANATIC MAGNIFIER

The triple lens form most frequently employed in modern pocket magnifiers for magnification up to × 20

The majority of simple magnifiers are for hand use. They are preferably held quite close to the eye in order to secure the widest possible field in this "keyhole" type of observation. For dissecting purposes, however, "aplanatic" magnifiers of somewhat longer focus are mounted in simple mechanical stands, one of which is shown in Fig. 8.

In order to overcome the difficulty of the short working distance with the higher powers, *Chevalier* proposed, in 1839, to place a concave achromatic lens above the magnifying glass. A convenient form of the arrangement is shown in Fig. 9 (*a*); this is known as the Brücke lens, but is not often encountered; it is essentially a combination of a Galilean telescope with a long focus microscope objective. A more modern arrangement is to employ a prism erecting telescope system, similar to one member of an ordinary field glass, in combination with a microscope objective as "front lens attachment." Such systems are very convenient for naturalists and others. The use of the prism erecting system allows of a larger field than is possible with the Brücke system. The theory of the telescope systems will be dealt with in the following chapter.

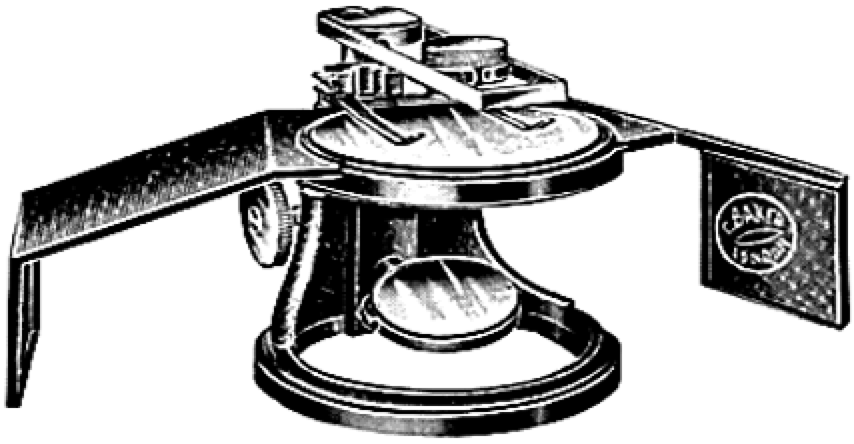


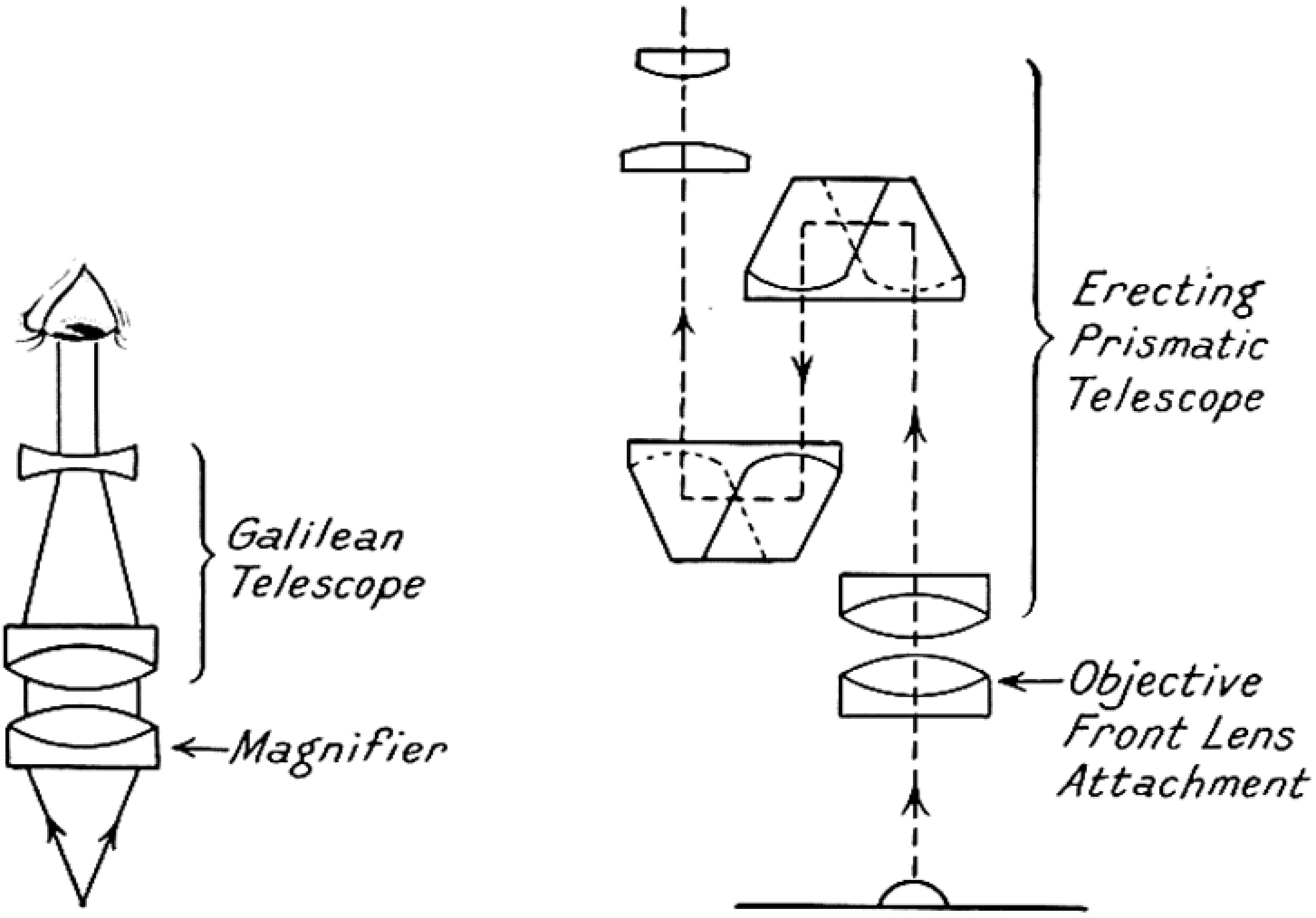
FIG. 8. DISSECTING STAND

(*C. Baker*)

Such telescopic magnifiers are adaptable for binocular vision, sometimes with the aid of suitable achromatic prism systems.

Depth of Focus and Depth of Field. It is convenient to consider some general properties of optical systems in this connection, such as depth of focus (image space) and depth of field (object space) because related questions arise not only with magnifiers but with the majority of optical instruments. In general optical systems are used to form images of objects or groups of objects in an extended field of view, projecting such images into a convenient field of vision or physical registration.

Sometimes the object points may lie in a single surface or approximately so, as when a picture is viewed by an instrument; more usually the objects are dispersed in three-dimensional space. The free image points are then correspondingly dispersed in three dimensions, but the light may be intercepted by a plane screen such as a photographic plate, etc. Each image point is illuminated by a cone of rays all originally derived from the conjugate object point, the cone having its base in the exit pupil of the system (Vol. I, p. 98) and its apex in the geometrical image; thus unless the plate intersects the image exactly there will be a finite ray-intersection disc in this plane. However, diffraction spreads the image even at the true focus, and the presence of the physical "depth of focus" in the image allows of some toleration in the position of the plate consistent with the appearance of "sharp" focus; such a tolerance has its counterpart in the finite range of object-distance within which there is substantially sharp focus on a fixed screen. (In the case of a magnifier we may think of the combination of the lens and the optical system of the unaccommodated eye; the fixed screen is then the retina.)



(a) Brücke Lens (b) Telescope Magnifier Combination

FIG. 9

If a real image is under very high magnification it is probable that the "Rayleigh limit" (Vol. I, p. 140) sets the limit of the depth of focus. If x' is the distance of the image from the second principal focus of the optical system, the tolerance $\Delta x'$ for the image distance can be written when α' is not too large (see equation (438), p. 140, Vol. I)—

$$\Delta x' \approx \pm \frac{\lambda}{2n'\alpha'^2}$$

where λ is the wave-length of the light, n' is the refractive index of the medium of the image space, and α' is the maximum inclination of a ray with respect to the principal ray. Using the law of axial magnification (Vol. I, p. 205) valid when Δx and $\Delta x'$ are very small we have (see Vol. I, p. 46)

$$\frac{\Delta x}{\Delta x'} = \left(\frac{h}{h'}\right)^2 \frac{n}{n'} = \left(\frac{1}{m}\right)^2 \frac{n}{n'}$$

where m is the image magnification h'/h and, by using the optical sine relation,

$$\frac{1}{m} = \frac{n'}{n} \frac{\sin \alpha'}{\sin \alpha}$$

Thus

$$\frac{\Delta x}{\Delta x'} = \frac{n'}{n} \cdot \frac{\sin^2\alpha'}{\sin^2\alpha}$$

The depth of field Δx corresponding to the "Rayleigh-limit" depth of focus is thus, when α is not very small

$$\Delta x \approx \pm \frac{\lambda}{2n \sin^2 \alpha} \quad . \quad . \quad . \quad (1.08)$$

and the total depth of field is $n\lambda/(NA)^2$, where (NA) is the "numerical aperture" defined as $n \sin \alpha$. However, if the object distance is large the tolerance is not correctly given by this last equation.

If the image is not viewed under great magnification it often happens that the depth of field and depth of focus depend upon the tolerable diameter of the "ray patch" presented to the eye.

Depth of Field. Let us consider with the help of Fig. 10 an optical system, having an entrance pupil of diameter a, which is forming, in the plane B′, a sharp image of all points in the plane B. Consider now a point P situated in the plane C. Rays from a limited area (diameter c) of the plane B can reach the entrance pupil by passing through P. Hence we see that the rays from the point P will intersect the image plane B′ in all parts of a disc corresponding to the image of the disc c. If the magnification is m, the diameter of this image $= mc$.

Similarly, the image of a point Q in the plane A will correspond to that of a finite disc in the plane B.

First Criterion. If the criterion for sharpness of definition is merely such that the dimension of mc must not exceed a certain limit, the distances of the planes A and C from B will be limited also.

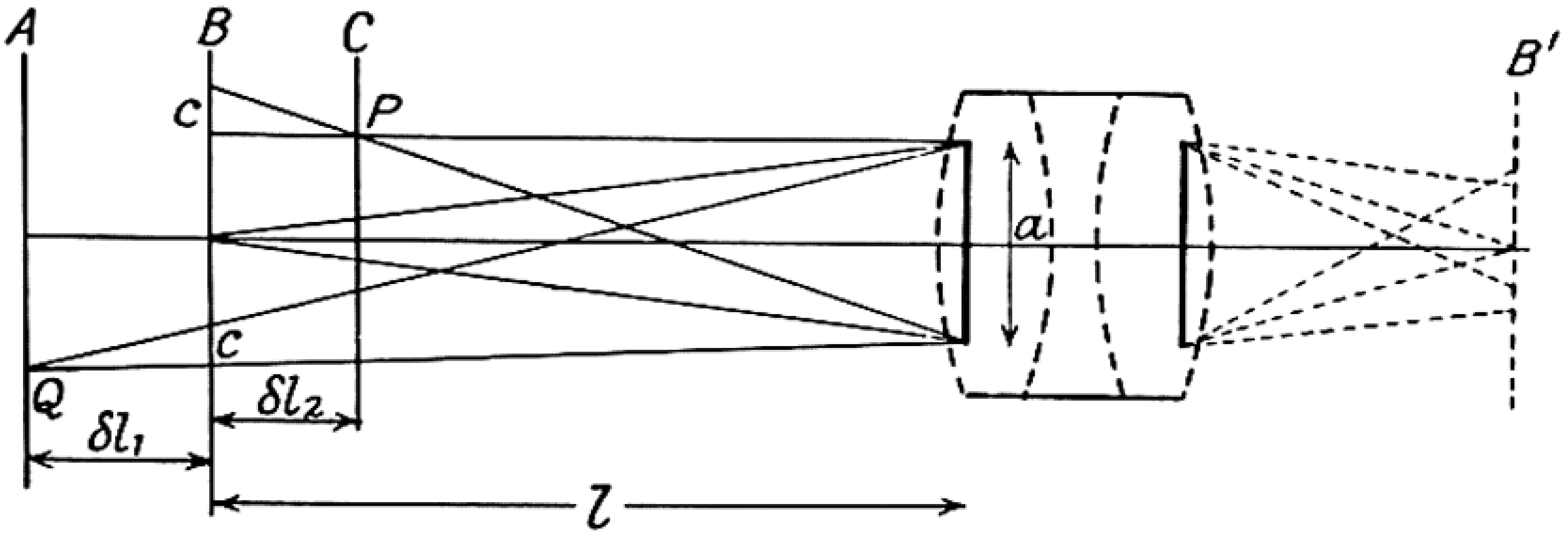


FIG. 10. THEORY OF FOCAL DEPTH IN THE OBJECT SPACE

Taking the dimensions shown in the diagram, we obtain from the similar triangles with a common apex in Q the relation

$$\frac{\delta l_1}{c} = \frac{l + \delta l_1}{a}$$

giving $$\delta l_1 = \frac{l}{a}\bigg/\left(\frac{1}{c} - \frac{1}{a}\right) = \frac{lc}{a - c} \quad . \quad . \quad . \ (1.09)$$

and from the triangles with their apex in P,

$$\frac{\delta l_2}{c} = \frac{l - \delta l_2}{a}$$

giving $$\delta l_2 = \frac{l}{a}\bigg/\left(\frac{1}{c} + \frac{1}{a}\right) = \frac{lc}{a + c} \quad . \quad . \quad . \ (1.10)$$

These equations give the depths of field on each side of the exact focus. The combined depth of field is given by adding the equations. Thus

$$\delta l_1 + \delta l_2 = lc\left(\frac{1}{a - c} + \frac{1}{a + c}\right) = lc\,\frac{2a}{a^2 - c^2}$$

$$\text{Depth of field} = \frac{2alc}{a^2 - c^2} \quad . \quad . \quad . \ (1.11)$$

In a case where c^2 is very small in comparison with a^2,

$$\text{depth of field} = \frac{2lc}{a} = \frac{c}{a/2l} \approx \frac{nc'}{m(NA)} \quad . \quad . \quad (1.12)$$

where c' is the tolerable diameter of the image disc, m is the magnification of the system, and (NA) is the numerical aperture, the refractive index of the object space being n.

Depth of Field Associated with Possible Changes of Accommodation of the Eye. The results of changes of accommodation are easily written down for the case of a magnifier held close to the eye. It may then be considered that the entrance pupil of the eye coincides with the posterior principal focus of the magnifier (or sufficiently nearly so). In Vol. I, p. 287, the amplitude of accommodation of the eye is expressed as

$$\frac{1}{k} - \frac{1}{b}$$

where k and b are, respectively, the distances of the far point and the near point. If, then, the object is held at such distances x_k and x_b from the anterior principal focus of the magnifier that the corresponding two values of x' are equal to k and b respectively, we have by Newton's equation

$$x_k = \frac{ff'}{k}$$

and

$$x_b = \frac{ff'}{b}$$

Hence

$$x_k - x_b = ff'\left(\frac{1}{k} - \frac{1}{b}\right) \quad . \quad . \quad . \quad (1.13)$$

Now if we take the "magnification of the lens" to be that obtained when the image occupies the near point we shall now have for the magnification

$$m = -\frac{b}{f'}$$

thus

$$f' = -\frac{b}{m}$$

and since

$$f = -\frac{nf'}{n'}$$

$$ff' = -\frac{nf'^2}{n'} = \frac{n}{n'}\left(\frac{b^2}{m^2}\right)$$

Hence
$$x_k - x_b = \frac{n}{n'}\left(\frac{b^2}{m^2}\right)\left(\frac{1}{k} - \frac{1}{b}\right) \quad . \quad . \quad (1.14)$$

Total Depth of Field. While the *possible* total depth of field of an instrument may depend on all of the factors previously mentioned, viz. the physical focus tolerance, the tolerance for the disc of confusion, and possible changes of accommodation of the eye when the image is viewed directly by the eye, the relative extent to which they are effective in practice varies greatly with the actual instrument employed.

In order to compare the results of these factors influencing the depth of field we may take the case of a simple magnifier of nominal magnifying power 12·5 (focal length 20 mm.). This is held conveniently so that the pupil of the eye (diameter 4·0 mm.) is at its posterior principal focus and the effective numerical aperture will thus be 0·1. For an object in air, assuming $\lambda = 0{\cdot}5\mu$, the "physical" depth of field suggested by the "Rayleigh limit" criterion will be 0·05 mm.

Now consider the depth of focus due to the tolerance for the disc of confusion. It is commonly taken that an image will appear tolerably sharp if the disc does not subtend an angle greater than about two minutes of arc (2′), or 1′ for very critical definition. The limiting disc at the nominal near point distance (250 mm.) must therefore be 0·145 mm. for the 2′ criterion, and the depth of field will be 0·116 mm. As against this, taking $k = \infty$ and $b = 250$, the depth of field due to possible changes in accommodation can rise to 1·6 mm. This is so great in comparison with either of the foregoing amounts that it is the only factor of real importance in the use of a mounted magnifier as in a dissection stand. Of course in using a simple magnifier it is usually very easy to adjust the position of the lens or object to bring any required detail into sharp focus.

REFERENCES

1. WHITWELL, A. Reference may be made to a series of papers published in *The Optician* from 1916 onwards. See also EMSLEY and SWAINE, *Ophthalmic Lenses*, p. 243 (Hatton Press).
2. *Brit. Pat.* 24, 009 (1953).
3. ALBADA, *Trans. Opt. Soc.*, **25,** 249 (1923–24).

CHAPTER II

THE TELESCOPE

Roger Bacon (1216–1294) was familiar with convex lenses, which were about this time beginning to be employed for spectacles. He states in his writings that—

> ". . . we can give such figures to transparent bodies, and dispose them in such order with respect to the eye and the objects, that the rays shall be refracted and bent towards any place we please, so that we shall see the object near at hand, or at any distance under any angle we please. And thus from an incredible distance we may read the smallest letters."

The above words may be interpreted as a veiled allusion to a telescope. Undoubtedly, they contain the germ of the scientific idea of the telescope, which is usually an instrument to project an image, of a distant object, which can be viewed under a greater angle than is possible with the unaided eye. Medieval philosophers were not in the habit of giving very explicit descriptions of what discoveries they made, for the ever present danger of being suspected for witchcraft or necromancy was associated with the exhibition of any unfamiliar phenomena.

Somewhat similar hints and allusions appear in the writings of Robert Recorde (1551), Battista Porta (1558), and others. The earliest circumstantial account of the actual construction of a telescope dates from 1590, when Zacharias Jansen, the son of a spectacle maker, Hans Jansen of Middelburg in Holland, is supposed to have invented the instrument. This testimony is due to the son of Zacharias.

There was also, however, another spectacle maker, Hans Lippershey, in the same town. It appears certain that he was in possession of the invention in the year 1608. A popular story ascribes the invention to the children of these two spectacle makers; they mounted a pair of lenses on a piece of wood in play, accidentally securing the correct distance between the lenses. However that may be, Lippershey undoubtedly pushed forward the invention and actually constructed binocular telescopes.

In June, 1609, Galileo heard of the invention without knowing any details of the construction. He returned to his laboratory at Padua, and in one day had made his first erecting telescope by mounting a plano convex and a plano concave lens in a short

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leaden tube. Subsequently, he made a number of telescopes of greater length and increased magnification of which examples are preserved in the Museo di Fisica, Florence[1].

Further historical points will best be given in the development of the theory of the instrument.

Elementary Theory. In its simpler form the telescope consists of two lenses mounted coaxially, but the single lenses may be replaced by systems of greater complexity for various reasons, as will be seen. We will, however, treat the telescope as consisting essentially of two main parts: the objective (*a*), and the eyepiece (*b*).

Imagine an object perpendicular to the axis which subtends a small angle α at the anterior principal focus F_a of the objective. Having given this angle we know that, wherever the image formed by the objective may be, its height will be given by

$$h'_a = f_a\alpha$$

where f_a is the first focal length of the objective.

The eyepiece acts as a simple magnifier. Turning to equation (1.02), Chapter I, we find that the angle under which the above image will be viewed (having given the power $\boldsymbol{F}_b$ of the magnifier, the refraction $\boldsymbol{K}$ of the eye, and the distance d between the adjacent principal points of magnifier and eye) will be

$$\omega = f_a\alpha\{\boldsymbol{F}_b(1 + d\boldsymbol{K}) - \boldsymbol{K}\}$$

Since, as before,

$$\text{Magnifying power} = \frac{\text{Angular subtense of image}}{\text{Angular subtense of object}}$$

we may write (if α_o is the angular subtense of the *object* at the eye)

$$\text{Magnifying power of telescope} = \frac{f_a\alpha\{\boldsymbol{F}_b(1 + d\boldsymbol{K}) - \boldsymbol{K}\}}{\alpha_o}$$

This expression becomes greatly simplified if it is assumed that the object is sufficiently distant to make α and α_o sensibly equal, as is the case in the great majority of applications of the telescope. The expression then becomes

$$\text{Magnifying power} = f_a\{\boldsymbol{F}_b(1 + d\boldsymbol{K}) - \boldsymbol{K}\} \quad . \qquad (2{\cdot}01)$$

The distance l' of the image from the second principal point of the eyepiece or magnifier is $k + d$; hence the corresponding distance l of the conjugate point for the magnifier system is given by

$$\frac{1}{k + d} - \frac{1}{l} = \boldsymbol{F}_b$$

This gives the position of the intermediate image formed by the objective and viewed by the eyepiece. We take the case when it coincides with F'_a. Then the distance

$$F'_aF_b = g = -(l - f_b)$$

In the usual case $f'_b = -f_b$, so that

$$g = -(l + f'_b)$$

The above equation gives

$$\frac{1}{l} = \frac{1}{k+d} - \mathbf{F}_b = \frac{1 - \mathbf{F}_b(k+d)}{k+d}$$

$$g = -\left\{\frac{k+d}{1 - \mathbf{F}_b(k+d)} + \frac{1}{\mathbf{F}_b}\right\}$$

$$= -\left[\frac{1}{\mathbf{F}_b\{1 - \mathbf{F}_b(k+d)\}}\right]$$

$$= -\left[\frac{\mathbf{K}}{\mathbf{F}_b\{\mathbf{K} - \mathbf{F}_b(1 + d\mathbf{K})\}}\right]$$

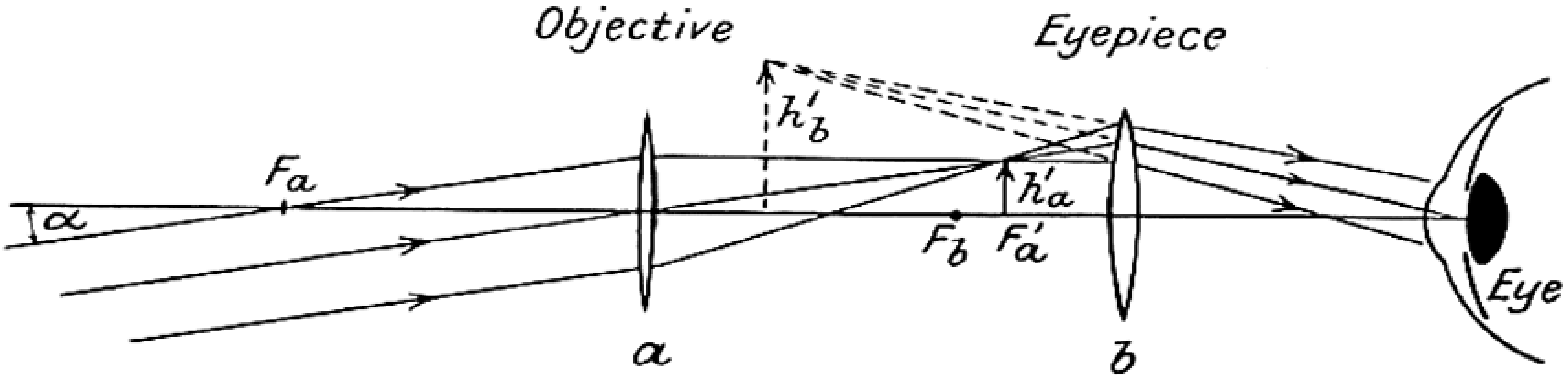

FIG. 11. COURSE OF RAYS IN THE TELESCOPE

But the Gaussian "power" of the instrument, equation (209), Vol. I, is equal to

$$\mathbf{F} = -\mathbf{F}_a\mathbf{F}_b g = \frac{\mathbf{F}_a\mathbf{K}}{\mathbf{K} - \mathbf{F}_b(1 + d\mathbf{K})}$$

Hence from equation (2.01) above,

$$\text{Magnifying power} = \frac{\mathbf{K}}{\mathbf{F}} = \frac{\text{Principal point refraction of eye}}{\text{Power of the system}} \qquad (2.02)$$

The "Infinity Adjustment." Further, an important case arises in which the instrument is adjusted to present a sharp image to the normal unaccommodated eye. In this case $\mathbf{K} = 0$. Hence

$$\text{Magnifying power} = f_a\mathbf{F}_b$$

Almost invariably, the objective is in air, so that $f'_a = -f_a$. Hence

$$\text{Magnifying power} = -\frac{f'_a}{f'_b} \quad . \quad . \quad . \quad (2.03)$$

The above formula is easily derived in the simplest case from inspection of a Gaussian diagram.

Fig. 11 shows the course of the rays in the telescope when a virtual image at a finite distance is presented to the eye. If, however, the refraction of the viewing eye is to be zero, it must receive parallel light. This case is shown in Fig. 12, in which F'_a and F_b are coincident. A parallel bundle of rays enters the objective, is focused in the common focal plane, and diverges to the eyepiece from whence it emerges once more parallel.

Tracing one ray through F_a it is seen to pass through F'_b; it is evidently parallel to the axis between objective and eye. The angle between the axis and the incident bundle of rays is

$$\tan \alpha = \frac{P_aQ_a}{P_aF_a} \text{ (numerically negative in the figure)}$$

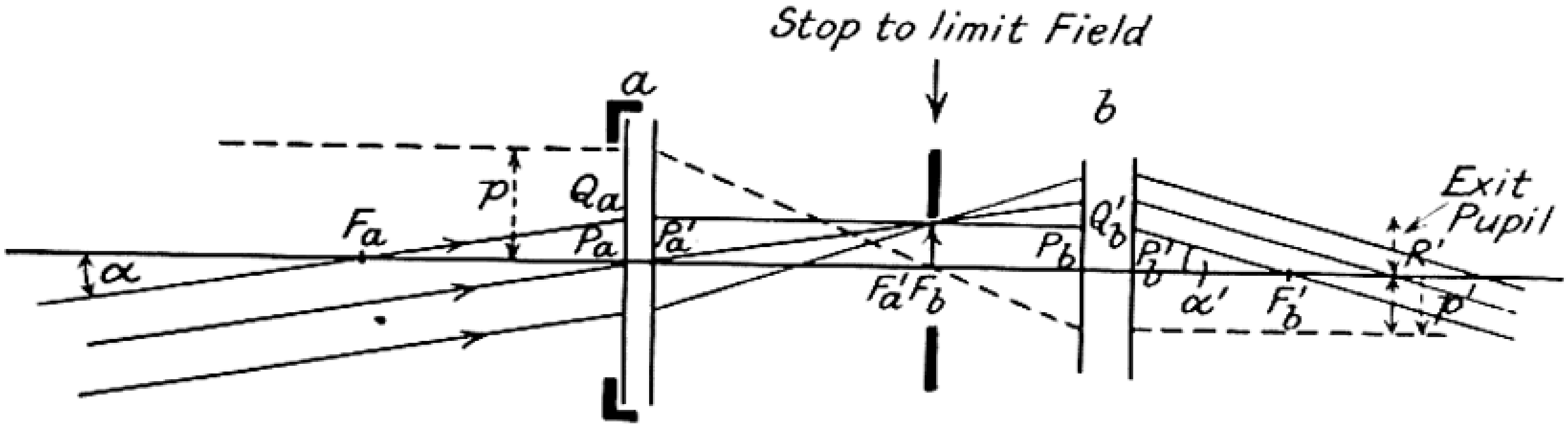

FIG. 12. ASTRONOMICAL TELESCOPE
(Diagrammatic)

The angle between the axis and the emergent bundle is

$$\tan \alpha' = \frac{P'_bQ'_b}{P'_bF'_b} \text{ (numerically positive in the figure)}$$

The ratio of the tangents of the angles becomes equal to the ratio of the angles themselves when they are small, and clearly gives an equally valid measure of the magnification; it is as good an approximation to relate the impression of "sizes" of the retinal image to the tangent of the angle of subtense as to the angular measure; neither is strictly accurate in the general sense; it is ultimately a matter of verbal definition.

$$\frac{\tan \alpha'}{\tan \alpha} = \frac{P_aF_a}{P'_bF'_b} = -\frac{f'_a}{f'_b}, \text{ in the usual case.}$$

The ratio of the two focal lengths is therefore a measure of the magnifying power. It follows that—

1. To obtain high magnification the focal length of the objective must be great in comparison with that of the eyepiece.

2. If the second focal lengths or the dioptric "powers" of the objective and eyepiece in such a simple instrument are of the same sign, the image will be inverted. This is the case in the astronomical telescope.

3. If the second focal lengths or the dioptric powers of the eyepiece and objective have opposite signs the image will be erect. This is the form of telescope invented by Galileo, and also, presumably, in Holland, though the details of the earliest Dutch instruments are not certainly known.

The Galilean Telescope. The action of the Galilean glass is illustrated in Fig. 13. The eyepiece *b* is now a negative or diverging lens, so that the common focal plane containing F'_a and F_b is shown behind the eyepiece. A parallel bundle of rays is focused towards the image point B′ in the common focal plane, but is intercepted by the eyepiece before reaching the focus and rendered parallel once again. The image will evidently be erect. If the separation of the lenses be diminished, the virtual image seen by the aid of the instrument can be formed at a finite distance. The formulae for magnification hold good in the present case.

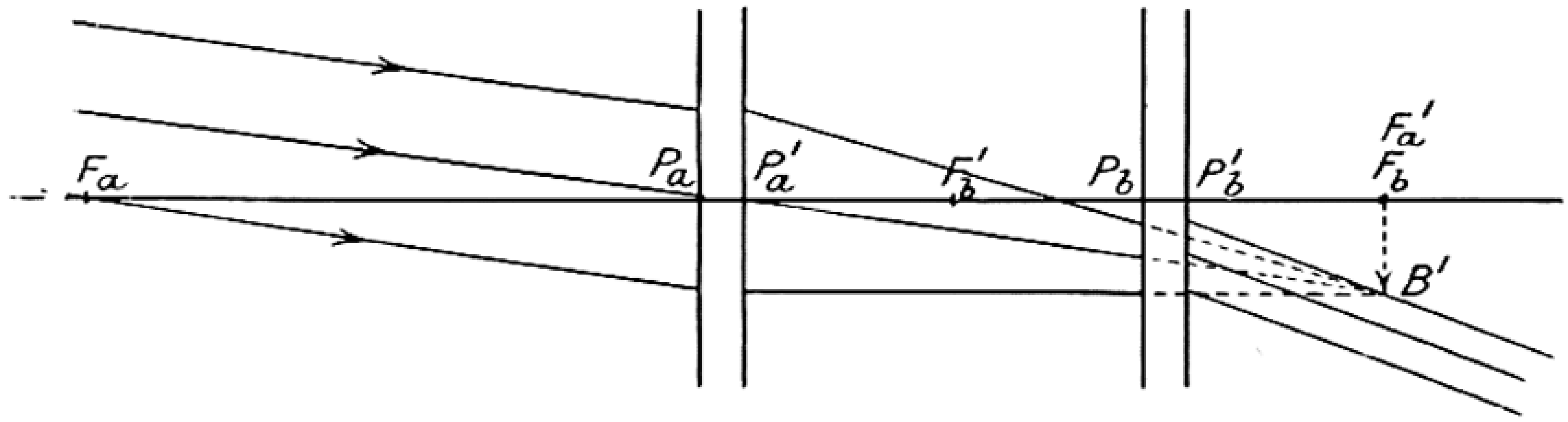

FIG. 13. GALILEAN TELESCOPE
(Diagrammatic)

The Pupils of the Telescope System. The entrance pupil of the system is usually considered with reference to an infinitely distant object point. Referring back to Fig. 12, the objective "*a*" is shown diagrammatically as limited in diameter. It is usually the case that a ray parallel to the axis passing, as shown, through the extremity of the objective, is transmitted unhindered by the eyepiece. If, therefore, we follow the usual plan for finding the entrance pupil by determining the images of the various diaphragms and lens rims, etc., formed by the parts of the instrument lying to the left of each such diaphragm, it will be found that all these images have a greater diameter than the boundary of the objective if the above condition regarding the ray is fulfilled.

The entrance pupil is therefore usually represented by the boundary of the objective itself, and the exit pupil is the image of this rim formed by all parts of the system lying to the right of it. The axial position of the exit pupil will be found by tracking through the system a ray from the centre of the entrance pupil.

Such a ray is exemplified in the diagram by that through P_a. It cuts the axis in R′ beyond the eyepiece. The radius of the exit pupil is found by tracing a ray from the boundary of the entrance pupil through the system, and finding the axial distance of its intersection of the plane through R′ perpendicular to the axis.

Let p and p' be the radii of the entrance and exit pupils respectively, then inspection of Fig. 12 shows that (for systems in air)

$$\frac{p}{p'} = -\frac{f'_a}{f'_b}$$

in the case of the ordinary adjustment when F'_a coincides with F_b.

But we found above that

$$\text{Magnifying power} = -\frac{f'_a}{f'_b}$$

thus we get also the new result (using diameters now instead of the radii of the pupils)

$$\text{Magnifying power (numerical)} = \frac{\text{Diameter of entrance pupil}}{\text{Diameter of exit pupil}}$$

For purposes of actual measurement, it may be noted that a real object of height p_1 anywhere in the object space, measured perpendicular to the axis of the telescope, must have an image of height p_1', where

$$-p_1' = p_1\left(\frac{f'_b}{f'_a}\right)$$

which is seen at once by tracing a ray through the top of the object parallel to the axis in the object space. Hence the magnifying power is $\frac{p_1}{p_1'}$. In one very useful practical method of measuring the magnifying power, the real object may be conveniently represented by the points of a pair of dividers opened out to a convenient extent (p_1), and placed immediately in front of the objective. The images of the points will be found just within the position of the exit pupil, and their separation (p_1') can be measured with the aid of a suitable scale mounted with a magnifier, or with a small travelling microscope. This is a very reliable and convenient method of measuring the magnifying power of any small telescope.

In another method the whole objective is illuminated by light diffused from a sheet of paper which more than covers its aperture, when the exit pupil can be observed as a uniformly illuminated disc behind the eyepiece, and its diameter can be measured (as above) with a small scale observed by a magnifier.

Object at Finite Distance. It may be the case that the instrument is being used with an object at a finite distance, and that the image is formed in the accommodation point at a distance k from the exit pupil or eye-ring *where the observing eye is situated.* This condition is represented in Fig. 14.

Trace a ray from the axial point of the object through the boundary of the entrance pupil. It must, therefore, pass through the boundary of the exit pupil, and its axial intersection point (real or apparent) must be in the final image plane at a distance k. Let h and h' be the perpendicular dimensions of object and image respectively.

The Lagrange relation gives

$$nha = n'h'\alpha'$$

where α and α' are the angles made with the axis by the above incident and emergent rays through the axial points of object and image. If p and p' are the radii of entrance and exit pupils respectively, and l is the distance of the object from the entrance pupil,

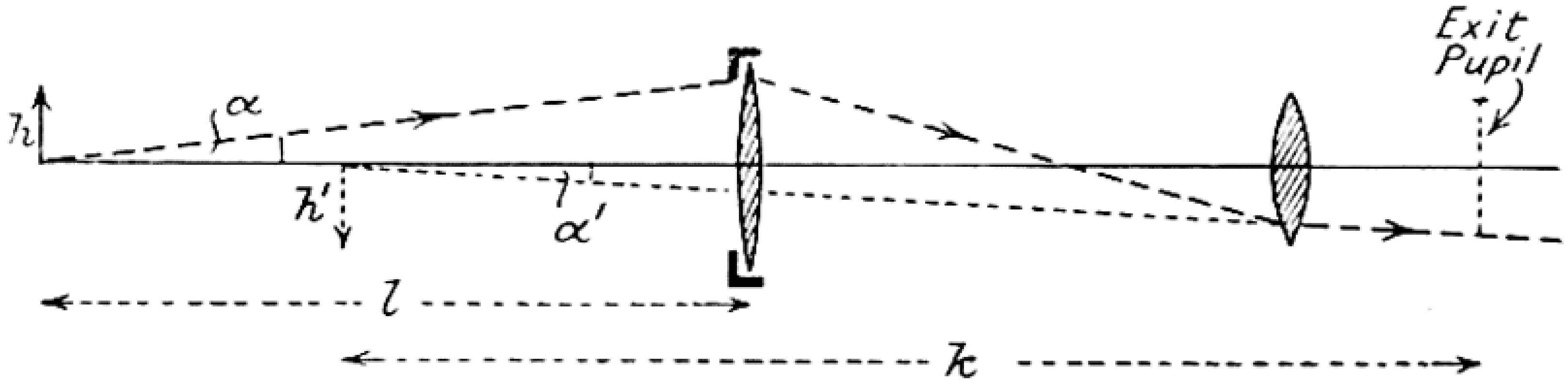

FIG. 14

$$\alpha = \frac{p}{l} \text{ (very nearly)}$$

$$\text{and } \alpha' = \frac{p'}{k}$$

Hence (since n and n', the refractive indices, are almost invariably unity, and will now be assumed to be so)

$$\frac{hp}{l} = \frac{h'p'}{k}$$

$$\text{or } \frac{p}{p'} = \frac{h'/k}{h/l}$$

$$= \frac{\text{Angular subtense of image at the eye}}{\text{Angular subtense of the object at the entrance pupil of the instrument}}$$

We may, therefore, look upon the ratio of the diameters of the entrance and exit pupils as representing the "magnifying power" in the above sense, even though the telescope may not be in the adjustment for infinitely distant object and image.

Telescopes using Prismatic Magnification. When a parallel beam of light is refracted obliquely at a refracting surface the perpendicular distance t between corresponding rays is changed according to the relation

$$\frac{t}{\cos i} = \frac{t'}{\cos i'}$$

where i and i' are the corresponding angles of incidence and refraction. Let AB, Fig. 15, be a distance t; AB represents a part of an incident wave front and $A'B' = t'$ represents a similar part of the refracted wave. Now suppose a new wave has an angle of incidence greater than i' by an amount di, and is represented by AC, the corresponding refracted wave as it passes through A′ will have some section A′C′ with an angle of refraction increased by di'.

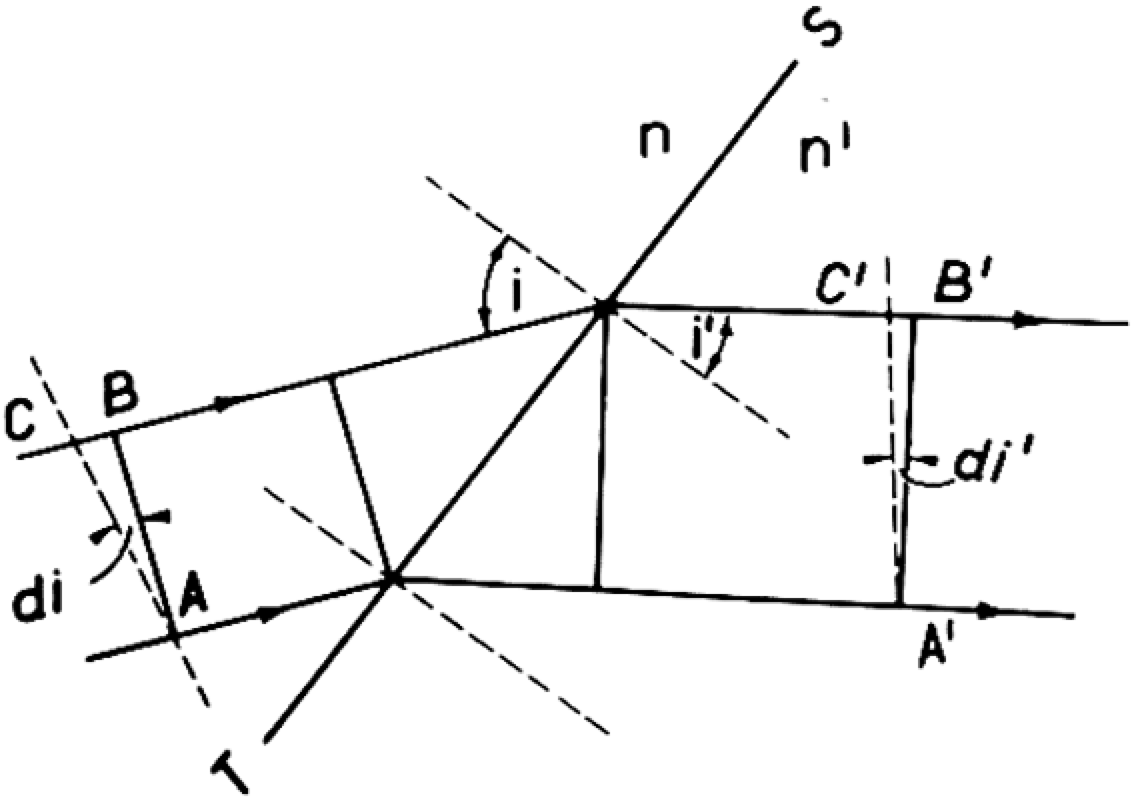

FIG. 15

Now it can readily be shown on the basis of Fermat's principle (Martin, *Geometrical Optics*, p. 113), that if C and C′ are points where the ray through B and B′ intersect the new wave front, then provided di and di' are very small, the optical path CB is equal to C′B′; this gives

$$nt\,di = n't'\,di'$$

and the angular magnification ratio produced by refraction at the surface is

$$\frac{di'}{di} = \frac{nt}{n't'} = \frac{n \cos i}{n' \cos i'} \quad . \quad . \quad . \quad (2.04)$$

Hence after any number of refractions (p, say), the angular magnification will be

$$\frac{di_p'}{di_1} = \frac{n_1}{n_p} \prod_1^p \left(\frac{\cos i}{\cos i'}\right) = \frac{n}{n'_p} \frac{t_1}{t'_p} \quad . \quad . \quad (2.05)$$

where the product of all the cosine ratios is involved.

Symmetrical refraction of a beam of parallel light by a prism does not produce any angular magnification, since the widths of the beam on entering and leaving are identical. However, if the angle of incidence on the first face of a prism of small angle is less than that for minimum deviation it will readily be seen that the width of the beam is diminished; consequently the prism produces an angular magnification in the tangential direction. The consequent angular *deviation* (and accompanying dispersion) tend to be annulled by a suitable prism placed with its apex in the reversed direction (Fig. 16A).

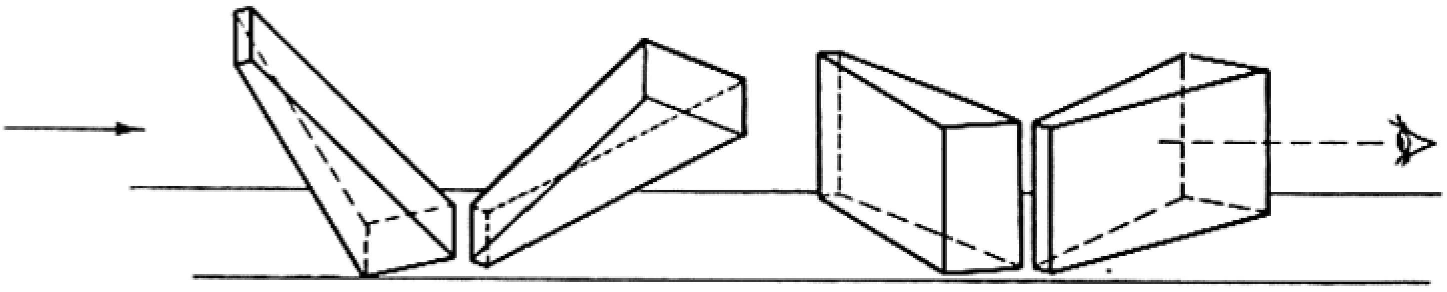

FIG. 16A. BREWSTER'S TEINOSCOPE

Sir David Brewster (*Treatise on Optics*, 1831) described a "teinoscope," in which two such pairs of prisms are employed; one pair magnifies in the horizontal direction, the other pair in the vertical. Each pair gives a lateral shift of the image. If the angular *direction* is to be maintained it is preferable to make each prism achromatic; then mutual rotation of the members of a pair can be made so to alter the angles of incidence that their magnification is variable. It is worth noticing that similar magnification laws are associated with the images characteristic of the diffracted beams produced by diffraction gratings.

Anamorphotic Systems. The demand for wider screen pictures in cinematographic projection has been met without discarding standard photographic equipment by the use of "anamorphs"; these are auxiliary "telescopes" arranged, however, to minify or magnify in the horizontal direction only. Thus, if the original photograph is made by the use of a minifying system (magnification, say, ½) then twice the usual horizontal width of view can be photographed on the standard film; projection is done with the corresponding auxiliary system reversed, so that the distortion of the film picture is cancelled and the projected picture has a width twice as great as usual. This scheme is used in the "Cinemascope."

Prism anamorphotic systems are generally employed, in spite of the slight disadvantage of the lateral shift of axis. It is essential that where high definition is required the rays from any object point shall be strictly parallel when passing through the system; hence the light from a near object has to be collimated; in the system used by Wray (Fig. 16B) this is done by a very lower power two-lens system; focus is effected by the varying of the separation of the lenses. The camera objective proper is kept focused "for infinity."

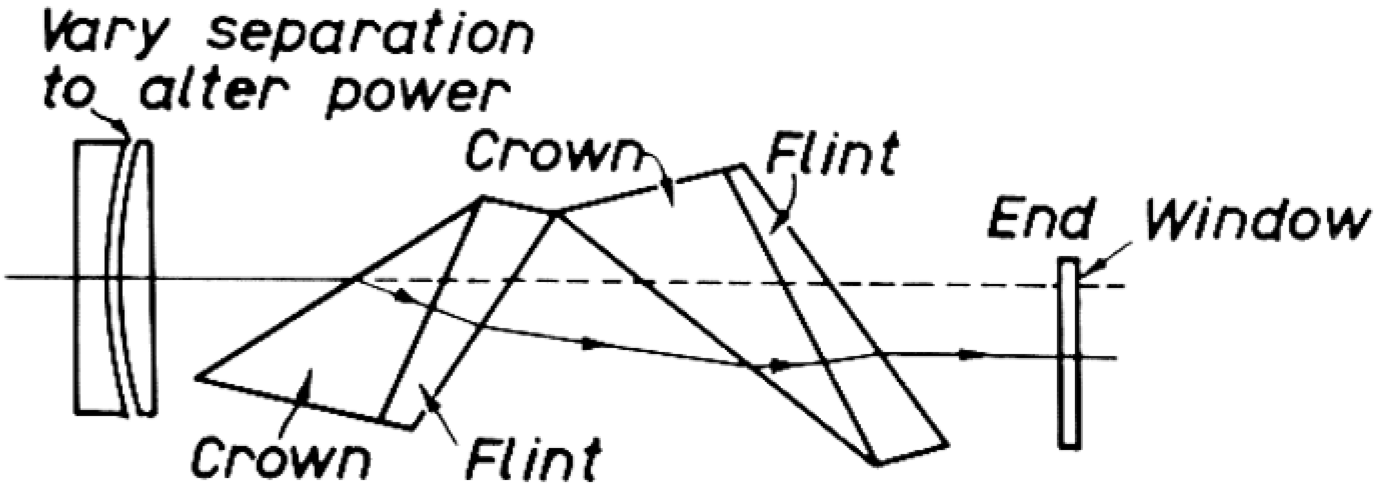

FIG. 16B. PRISM ANAMORPH

Cylindrical anamorphs are usually low-power auxiliary Galilean telescopes made with cylindrical lenses with their cylinder axes parallel. These have the advantages that there is no lateral shift of the axis, and the focusing can of course be readily effected by the separation of the positive and negative members. The original description of such a system was given by Chrétien.[2] References to the early literature of cylindrical lens systems are given by von Rohr,[3] and the primary aberrations of cylindrical lens systems have been discussed by Wynne,[4] who showed that instead of the five primary aberrations characteristic of a system of coaxial spherical surfaces, the cylindrical system (with cylinder axes co-planar) will in general have sixteen aberrations for object and image at finite distances; however, ten of these are zero for the special case of the auxiliary anamorphotic telescope. It is possible of course to produce real images with magnifications differing in the principal azimuths by systems of crossed cylinders, but the correction of such systems is a difficult matter.

In spite of the symmetry of the cylinder anamorphs, they are not so successful as the prism type, as the field aberrations are troublesome; moreover the production of optically accurate cylinders is attended with much difficulty. A type of mirror anamorph has been described by the firm De Oude Delft.[5]

Position of Principal and Focal Points of the System. The formulae of Chapter II, Vol. I, enable us directly to calculate the position of the principal and focal points of a combination of two optical systems.

If a telescope is in the adjustment when F'_a coincides with F_b, then the intercept "g" between the adjacent foci is zero. The formulae

$$f = \frac{f_a f_b}{g} \qquad P_aP = \frac{f_a d}{g} \qquad P_aF = \frac{f_a}{g}(d + f_b)$$

$$f' = -\frac{f'_a f'_b}{g} \qquad P_b'P' = \frac{f'_b d}{g} \qquad P'_bF' = \frac{f'_b}{g}(d - f'_a)$$

show that the focal lengths of the system are infinite, and also that the principal and focal points are at an infinite distance. The combination is said to be in *afocal adjustment*. Hence the general methods of discussing the optical performance of the system (position of principal planes, focal length, etc.) find very little application.

Telescope Objectives. The early work of *Chester Moor Hall* (1733) and *Dollond* (1758) on the achromatic object glass, and the development of the telescope up to the time of Fraunhofer, can only be mentioned here. Fraunhofer's re-discovery of the dark lines of the solar spectrum and their use for exact measurements of refractive index first placed the matter of achromatism on a definite basis. Fraunhofer achromatized his telescope objectives in the following way. He divided the solar spectrum into regions bounded by the lines A, B, C, D, E, F, and G, and then measured the ratio of the flint dispersion to the crown dispersion for each region. He adopted a mean value in which the ratios were weighted according to the amount of light in each region; thus this work involved one of the earliest essays in heterochromatic photometry. The mean ratio gave a measure of the ratios of the total curvatures $(\mathbf{R}_1 + \mathbf{R}_2)$ for each component. This condition resulted in lenses which had their minimum foci somewhat towards the blue end of the spectrum, and Fraunhofer found empirically that a better result was obtained by a slight change in the ratio which resulted in a minimum focus nearer the red. The condition of bringing together the foci for C and F (introduced somewhat later) ensures that the minimum focal length of an ordinary objective falls in the apple-green region of the spectrum, and produces the maximum bunching together of the radiations which are brightest to the eye. Subsequent systematic work[6] reveals little to be desired in this provision for visual observation.

The type of objective manufactured by Fraunhofer is shown in Fig. 17. It consists of a double convex crown and a nearly plano-concave flint. Primary chromatic aberration is corrected in the above manner, and spherical aberration is corrected for one zone. The lens is practically free from coma, although it must be doubted whether Fraunhofer knew of the "sine condition." It is the leading type of all small telescope objectives.

If such lenses are designed with the same radius on the adjacent faces, the contact may be cemented. This helps cleanliness and avoids internal reflections. Chromatic and spherical aberration can still be removed, but freedom from coma can only be secured by the choice of suitable glasses.[7]

The absolute elimination of coma is not always considered essential by manufacturers of objectives for small theodolites where good central definition suffices. If the axis of the lens happens, however, to be slightly out of its proper alignment, troublesome coma may appear in the middle of the field. It is not usual to cement lenses with diameters greater than 2 in.

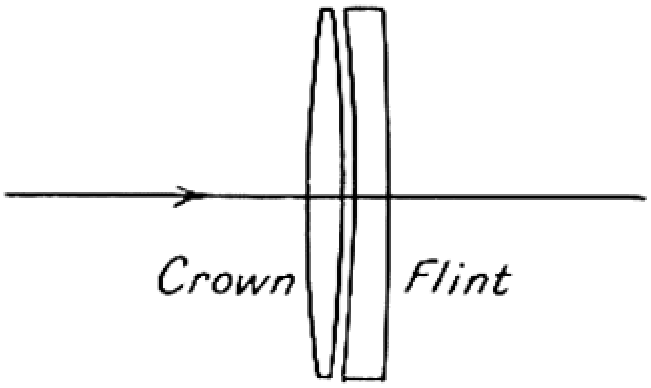

FIG. 17. FRAUNHOFER OBJECTIVE

Crown lens $N_D = 1{\cdot}531$
Flint lens $N_D = 1{\cdot}616$
$r_1 = +340$, $r_2 = -139$
$r_3 = -136$, $r_4 = -620$

Variations of the above general design are sometimes used for particular purposes. The "Steinheil" form of objective is shown in Fig. 18 (*a*). The flint iens faces the object, and somewhat steeper curvatures are necessary, but the freedom in removing aberrations is much the same as with the Fraunhofer type.

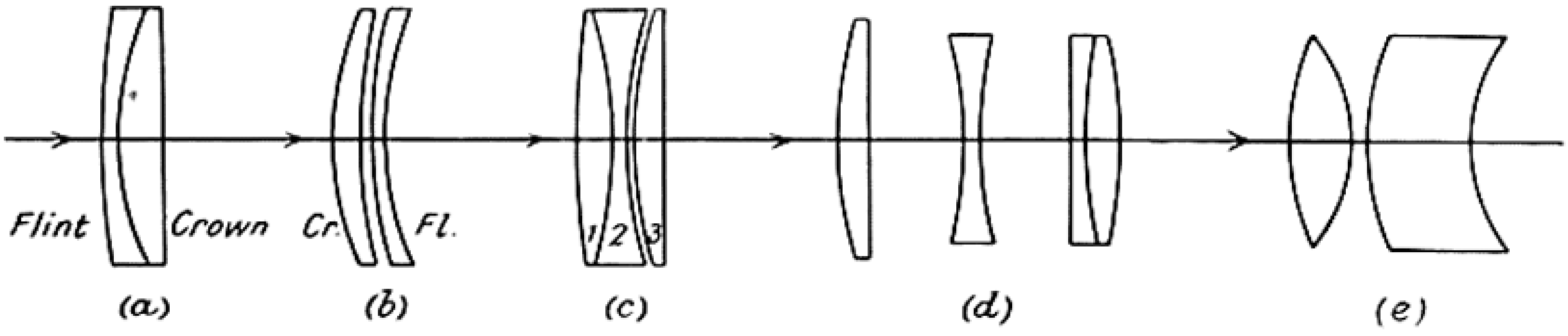

FIG. 18. TYPICAL TELESCOPE OBJECTIVES

(*a*) Steinheil
(*b*) Gauss
(*c*) Cooke
(*d*) Astrographic objective
Tessar type (Zeiss)
(*e*) Objective by Busch

Glasses in Cooke objective—
(1) Baryta light flint (O, 543)
(2) Borosilicate flint (O, 658)
(3) Crown (O, 374)
Jena glass numbers

The removal of spherical aberration for two wave-lengths is possible by bending both components of the achromatic combination, as suggested by Gauss. The profile of such an objective is shown in

Fig. 18 (*b*). Such a condition is, generally, secured only by the loss of freedom from coma. Such lenses have been used in large theodolites.

Herschel proposed objectives calculated for the removal of spherical aberration for two object distances, but this provision is not often deemed necessary.

An unusual telescope objective, Fig. 18 (*e*), has been used in a range-finder by Busch. It consists of two thick and well-separated components. The manner in which the thickening of a lens allows the increase of the power of the surfaces without a corresponding increase of the power of the lens, and the associated possibilities for the correction of the field, is discussed in the chapter on the Photographic Objective.

Large and Small Objectives. The modern manufacture of small telescope objectives of apertures up to 2 in. is now largely a matter of mass production. The data for the system will be computed before manufacture begins, and the optical performance will be expected to agree with calculation.

The manufacture of large objectives involves much greater uncertainty, and figuring by hand, sometimes with local rubbing, is usually necessary to correct (*a*) zonal spherical aberration, (*b*) errors in the regularity of the surfaces, (*c*) effects of lack of homogeneity in the glass.

The residual secondary spectrum of an astronomical objective is prejudicial in exacting observations. Considerations of cost and the chemical stability of glass usually call for a doublet of crown and flint for the largest refractors of aperture 10 in. or over, and the secondary spectrum must perforce be tolerated, but the attempts to reduce this in smaller lenses must be noted.

Reduction of Secondary Spectrum. The attempt to produce glasses from which a pair could be selected for the similarity of their run of partial dispersions and so eliminate the residual secondary spectrum of a doublet has already been discussed (Vol. I, p. 245). Failing the requisite chemical stability of suitable glasses, interest is given to the possibilities involved in separating the crown and flint components of a doublet so as to have a space between them. The partial dispersions of a typical "hard crown" and "dense flint" are—

	V	$b - C$	$e - F$	$g - h$
Hard crown . . .	60·4	0·175	0·456	0·449
Dense flint	36·0	0·163	0·468	0·504

Notice that the dispersion of the flint is relatively too low in the red, but too high in the violet. Separation of the crown and flint components of a telescope objective (Fig. 19) would allow of the use of a smaller flint lens of shorter focal length. At the same time the violet ray is more deviated than the red in passing the crown lens, so that the violet meets the "diverging lens" nearer the axis than the red. This clearly acts in the sense of reduction of the secondary spectrum, but the correction so attainable is not sufficient with a single lens as the rear member without making the curvatures of the lenses so great as to negative the advantage gained. A proposal by Rogers to use a doublet correcting lens which is of zero power for mean wave-lengths, and yet acts in a "divergent" sense as between the red and blue, solves the problem of the secondary spectrum only to introduce troublesome spherical aberration and chromatic differences of magnification. Lenses of similar type have been produced by Plössl and others under the name "dialyte objectives," but they have not come into very common use.

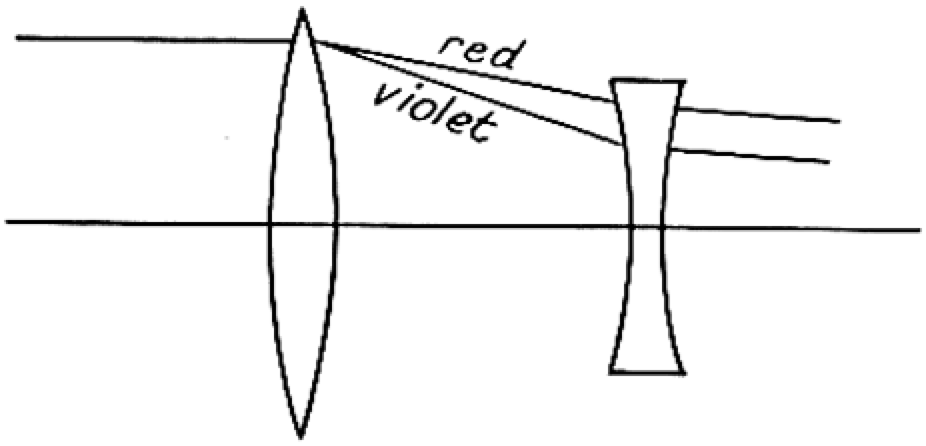

FIG. 19. ACTION OF SEPARATED COMPONENTS

There are certain advantages attainable by a *modest* separation of the lenses of a doublet objective, in spite of the failure appreciably to reduce the secondary spectrum. The objective follows the changes of external temperature with greater facility, and a suitably adjusted lens can be freed from spherical aberration for two colours. As will be understood from the formula quoted below, the shorter focal length of the flint lens tends in the direction of a flatter image field.

If d_1' is the separation, and F_1 and F_2 are the powers of the two lenses respectively, and V_1 and V_2 their constringence values; moreover y_1 and y_2 are the relative heights of intersection of a "marginal ray" derived from an axial object point at a distance l_1 from the first lens so that the vergence of the incident light is L_1, then we have

$$\frac{y_2}{y_1} = 1 - d_1' \, (F_1 + L_1)$$

$$\frac{F_1}{V_1}\left(\frac{y_1}{y_2}\right)^2 + \frac{F_2}{V_2} = 0. \qquad (2.06)$$

See Appendix IV, p. 381. This gives the necessary relation between the powers and constringences. Compare this with the equation on p. 242, Vol. I.

The most successful mode of securing the practical elimination of the secondary spectrum is by the use of a three-component lens in which we may have, for example, two positive components which together represent one lens of a glass having a run of partial dispersions similar to that of the negative component. Fig. 18 (*c*) shows the section of a Cooke-Taylor photo-visual objective.

The relative partial dispersions of the glasses used for the two positive components of the Cooke photo-visual objective are given in the following table. The mean of their partial dispersions is shown in the third line, while the fourth line gives figures for the boro-silicate flint used in the negative lens. The figures are taken from the original paper by H. Dennis Taylor, and give the refractive indices and partial dispersions on the older system. The glass numbers referred to the Jena Catalogue. The powers of the two positive components are very nearly equal, so that the effect of these lenses combined has a run of partial dispersion exceedingly close to that of the flint negative component.

Glass	n_D	V	Dispersion	Relative Partial Dispersions				
			C–F	A–C	D–F	E–F	F–G′	F–H
O 543	1·564	50·7	0·01115	0·3354	0·7085	0·3309	0·5830	1·1857
O 374	1·511	60·8	0·00844	0·3507	0·7026	0·3247	0·5675	1·1564
Mean				0·3420	0·7059	0·3282	0·5763	1·1730
O 658	1·546	50·1	0·01090	0·3425	0·7052	0·3278	0·5767	1·1745

The limits of size of astronomical refracting objectives are set in practice by the difficulties attendant on the production of discs of optical glass of large diameter. The Yerkes objective has a diameter of 1·02 metres and a focal length of 18·9 metres, and the Lick refractor a diameter of 0·91 metres and focal length 17·6 metres; these are large doublets. Generally speaking, the aperture ratio (aperture : focal length) decreases with the diameter. With very small lenses 1 : 4 may be reached, but 1 : 18 is as much as can be allowed with large refractors on account of the prominence of the secondary spectrum.

When the triplet lenses are used the necessary curvatures are rather large, and it is not easy to obtain a satisfactory performance with an aperture ratio better than 1 : 15. The Gauss condition (spherical correction for two wave-lengths) can be secured by separating two of the components. Such lenses have been made up to 12½ in. clear aperture.

The image given by a triplet apochromatic objective presents practically no trace of colour, and such a lens, if used for astronomical photography, may be focused visually.

Astrographic Objectives. If the correction is not to be apochromatic as in the photo-visual objectives mentioned above, the lens will be best adapted for astronomical photography if the "F" line focus is united with that for Hδ (wave-length = $0{\cdot}4101\mu$). This ensures a better bunching of the most "actinic" regions of the spectrum, using the term to refer to those wave-lengths which most affect an ordinary photographic plate. The use of such plates confines the effective light to a limited spectral band, and tends to secure better definition on that account. Modern photographic work in astronomy includes, however, work both with ultra-violet and infra-red radiations, so that the exact requirements for specialized work may be very varied. The optical design of astrographic objectives aims naturally at securing the largest possible *flat field* together with the indispensable high definition required for astronomical photography, which calls for freedom from spherical aberration, coma, and astigmatism. These requirements are met by lenses of which the design resembles photographic lenses, except in so far as they work at a very much smaller aperture ratio. While they imitate the flat field of the photographic lens, they allow of much better central definition. Lenses of the type of the Taylor triplet, and the Tessar (Carl Zeiss) (Fig. 18 (*d*)) are in use for this purpose. It is frequently the case, however, that astrographic work is done with doublet lenses very carefully designed, and made to be as free as possible from spherical aberration and coma.

The Image Field and its Curvature. The diagrams so far used show the ordinary flat image field of the elementary theory. In practice the phenomena of curvature of field and the accompanying degree of astigmatism are of the greatest significance. Little or nothing can be done to remove the astigmatism of an ordinary doublet telescope objective, and the curvature of the field and astigmatism with those objectives constructed of the usual types of glass, such as hard crown and dense flint, are finite and comparable with those of a single lens of the same focal length.

Thus for thin lenses in air the radius of the Petzval surface is

$$\frac{1}{R} = \Sigma\left(-\frac{1}{nf'}\right)$$

where n is the refractive index of a lens and f' the focal length. E. W. Taylor records an achromatic objective having—

Positive lens: Medium barium crown, $n_D = 1{\cdot}5736$, focal length = 3·543
Negative lens: Light flint, $n_D = 1{\cdot}6039$, focal length = 5·391

The focal length of the combination is 10·06; and the radii which can be found by calculation and checked by experiment are—

Tangential image field	2·85
Sagittal image field	6·05
Petzval surface	15·6

In accordance with theory, the tangential surface is three times the distance of the sagittal surface from the Petzval surface.

Such a curvature will not be of much significance in the small diameter intercepted by an eyepiece, but the curvature of the image field for the eyepiece has also to be taken into consideration.

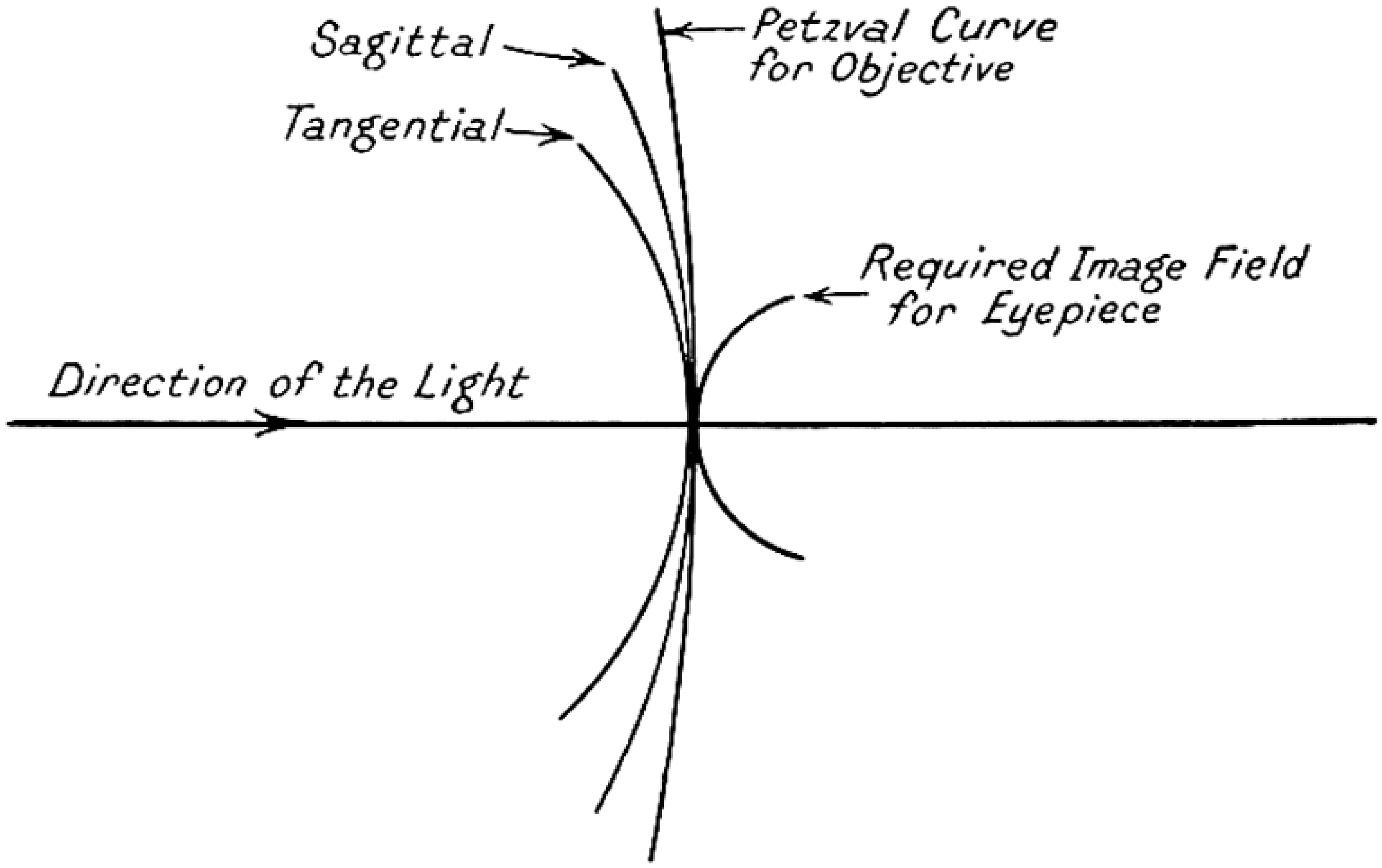
Sagittal
Tangential
Petzval Curve for Objective
Required Image Field for Eyepiece
Direction of the Light

FIG. 20

The cheaper forms (see below) are constructed of two separated lenses made out of the same glass. Take, for example, an Huygenian form in which the focal length of the field lens may be $3q$, say, while that of the eye-lens is q. Applying the formula above and assuming $n = 1{\cdot}5$ approximately

$$\frac{1}{R} = -\frac{1}{4{\cdot}5q} - \frac{1}{1{\cdot}5q}$$

The focal length for the Huygenian combination (p. 50) at a distance of $2q$ proves to be $\frac{3q}{2}$. This shows that $R = 0{\cdot}75f$, so that the

Petzval radius for an eyepiece of focal length 1 in. would be about three-quarters of an inch.

Object points on a surface of such curvature (assuming astigmatism absent) would give the parallel rays required to emerge from the eyepiece. In Fig. 20 is shown in full size the section of the image surfaces for the objective discussed above, assuming a focal length of 10 cm. The right-hand curve is that on which the image should lie if the field of the instrument is all to be "at infinity" when viewed with an eyepiece of focal length 1·3 cm. approximately (if we may assume the above theory, and also neglect the astigmatism of the eyepiece).* The gap between the two increases very rapidly with distance from the axis, but the physical depth of focus permits a certain limited region of the field to appear sharp. If the outer parts of the field are to be viewed in sharp focus, the eyepiece may be pushed inwards. The central parts of the field may then be viewed clearly by the exertion of accommodation. In attempting to observe this with actual telescopes, allowance will have to be made for the residual astigmatism which will be present. In eyepieces like the Huygenian, where the light has to pass through a larger field lens and then a smaller eye lens, it is possible to modify the amount and direction of the astigmatism by suitable curvatures of the lenses, and to produce over-correction.

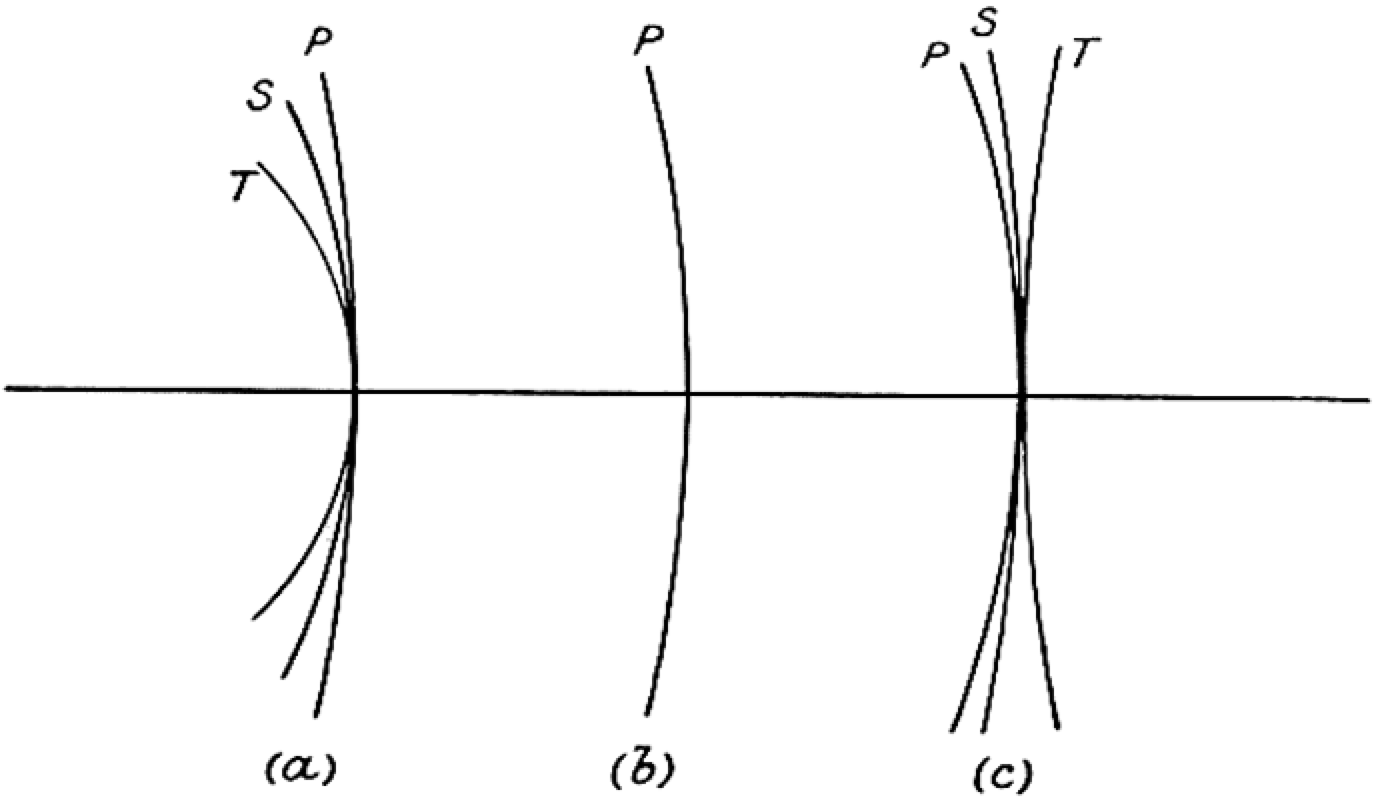

FIG. 21

(*a*) Under-corrected astigmatism
(*b*) Correction; images on Petzval surface
(*c*) Over-corrected astigmatism

N.B. The lens system is imagined to the left of these figures

Take Fig. 21 (*a*) to represent an ordinary case of curvature of the field and "under-corrected" astigmatism (as of the type which arises with a simple lens). If we change the curvatures or separations of the component lenses of a complex system without altering their focal lengths or refractive indices, we shall sometimes be able to alter the amount of the astigmatism, but not the Petzval curvature. In Fig. 21 (*b*) the astigmatism is eliminated, and the field has now the curvature of the Petzval surface. In Fig. 21 (*c*), the astigmatism is over-corrected in such a way as to make the mean curvature of tangential and sagittal surfaces zero, so that the field is "flattened" in that sense. The tangential field remains at three times the distance of the sagittal field from the Petzval surface. The possibilities mentioned in this paragraph do *not*, however, apply to ordinary close doublet telescope objectives. The subject will be again discussed in connection with photographic lenses.

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* Actually the eyepiece will be designed to have some over-corrected astigmatism which will flatten the field to some extent.

Design of a Doublet Telescope Objective. The rough design will be worked out for a doublet which is to satisfy (1) the ordinary condition of achromatism, (2) freedom from spherical aberration for some wave-length of the spectrum. If the lens is to be cemented the degrees of freedom will then be exhausted, but if the components may have different radii for the adjacent surfaces, the condition of freedom from coma may be added. Take the following glasses—

	n_D	V
Hard crown	1·5186	60·3
Dense flint	1·6041	37·8

the use of equation (803), Vol. I, then gives for the necessary powers of the components, denoted by the suffixes $_a$ and $_b$.

$$F_a = 2{\cdot}68 \qquad\qquad F_b = -\,1{\cdot}68$$

The formula for the spherical aberration coefficient of a thin lens, (412), Vol. I, shows it is proportional to A, where

$$\mathrm{A} = F^3\left(\frac{n}{n-1}\right)^2 + F^2L_1\left(\frac{3n+1}{n-1}\right) - F^2R_1\left(\frac{2n+1}{n-1}\right) + FL_1^2\left(\frac{3n+2}{n}\right) - FR_1L_1\left(\frac{4n+4}{n}\right) + FR_1^2\left(\frac{n+2}{n}\right)$$

In the case of the first lens we take $L_1 = 0$, and hence

$$\mathrm{A}_a = F^3{}_a\left(\frac{n}{n-1}\right)^2 - F^2{}_aR_1\left(\frac{2n+1}{n-1}\right) + F_aR_1^2\left(\frac{n+2}{n}\right)$$

Calculating the requisite numerical coefficients (by four-figure logs).

$$\mathrm{A}_a = 165{\cdot}0 - 55{\cdot}91\ R_1 + 6{\cdot}21\ R_1^2$$

It is more convenient, however, to express the relation in terms of $\mathbf{R}_2$ where

$$\mathbf{F} = (n_a - 1)\ (\mathbf{R}_1 - \mathbf{R}_2)$$

and
$$\mathbf{R}_1 = \mathbf{R}_2 + 5{\cdot}168$$

We then obtain

$$A_a = 42{\cdot}0 + 8{\cdot}27\ \mathbf{R}_2 + 6{\cdot}21\ \mathbf{R}_2{}^2 \quad . \quad . \quad . \quad (a)$$

For the second lens, $\mathbf{L}_1 = \mathbf{F}_a = 2{\cdot}68$, and $\mathbf{F}_b = -\ 1{\cdot}68$. The equation, when simplified, reduces to

$$A_b = -\ 11{\cdot}91 + 9{\cdot}58\ \mathbf{R}_3 - 3{\cdot}77\ \mathbf{R}_3{}^2 \quad . \quad . \quad (b)$$

In order that the lens may be cemented it will be necessary for $\mathbf{R}_2$ to be equal to $\mathbf{R}_3$. Hence if the sum of the aberrations is to be zero

$$A_a + A_b = 0 = 30{\cdot}09 + 17{\cdot}85\ \mathbf{R} + 2{\cdot}44\ \mathbf{R}^2$$

a simple quadratic equation in $\mathbf{R}$ having roots $-\ 4{\cdot}67$ and $-\ 2{\cdot}64$. This determines the forms of two possible lenses for cementing, although one (for reasons given below) is much preferable to the other.

First lens:

$$\mathbf{R}_1 = 0{\cdot}498 \qquad\qquad \mathbf{R}_1 = 2{\cdot}52$$
$$\mathbf{R}_2 = \mathbf{R}_3 = -\ 4{\cdot}67 \qquad\qquad \mathbf{R}_2 = \mathbf{R}_3 = -\ 2{\cdot}64$$
$$\mathbf{R}_4 = -\ 1{\cdot}885 \qquad\qquad \mathbf{R}_4 = 0{\cdot}145$$

It may be deemed that the contact should be left uncemented in order to satisfy the condition for the absence of coma by a possible difference of curvature of the adjacent faces. The most instructive method to proceed is to plot the values of the aberration found from equations (*a*) and (*b*) above in a graphical diagram. The following table gives points for plotting—

R_2 or R_3	$-\ 5$	$-\ 4$	$-\ 3$	$-\ 2$	0	$+\ 2$	$+\ 4$
A_a .	155·75	108·22	73·08	50·30	42·0	83·38	174·38
A_b .	− 154·01	− 110·55	− 74·58	− 45·15	− 11·91	− 6·83	− 33·91

As is clearly seen from the equations, we obtain two parabolas (Fig. 22) with vertical axes. It is convenient to plot A_b values with reversed sign. The parabolas then intersect in the two points with abscissae $-\ 4{\cdot}67$ and $-\ 2{\cdot}64$ corresponding to the roots of the equation above.

Coma. By an extension of the method given in Chapter IV, Vol. I, we can investigate the value of the coefficient a_2 in equation (425) of that chapter, i.e. the "coma" coefficient. Conrady has shown that it is proportional to

$$\frac{N+1}{N}\,\boldsymbol{FR}_1 - \frac{2N+1}{N}\,\boldsymbol{FL}_1 - \frac{N}{N-1}\,\boldsymbol{F}^2$$

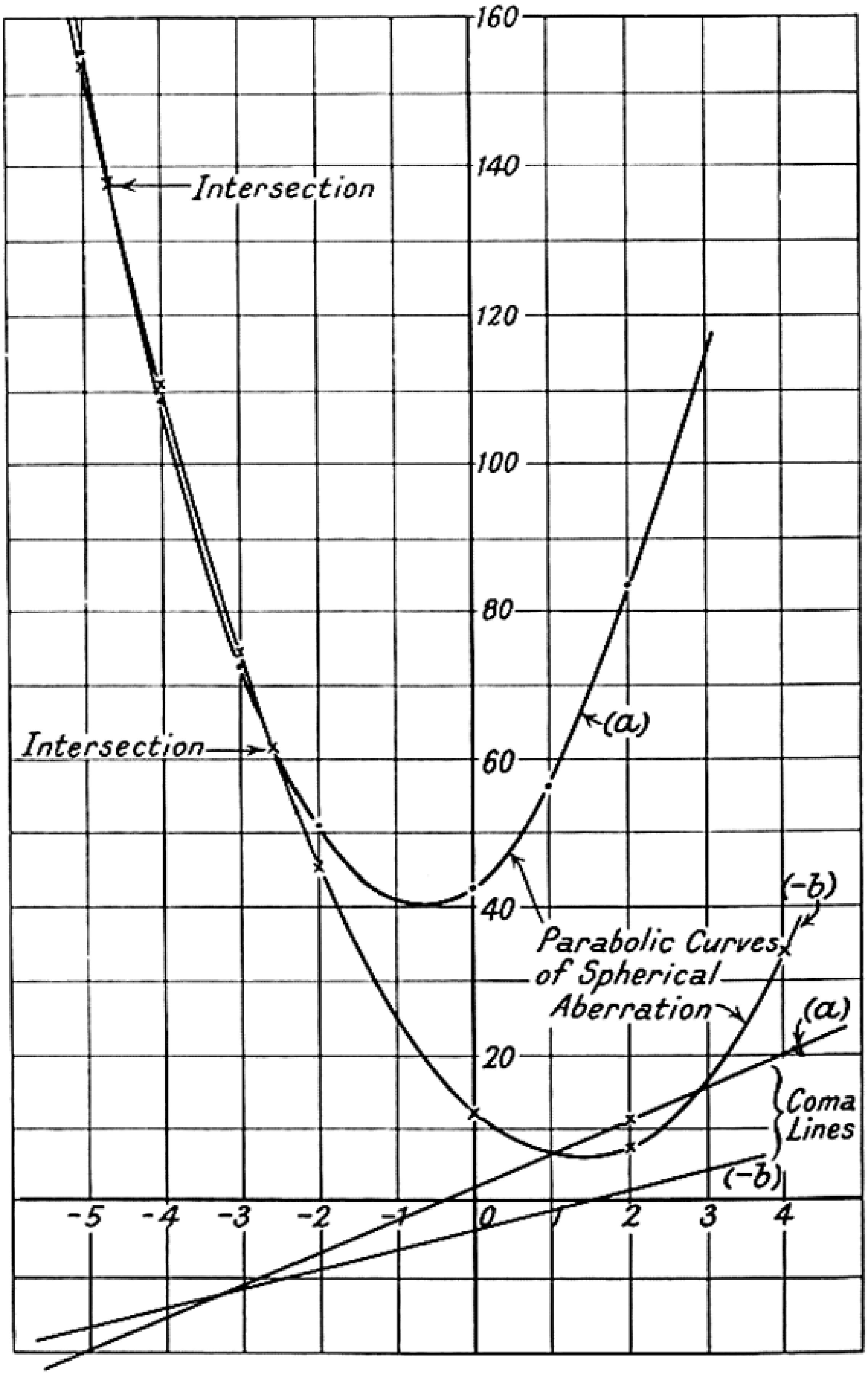


FIG. 22. GRAPHICAL PRESENTATION OF SPHERICAL ABERRATIONS AND COMA FOR A DOUBLET OBJECTIVE

for the case of a thin lens. We insert the necessary numerical quantities, and find for the above coma coefficients of lenses *a* and *b*.

$$C_a = 4\cdot445\ \boldsymbol{R}_2 + 1\cdot94$$
$$C_b = -\ 2\cdot73\ \boldsymbol{R}_3 + 4\cdot32$$

These lines are drawn in the diagram—again plotting C_b values with reversed sign; it will thus be seen that the difference between C_a and C_b for the abscissa of the lower intersection point of the two parabolas is about the same as with that for the higher intersection point. The lower intersection point with the shallower curve for the contact would, however, be preferable. (We can calculate the intersection point of the coma lines as $-3{\cdot}66$.)

If an uncemented doublet is allowable, we can choose a pair of lenses to be free from both spherical aberration and coma.

If $C_a + C_b = 0$, we get by adding the equations above

$$4{\cdot}445\ \boldsymbol{R}_2 - 2{\cdot}73\ \boldsymbol{R}_3 + 6{\cdot}26 = 0$$

$$\boldsymbol{R}_3 = 1{\cdot}63\ \boldsymbol{R}_2 + 2{\cdot}29$$

This relation between $\boldsymbol{R}_2$ and $\boldsymbol{R}_3$ will ensure absence of coma.

Substituting this value for $\boldsymbol{R}_3$ in the expression for A_b above, we obtain

$$A_b = -\ 10{\cdot}02\ \boldsymbol{R}_2^{\,2} - 12{\cdot}52\ \boldsymbol{R}_2 - 9{\cdot}73$$

We had

$$A_a = 6{\cdot}21\ \boldsymbol{R}_2^{\,2} + 8{\cdot}27\ \boldsymbol{R}_2 + 42{\cdot}0$$

if the sum of the aberrations is to be zero, then we get on adding

$$0 = 3{\cdot}81\ \boldsymbol{R}_2^{\,2} + 4{\cdot}25\ \boldsymbol{R}_2 - 32{\cdot}27$$

This quadratic equation yields two solutions, i.e. $\boldsymbol{R}_2 = -\ 3{\cdot}52$ or $+\ 2{\cdot}405$. The first of these is the one of practical interest, since the second would give a pronounced meniscus form to the lens as a whole. The figures therefore become

$$\boldsymbol{R}_1 = 1{\cdot}65$$
$$\boldsymbol{R}_2 = -\ 3{\cdot}52$$
$$\boldsymbol{R}_3 = -\ 3{\cdot}45$$
$$\boldsymbol{R}_4 = -\ 0{\cdot}67$$

Suitable thicknesses have now to be assigned, and the system perfected by trigonometrical trials.

The foregoing account explains the approximate method of designing a telescope objective which is to be corrected in itself. If the objective is to be used for visual observation, it is desirable to correct the spherical aberration for the brightest light in the spectrum ($\lambda = 0{\cdot}55\mu$). However, it is usually satisfactory to give trigonometrical correction using the refractive indices for the "e" line, $0{\cdot}5461\mu$. See Vol. I, p. 240.

When an objective is intended for use with a definite eyepiece, the axial chromatic and spherical aberration of the latter can be found (as the eyepiece design is usually completed first) by tracing some rays of an axial parallel beam backward through the system.

The objective is then designed so as to compensate the axial aberrations of the eyepiece, and will need to be slightly over-corrected if intended for use with an ordinary Huygenian or Ramsden eyepiece. The condition for chromatic correction can be adjusted to allow for this, even in getting the rough design.

The approximate solution is turned into a trial formula by finding the radii, and assigning a suitable thickness and diameter necessary to obtain the required aperture ratio. The trigonometric formulae are then employed in tracing a group of parallel rays through the objective, a paraxial ray, a marginal ray, and one or more rays in mid-zones, preferably using the wave-length for brightest light. "Red" and "blue" rays can also be traced through the mid-zone. The last radius can be adjusted to produce the exact chromatic correction required, and the state of correction as regards spherical aberration and coma can be examined. Chapter IV of Vol. I contains an explanation of the methods by which the phase differences of disturbances arriving in the image can be deduced from a knowledge of the geometrical aberrations of ray paths. The state of correction as regards coma is examined by the so-called "offence against the sine condition."

The optical sine relation (equation (401), Vol. I)

$$nh \sin \alpha = n'h' \sin \alpha'$$

and the corresponding paraxial form

$$nh\alpha_o = n'h'\alpha'_o$$

give two values of the magnification deduced from the marginal and paraxial ray paths, viz.

$$m_m = \left(\frac{h'}{h}\right)_{marginal} = \frac{n \sin \alpha}{n' \sin \alpha'}$$

and

$$m_p = \left(\frac{h'}{h}\right)_{paraxial} = \frac{n\alpha_o}{n'\alpha'_o}$$

The "sine" condition for freedom from coma (valid in the absence of spherical aberration) is that m_m and m_p shall be identical.* The

---

* Note that the magnification for a marginal ray is

$$m_m = \frac{n \sin \alpha}{n' \sin \alpha'} = \frac{n\left(\frac{y_m}{l}\right)}{n' \sin \alpha'},$$

where y_m is the incidence height of the ray and l is the distance of the object when this distance is large. Hence when the incident light is parallel to the axis, the sine condition may be expressed as the necessary constancy of $\frac{y_m}{\sin \alpha'}$.

offence against the sine condition is given by $\left(\frac{m_m}{m_p} - 1\right)$. The computing schedule (Vol. I, p. 17) will usually begin with the same numerical value for sin α and α_0. Hence

$$\text{Offence against the sine condition} = \frac{m_m}{m_p} - 1 = \frac{\alpha'_0}{\sin\alpha'} - 1 \quad . \ (2.07)$$

The numerical result thus obtained is clearly a measure of the coma, taken as the radial distance between the focusing points of paraxial and marginal zones divided by the radial distance of the image point from the centre of the field. It is the general experience with ordinary telescope and microscope objectives that it must not be allowed to rise above one part in 400, or 0·0025.

It is not within the scope of the present book to describe in detail the systematic trials by which the design of an objective is finally completed. The approximate thin lens method saves much time in the early stages. Modern advances in design have arisen through the ability to interpret the aberrations obtained from the numerical work in terms of optical path differences at the focus of the lens, and through an exact knowledge of the tolerances allowable. An introductory account of the general principles involved is given in Vol. I, Chapter IV. See also Conrady[8] and H. H. Hopkins.[9]

Resolving Limit of a Telescope. In Chapter IV, Vol. I, it was explained that the closest approach of two elementary "star" images which still permits of the recognition of the double nature of the concentration is approximately such that the centre of one Airy disc falls on the first dark ring of the neighbouring image.

The angle subtended by the radius of the Airy disc at the second nodal point of a telescope objective therefore represents the angular resolving power. We have the approximate formula for an image in air

$$\rho = \text{Radius of Airy disc} = \frac{0{\cdot}61\lambda f'}{y} \quad \text{(See Vol. I, p. 94.)}$$

The angle w' subtended by this radius at the second nodal point is given by

$$w' = \frac{\rho}{f'} = \frac{1{\cdot}22\lambda}{a} \text{ radians}$$

where a is the diameter of the objective.

This is the same angle as that subtended by the objects at the first nodal point. If this is put into English units we easily find—

$$\text{Angular resolving limit in seconds} = \frac{5{\cdot}5}{\text{Aperture in inches}}$$

It has been made clear that the theoretical basis for this limit is only approximate; the exact figure depends upon the distribution of light in the actual "Airy disc," and this may be modified by spherical aberration and other causes in an actual lens. Dawes's Rule derived from experiment gives—

$$\text{Angular resolving limit in seconds} = \frac{4{\cdot}5}{\text{Aperture in inches}}$$

In order that it may be possible for the eye to perceive the doubling of the image of a close double star, the separation of the images must clearly subtend an angle at least as great as the *minimum separabile* for the eye, which Hooke found to be one minute of arc.

As viewed by the eyepiece of focal length f'_b, the angle under which the radius of the Airy disc is seen is

$$\begin{aligned}\frac{\rho}{f'_b} &= \frac{1{\cdot}22\lambda f'_a}{af'_b} \\ &= w'\,\mathrm{M}\;.\;\;.\;\;.\;\;\text{(from above)}\end{aligned}$$

Note that this is, of course, "angular resolving limit" multiplied by the "magnification." If the angle of view is to be 1′ of arc (= 0·00029 radians), then

$$0{\cdot}00029 = \frac{1{\cdot}22\lambda}{a}\,.\,\mathrm{M}$$

It is easy to get an approximate figure if we take $\lambda = 0{\cdot}00058$ mm., and thus

$$\mathrm{M} = \frac{a\ (\text{mm.})}{2{\cdot}44} = 10 \times (\text{aperture in inches})\;.\;\;.\;\;.\;\;\text{approximately.}$$

At such a magnification, however, the resolvable detail, even if just visible, would be unendurably small, and even though no fresh detail can be rendered it will make for freedom from visual strain to increase the magnification to three or four times the above figures. Further enlargement serves little or no useful purpose; it is "empty magnification."

The above equations give (for 1′ visual angle)

$$f'_b = \frac{1{\cdot}22\lambda}{0{\cdot}00029}\left(\frac{f'_a}{a}\right)$$

$$f'_b\ (\text{mm.}) = 2{\cdot}44 \times (\text{aperture ratio number})$$

As indicated, however, it will be better to take a focal length only one-third or one-quarter of the above value. Thus for an astronomical objective with an aperture ratio number $\frac{f'}{a} = 15$, the indicated value of f'_b is 37 mm., but we shall be able to use with visual advantage eyepieces with focal lengths as short as 12 mm. or 9 mm.

The above equation which gives the minimum magnification for the telescope is useful when we put

$$\mathrm{M} = \frac{\text{Diameter of entrance pupil in mm.}}{\text{Diameter of exit pupil in mm.}} = \frac{a}{2{\cdot}44}$$

It will be seen that in order to do justice to the resolving power of the objective the magnification should be *at least* sufficient to reduce the diameter of the exit pupil to 2·4 mm., and we can with visual advantage use exit pupils down to 0·6 mm. If a handy test of the effective state of magnification is required, the measurement of the diameter of the exit pupil gives the information at once. It will be understood that the majority of binoculars and instruments with large exit pupils come nowhere near making the fullest use of the resolving power of their objectives, although the large exit pupil is important in maintaining the illumination of the image.

While the "resolving limit" of a telescope enables us to predict its performance in forming the image of a known "double star," it is not true, conversely, that the observation of an intensity distribution of the kind we should expect of a double-star image enables us to infer the true character of the object, which might, for example, be an assembly of many point sources. To say that "the object is resolved" implies that a great deal is already known about its nature. Further remarks about the theory of image formation will be found in Chapters III and VII.

Similar results with regard to resolving power are found for objects other than stars. A discussion of the effect of the object-contrast on the performance of the system will be found in Chapter VII.

Depth of Field in Use of a Telescope. In binoculars, periscopes, etc., where the depth of field is important, the magnification is often so low in relation to the "resolution limit" of the instrument, that the "disc of confusion" will be far more significant than the physical depth factor. The discussion on p. 17 can be applied if we remember that the confusion disc c will be given by

$$c = \frac{l\sigma}{m}$$

where σ is the visual limiting angle and m is the angular magnifying power. Let us write $(\sigma/m) = \alpha$, then the equations for δl_1 and δl_2 become

$$\left.\begin{aligned}\delta l_1 &= \frac{l^2\alpha}{a - l\alpha} = \frac{l\alpha}{(a/l - \alpha)}\\ \text{and}\qquad \delta l_2 &= \frac{l\alpha}{(a/l + \alpha)}\end{aligned}\right\} \quad . \quad . \quad (2.08)$$

In the position when $\alpha = a/l$, the range of δl_1 will extend outwards to infinity; this gives

$$\delta l_2 = \frac{l}{2}$$

and

$$l - \delta l_2 = \frac{a}{2\alpha}$$

so that the depth of sharp focus will extend from a distance of $a/2\alpha$ to infinity. If a telescope has an object glass of diameter 0·029 metres, and we take $\alpha = 0{\cdot}00029$, the range of really sharp focus should be from 50 metres to infinity.

Effect of Change of Accommodation of the Eye. If the eyepiece is of fairly high power it may usually be assumed that, as before, the eye pupil coincides with the principal focus of the magnifier which has focal length f_e', say. Then the focal range in the intermediate field will be

$$f_e'^2\left(\frac{1}{k} - \frac{1}{b}\right)$$

If k is infinite the accommodation in dioptres is $1/b$ when b is in metres. Hence if the image given by the objective is in a fixed position the necessary linear movement of the eyepiece to restore focus when the accommodation is changed (or to allow for the "refraction" of an ametropic eye) is proportional to the change in dioptres. Eyepieces with helical focusing have an equi-spaced focusing scale usually marked in dioptres.

If the telescope is originally in afocal adjustment, the emmetropic eye obtains a sharp image when accommodation is relaxed. When the eye changes its accommodation (increasing its dioptric power) by $1/b$ dioptres, the linear shift of the intermediate focus is $f_e'^2/b$. In order to allow for this the object might shift from infinity to a finite distance x beyond the first principal focus of the objective.

Thus

$$x\frac{f_e'^2}{b} = f_0'^2$$

where f_0' is the focal length of the objective. Thus

$$x = b\left(\frac{f_0'}{f_e'}\right)^2 = \frac{m^2}{B}$$

where m is the ordinary magnification of the telescope when in afocal adjustment and B is the amount of the accommodation (in dioptres if x is in metres). Hence if a telescope magnifying ten times is used by an eye which increases its accommodation by 1 dioptre, a sharp image, in principle, could be obtained of a point at 100 metres distance. Note, however, that the limited acuity of the eye will, with a small telescope, such as cited above, already produce the impression of a sharp image at 100 metres so that the effect of the change of accommodation of this amount will not be very noticeable. Changes of accommodation can assist to some extent in increasing the total depth of field, but the matter may be complicated in the use of binocular instruments by the special relations between accommodation and convergence.

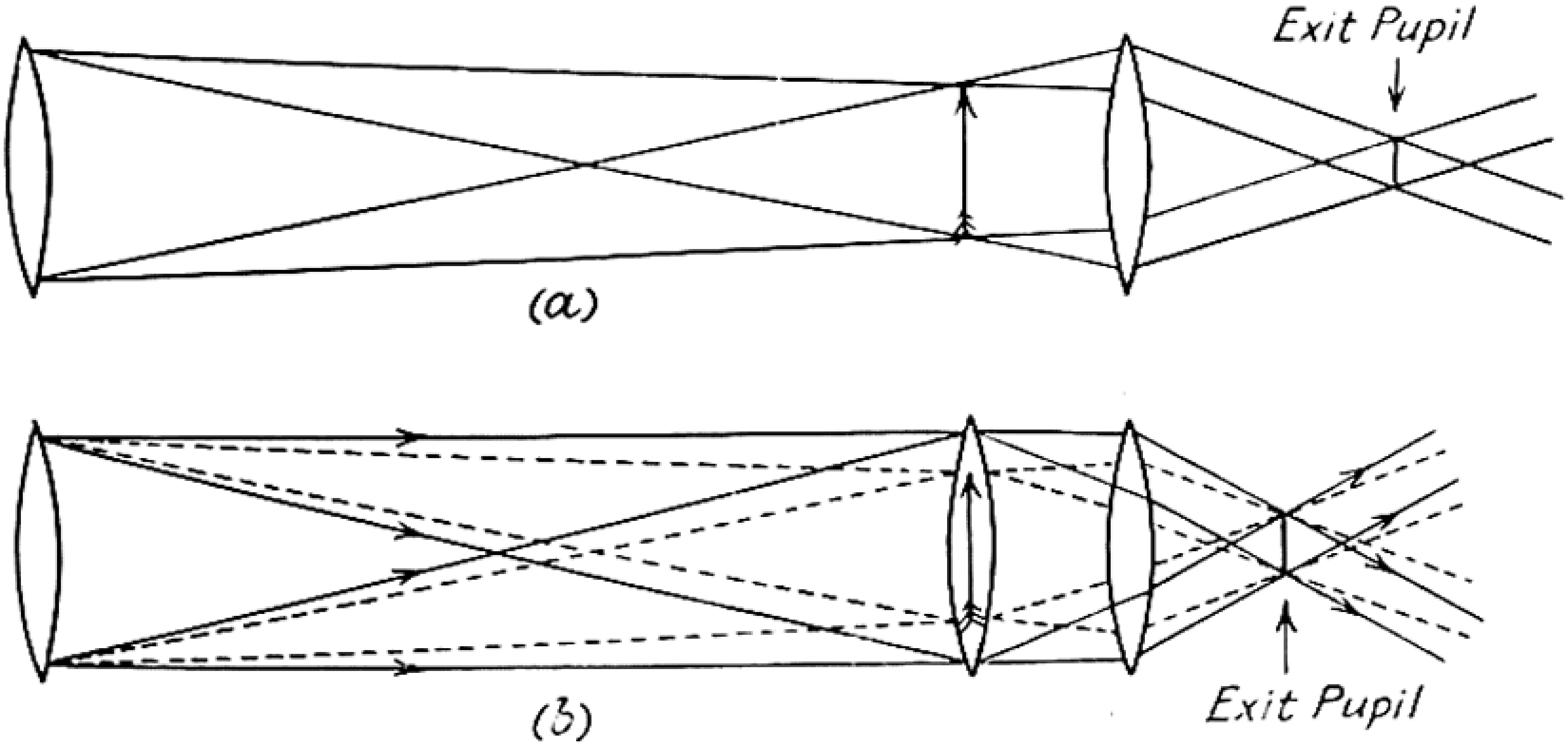
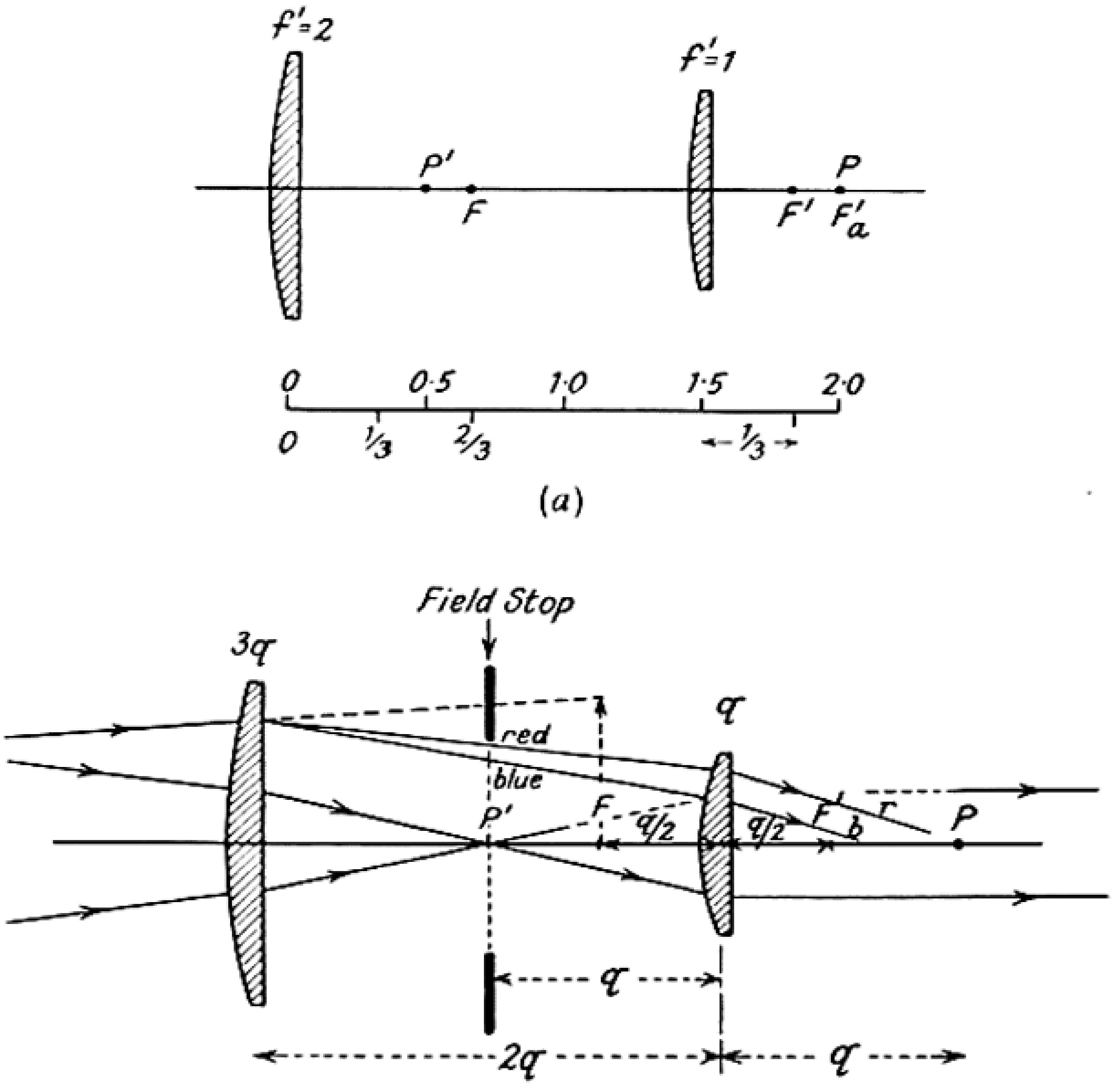


FIG. 23

(*a*) Telescope without field lens
(*b*) Telescope with field lens; note enlargement of field and shift of exit pupil

The Eyepiece. The early astronomical telescope as employed by Kepler about 1611 consisted of a convex object glass and a convex eye lens. Reference to Fig. 23 (*a*) shows that all the rays passing through the extremities of the image are just intercepted by the eye lens, but that if the image were larger, part of the rays would fall outside the eye lens. The illumination of the peripheral parts of the image would therefore be poor, and the optical performance of the instrument would suffer otherwise. It is not known with certainty to whom the suggestion of the field lens (Fig. 23 (*b*)) should be credited. It will be seen that the use of a convex lens in the plane of the image bends the rays towards the axis, although it can make no difference to the size of the image. Hence the peripheral parts of the image are now viewed by full pencils, the apparent field of view being controlled by the margin of the field lens in the case shown.



(a)

(b)

FIG. 24. TWO FORMS OF HUYGENIAN EYEPIECE

(a) Illustrates the position of the principal and focal points
(b) To illustrate the achromatism

It is worthy of note that the exit pupil is now moved considerably closer to the eye lens, although it remains of the same size. Most modern eyepieces, however, do not place the field lens in the focal plane, because any specks of dirt on the glass are then seen in focus in the field of view.

The addition of yet a third convex lens, as an erector, transformed this eyepiece into a terrestrial erecting eyepiece. This will be dealt with below, but we must first notice two important forms of two-lens eyepiece due to Huygens (1703) and Ramsden (1783). Their forms are illustrated in Figs. 24 and 26 respectively.

In dealing with the simple magnifier, it was mentioned that one serious difficulty arises in the variation of magnification with the wave-length (or with the colour) of the light. This may be regarded as arising from a variation of focal length with wave-length. Consider, however, the expression for the power of a combination of two thin coaxial lenses of the same glass, of refractive index n and separated by a distance d.

$$\mathbf{F} = \mathbf{F}_a + \mathbf{F}_b - d\,\mathbf{F}_a\mathbf{F}_b$$

This becomes

$$\mathbf{F} = (n-1)\,\mathbf{R}_a + (n-1)\,\mathbf{R}_b - d(n-1)^2\,\mathbf{R}_a\,\mathbf{R}_b$$

Let the refractive index of the glass change for some given change of wave-length and become $n + \delta n$; then the new power $\mathbf{F}_1$ is

$$\mathbf{F}_1 = (n + \delta n - 1)\,\mathbf{R}_a + (n + \delta n - 1)\,\mathbf{R}_b - d(n + \delta n - 1)^2\,\mathbf{R}_a\mathbf{R}_b$$

Subtracting

$$\mathbf{F}_1 - \mathbf{F} = \delta n\,.\,\mathbf{R}_a + \delta n\mathbf{R}_b - d\{2\delta n(n-1) + (\delta n)^2\}\,\mathbf{R}_a\mathbf{R}_b$$

The condition that the focal length may be unchanged, if δn is small enough for its square to be neglected, is thus

$$0 = \mathbf{R}_a + \mathbf{R}_b - 2d(n-1)\,\mathbf{R}_a\,\mathbf{R}_b$$

or

$$d = \tfrac{1}{2}\left\{\frac{1}{(n-1)\,(\mathbf{R}_a)} + \frac{1}{(n-1)\,(\mathbf{R}_b)}\right\}$$

$$= \tfrac{1}{2}(f'_a + f'_b) \quad . \quad . \quad . \quad . \quad . \quad (2.09)$$

This simple formula furnishes a rough guide to the separation of two lenses of the same glass necessary in order to secure "achromatism of the focal length" of the magnifier, and thus freedom from the most objectionable radial colour effects in the field of view.

The Huygenian Eyepiece. In the Huygenian eyepiece the design varies amongst different makers, being adjusted to suit the thickness of the lenses and the objective distance. In a common form the focal length of the "field" lens is about twice that of the "eye" lens; the separation then called for by elementary theory is about one and a half times the focal length of the eye lens. Working out (by the thin lens theory) the position of the principal and focal points of the combination, they are found as in Fig. 24 (*a*). The principal points are so situated that the first is found behind the system and the second within it. The focal length of the combination is $\frac{4}{3}$ times that of the eye lens, and the power is positive, although it is clearly impossible to put a real object into the first principal focus and obtain an erect magnified image.

The design usually quoted in textbooks is shown in Fig. 24 (*b*); although not of much importance, it will serve to explain some features of the arrangement; in this case the field lens has a focal length of $3q$, say, an eye lens of focal length q, and a separation of $2q$; where q is some suitable unit. The application of the usual formulae gives $P_aP = 3q$; $P'_bP' = -q$; $f = -\frac{3}{2}q$; $f' = \frac{3}{2}q$. The first focal point F is $\frac{3q}{2}$ behind the front lens; the second, F′, is $\frac{q}{2}$ behind the eye lens.

In the ordinary use of the Huygenian eyepiece it will be placed so that the image-forming rays from the objective converge towards a point in the first focal plane through F. They reach the field lens and form a real image between the two components in the plane of the anterior principal focus of the eye lens b, which is the plane of the stop limiting the field of view. The eye lens renders the rays parallel after refraction. By use of the plano-convex form of each component it is possible to produce "over-corrected" astigmatism, and thus to flatten the field to some extent. Distortion may also be reduced. A field of 40° may be attained.

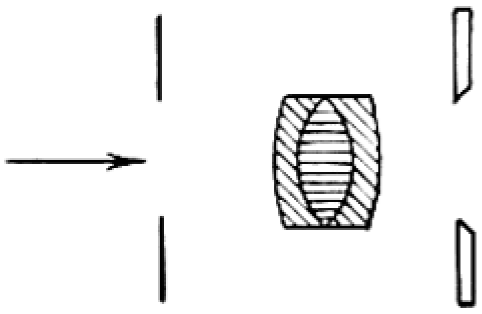


FIG. 25. TRIPLE CEMENTED LENS USED AS EYEPIECE

The achromatism of the system arises as follows. A ray directed towards the off axis image point is shown in the Fig. 24 (*b*). (It may be considered to be the principal ray from the centre of the objective.) It is refracted by the field lens, and the dispersion causes a greater deviation for the blue than for the red. The separation of the field lens and eye lens is, however, such as to make the blue ray intersect the eye lens nearer the axis than the red, and the greater deviation now produced in the red now renders both of them parallel on final emergence. Hence both the blue and red images will be seen under the same angle by the eye.

An achromatic lens can sometimes be used to get the same effect in avoiding lateral chromatism. Fig. 25 shows the use of a triple cemented lens for this purpose.

The Ramsden Eyepiece. Invented by Ramsden for the observation of the micrometer webs in reading microscopes, this eyepiece has two lenses of equal focal length disposed as shown in Fig. 26. The required theoretical separation for achromatism is equal to the focal length of either, but with a telescope of any practical length this brings the exit pupil too close to the eye lens; for this reason the separation is reduced and the residual chromatic differences are tolerated. In the case shown, the focal length of each lens is q, and the separation is $2q/3$; the focal and principal points are then disposed symmetrically as shown in the figure, the combined focal length being $3q/4$.

The Ramsden eyepiece as actually used does not thus strictly secure achromatism of magnification, but is extremely useful in "measuring instruments" and "optical sights," because it is possible to place cross threads or stadia lines in the common focal plane of the eyepiece and objective, and thus to measure directly the comparatively undistorted image produced by the latter. The angular field represented by the separation of two stadia lines is independent of the eyepiece, and we may thus change from one magnification to another, if required, by the choice of a new eyepiece.

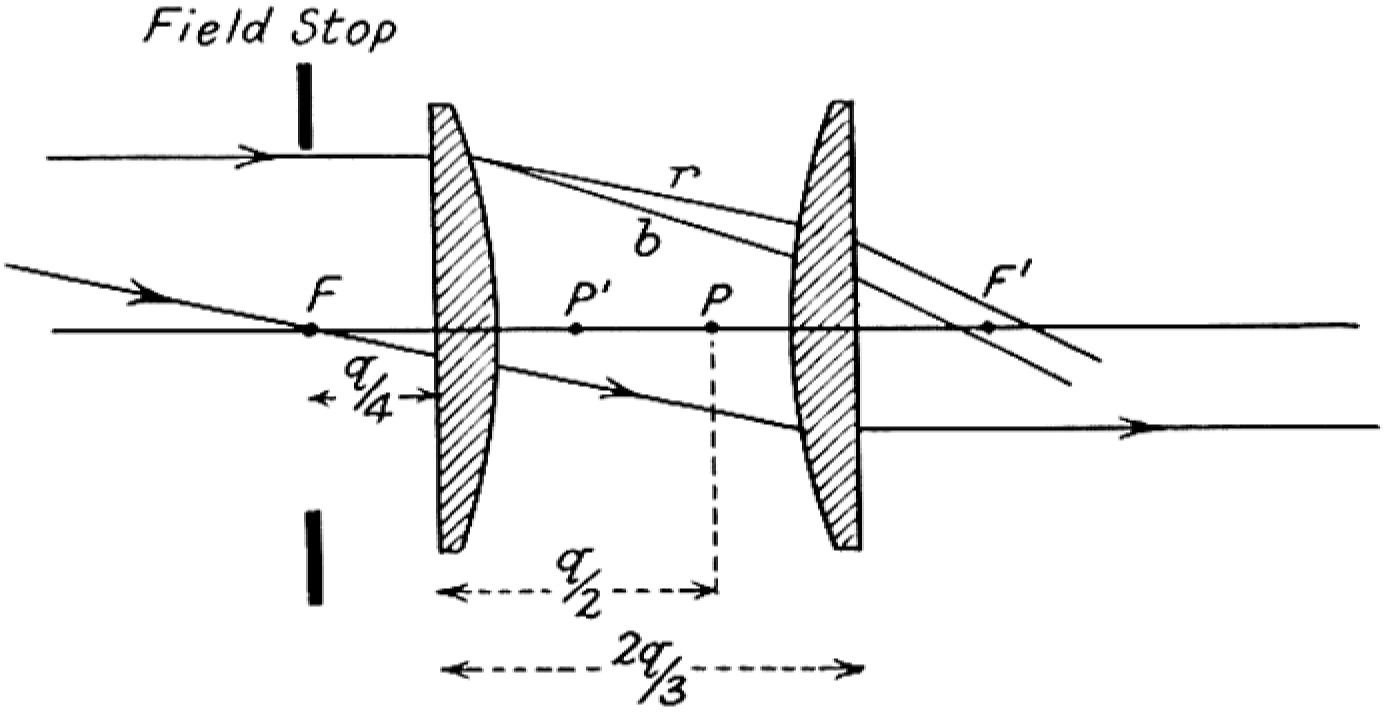


FIG. 26. RAMSDEN EYEPIECE SHOWING THE POSITION OF THE PRINCIPAL AND FOCAL POINTS

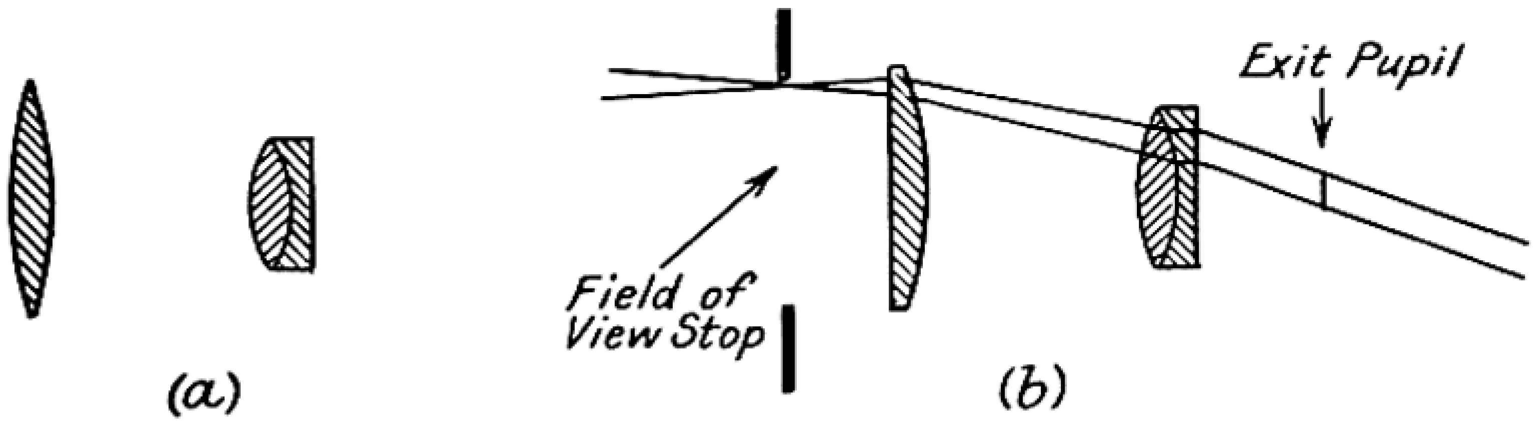


FIG. 27

(a) Kellner eyepiece (b) Achromatized Ramsden eyepiece

The Further Development of Eyepieces. Kellner (1849) invented an eyepiece (Fig. 27 (*a*)) with a double convex crown front lens and a cemented eye lens of the ordinary "crown and flint" type, which he called "orthoscopic," and for which he claimed a greatly improved colour correction in the outer regions of the field. In this case, however, the first focal plane lies in or very close to the front lens, so that any dust on the surface is seen in focus with the field of view. In a later type (Fig. 27 (*b*)), introduced by Zeiss, the advantages of the Ramsden type were retained, and a still better achromatism was effected by the use of barium silicate crown and silicate flint for the members of the cemented eye lens. This type is usually known as the "Achromatized Ramsden" form.

Requirements of Eyepiece Design. The following general requirements have (*inter alia*) to be studied in the design of eyepieces: (*a*) the axial chromatic and spherical aberration (together with the additions due to any prism system) will have to be compensated by the objective; (*b*) the coma of extra-axial images should be eliminated or similarly compensated; (*c*) chromatic difference of magnification should be eliminated for two suitable wave-lengths; (*d*) the Petzval sum (if negative as usual) should be as low as possible, and the astigmatism should usually tend towards over-correction; (*e*) the eye-point or exit pupil should be at an adequate distance from the last surface; in some cases the clearance must allow the use of spectacles, etc.; (*f*) the exit-pupil position should be as free as possible from spherical aberration; (*g*) the distortion should be kept as low as possible.

Modern Eyepieces. The traditional forms of eyepiece cannot well be designed to give adequately corrected fields exceeding 40° (Ramsden) or perhaps 50° (Huygenian), but the more complex types reach 60° to 70° in field diameter. There is room for a variety of compromise in the arrangement of the unavoidable residual defects, and this gives rise to a great variety of types, of which a useful survey is given by E. W. Taylor.[10]

By increasing the complexity of the Kellner eyepiece, Fig. 28 (*a*), Dennis Taylor was able in 1918 to obtain a field of view of 65°, with an eye clearance of 69 per cent of the focal length. See British Patent 125430. In some gun-sighting systems, e.g. dial sights, a large eye-clearance is obtained by the use of two achromatized lenses in a close combination, Fig. 28 (*b*), which gives a field of about 36°, with an eye clearance about 77 per cent of the focal length; the glasses used can be Hard Crown and Dense Flint.

An even greater relative eye clearance, 92 per cent, is obtained in a Zeiss eyepiece, Fig. 28 (*c*). The glasses used are 717295 and 516640. See B.P. 509990.

The so-called "orthoscopic" type (Abbe), Fig. 28 (*e*), had in early types a well-corrected field of 40°; more recently the field has been extended to 70°, the eye clearance being 82 per cent. The glasses are 569631, 728283, 569631, 701411. See B.P. 509585.

There are numerous other specifications in the patents; for example (Fig. 28 (*d*)) a Goerz eyepiece designed for use with a prism system and therefore under-corrected; the eye clearance is 59 per cent. The glasses used are 614564 and 673320. See B.P. 215337.

Telescope systems must often be used in faint light, and therefore a large exit pupil is necessary to maintain the brightness of the retinal image; both objective and eyepiece must then work at a high aperture ratio. One of the important difficulties in such a case is that the elimination of spherical aberration, both for the actual image and for the pupils of the system, is difficult. If the position of the exit pupil varies with the slopes of the principal rays from various points of the field, then movement of the eye pupil may produce undesirable shadows in the field of view. One way in which

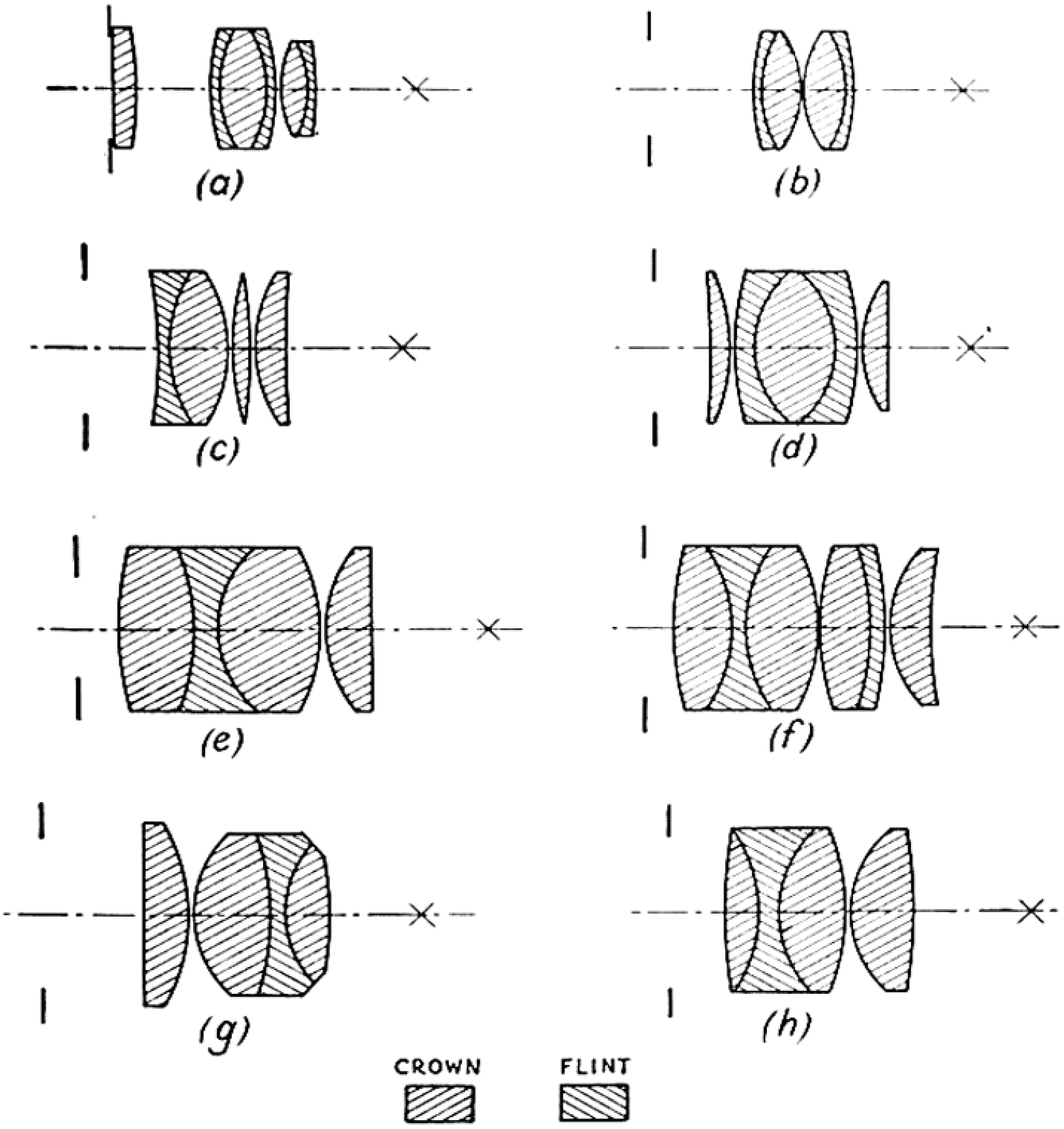


FIG. 28. VARIOUS FORMS OF MODERN EYEPIECES

(*a*) Cooke
(*b*) Dial sight
(*c*) Zeiss (wide eye-clearance)
(*d*) Goerz
(*e*) Abbe orthoscopic
(*f*) More recent design (Zeiss)
(*g*) Zeiss eyepiece with aspheric surface
(*h*) Barr and Stroud (with aspheric surface)

this may occur is indicated in Fig. 29, where S represents the last surface of the system, and A, B, and C suggest the marginal, zonal, and paraxial positions of the exit pupil. Suppose for a moment that the eye pupil coincides with the paraxial pupil, and is of the same size. Then clearly the area of the eye pupil illuminated by the zonal and marginal beams shows a progressive diminution, and thus, for a stationary eye, the illumination of the field decreases correspondingly from the centre outward. On the other hand, when the eye pupil coincides with A, the whole of the marginal beam enters the eye and the whole of the paraxial beam likewise, but the zonal beam is partly screened. This would produce a diminution of brightness in an intermediate zone of the field.

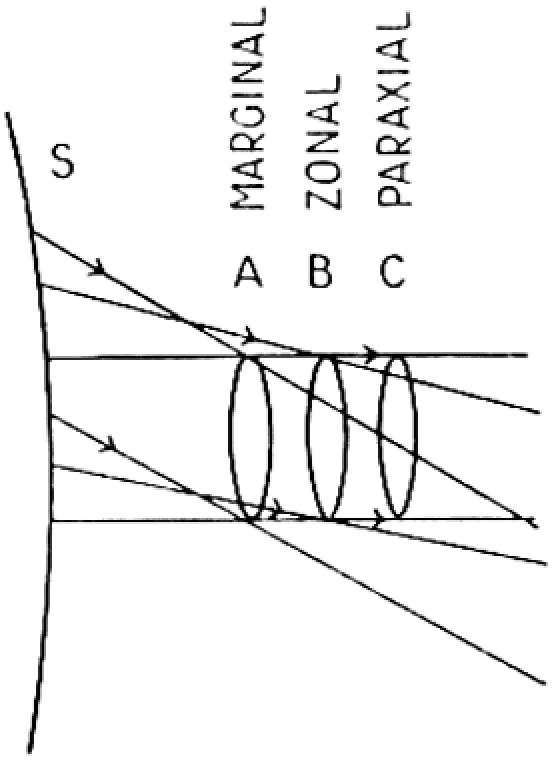


FIG. 29. ABERRATION OF PUPIL POSITIONS

If the eye pupil is smaller than the exit pupil of the instrument and if the eyeball rotates in order to bring different parts of the field to the fovea, the relative areas of the three beams intercepted by the eye pupil in various positions may differ very markedly, and thus the shadow effects will be correspondingly enhanced.

Similar effects may be caused by chromatic aberration of the pupil position, or even by the zonal variation of magnification (offence against the sine condition for the principal ray), but such causes are unlikely to be comparable with the spherical aberration in their results.

It is not possible in a general account of eyepieces to give any idea of the relative merits of different designs in the correction of the field, and in fact such information can only be gained by numerical studies. In the simpler eyepieces the Petzval curvature is severe; thus for the Huygenian or Ramsden of focal length unity, the curvature is 1·33, and the spherical aberration is under-corrected; under these conditions it is shown in discussion of the theory of primary aberrations that when the pupil position is such that the coma is corrected a maximum amount of over-corrected astigmatism is produced. The Petzval curvature is considerably reduced in some of the recent designs; there does not, however, seem to be any general agreement as to the most favourable conditions for the correction of the astigmatism.

The general method of securing a large eye clearance is to concentrate as much power as possible in the components close to the eye; this brings the second principal point of the system, and thus also the exit pupil, closer to the observer. The front components can provide the over-correction necessary to compensate for the under-correction of the rear lenses.

In some recent eyepiece designs, Zeiss, Fig. 28 (*g*), B.P. 399452, and Barr and Stroud, Fig. 28 (*h*), B.P. 530506, aspherical surfaces have been used. By such means it is possible to modify the corrections as explained in Chapter VIII, and a great number of possibilities are opened up which have not as yet been fully explored.

Erecting Systems. *The Terrestrial Eyepiece.* The possibility of erecting the image in his simple astronomical telescope by introducing an erecting lens between objective and eyepiece was known to Kepler. Rheita obtained improved definition by using two intermediate lenses. The essential form of the four-lens terrestrial eyepiece as made by Dollond, Ramsden, Fraunhofer and many others, down to the present day, was the outcome of many empirical trials. A typical construction is shown in Fig. 30, and the diagram will be sufficient explanation of the general mode of action; the erecting lens system is followed by a Huygenian eyepiece of usual form. The stop in the erector may be made to coincide with an intermediate image of the entrance pupil, thus removing any stray light such as that reflected from the interior of the tubes. A shift of the erector stop, however, cuts down the aperture of some of the oblique pencils, and some opticians use this method of improving the definition in the outer parts of the field. Improved erecting eyepieces have been devised in which the erecting lens is a cemented triplet. In Fig. 31 the field lens is employed to secure the necessary convergence of the principal rays.

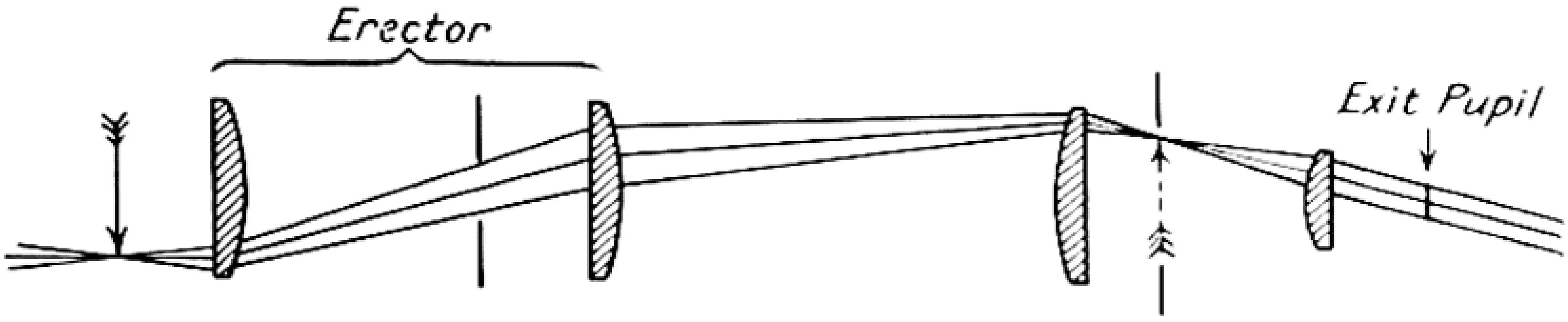


FIG. 30. ERECTING, OR TERRESTRIAL EYEPIECE

It is particularly important with these eyepieces to design the objective to suit them; a certain amount of spherical and chromatic over-correction is necessary.

Systems of Variable Power. The apparent angular field of most ordinary eyepieces being limited to 60° or so, the actual field of a telescope when using a variety of eyepieces is roughly inversely proportional to the magnification; we may view a small field at high power or vice versa according to requirements. Periscopes and various other instruments are often provided with two or three eyepieces easily switched into position. An alternative device useful in a variety of circumstances is to employ a low-power (afocal) telescope system which can be placed in position, when required, in front of the entrance pupil of the whole system. This can be an erecting system (for example a Galilean telescope) or an inverting system according to convenience. If the magnification factor is M, the angular magnification of the whole system will be increased in the same ratio; however, if the system is so arranged that the auxiliary telescope is reversed the magnification factor can be changed to 1/M. Such auxiliary systems are, for example, included in some forms of lower power stereoscopic microscopes, in this case behind the objective; Galilean erectors are favourable on account of their comparatively flat fields. In the arrangement due to Zeiss and used in the stereo-microscope there are two Galilean systems each of which can be reversed by rotation of the system (Fig. 32) and there is a "clear" position; thus five magnifications are possible in all.

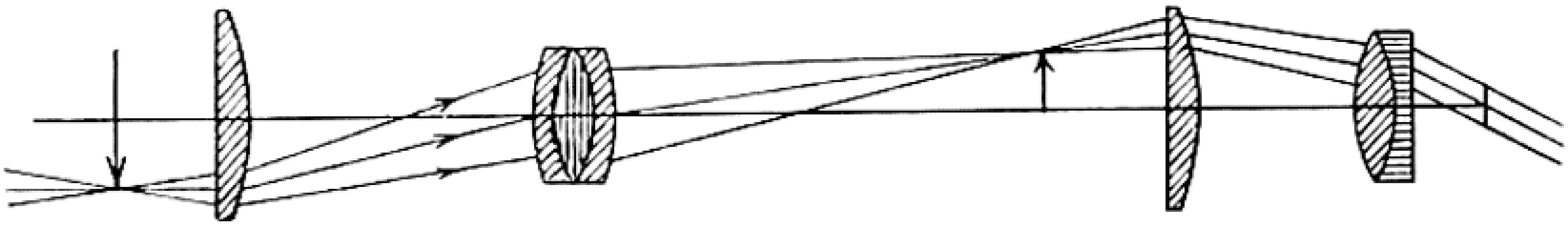


FIG. 31. TERRESTRIAL EYEPIECE WITH TRIPLE CEMENTED ERECTING LENS

An alternative manner of altering the power of, say, a terrestrial telescope, is to vary the relative conjugate distance ratio of the erecting lens system.

Assuming the usual notation, the distance k between object and image for the erector system is given by

$$k = l' + d - l$$

where d is the distance between the principal points of the erector.

Thus
$$k = d - \frac{l^2}{f + l}$$

Differentiating,

$$\frac{dk}{dl} = -\frac{l^2 + 2lf'}{(l + f')^2}$$

This is clearly zero (and the distance between object and image is a minimum) when $l = -2f'$, the condition when the specific magnification of the erector is − 1. Thus for short movements of the erector there will be no appreciable change of focus of the telescope; however, an increasing movement of the erector will demand some outward movement of the final eyepiece. Such a telescope made by the firm of Ross with a triple cemented lens as erector is described as having a range of powers from 7 to 21. Alternatively, it is possible to use an erecting lens with two components; the separation of these can be varied so as to alter the focal length of the combination; in this way, by suitable movements of the lenses the magnification of the whole telescope can be altered without a change in its overall length. Such a system has been manufactured by Ottway.

Especially for visual use, the required movements of the lenses can be produced with all necessary accuracy by mechanical means (cams and the like).

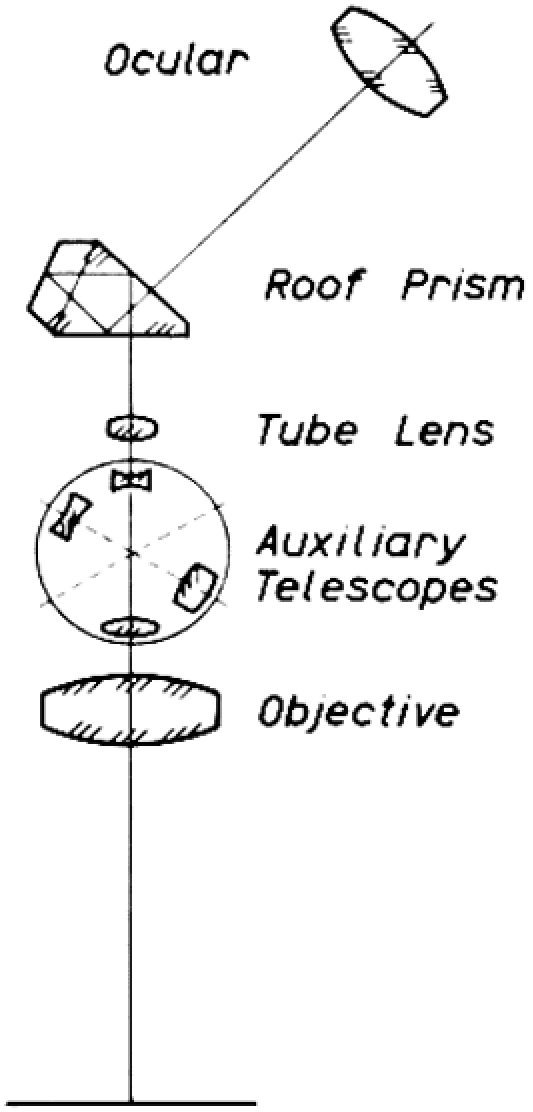


FIG. 32. LOW-POWER MICROSCOPE SYSTEM WITH VARIABLE MAGNIFICATION (FIVE POSSIBLE POWERS) BY ZEISS

Auxiliary Afocal Systems for Projection and Cinematography. In addition to such uses as are suggested in the previous section, auxiliary afocal systems can be of service in adapting a projection lens for use at various screen distances, where some available lens of fixed focal length would give a magnification too great or too small to fill the available screen. They can also be used in cinematography or television to give suitable variations of magnification to the picture or a part of it. However, when the focus has to be maintained precisely for photography or television the optical and mechanical construction has to be of high precision. The possible use of prismatic elements, capable of rotation, to provide variable magnification, should not be forgotten (see p. 27 above).

If the aperture of the lenses of the auxiliary afocal system is great enough, the diameter of the exit pupil is fixed by the diaphragm of the (fixed) photographic lens producing the final image; hence the effective stop-number (see below, p. 188) and the illumination of the image is independent of the magnifying power, though of course the presence of the afocal system will cause some loss of light.

A review by Hopkins[11] traces the development of such systems from the original proposals of Gramatski and others; the basic idea is that of two opposed Galilean telescopes, and Hopkins shows that the case where the positive lenses are outermost, Fig. 33 (*b*), is on the whole most favourable, though some commercial systems have used the negative lens outermost. That this statement is justified is shown as follows—

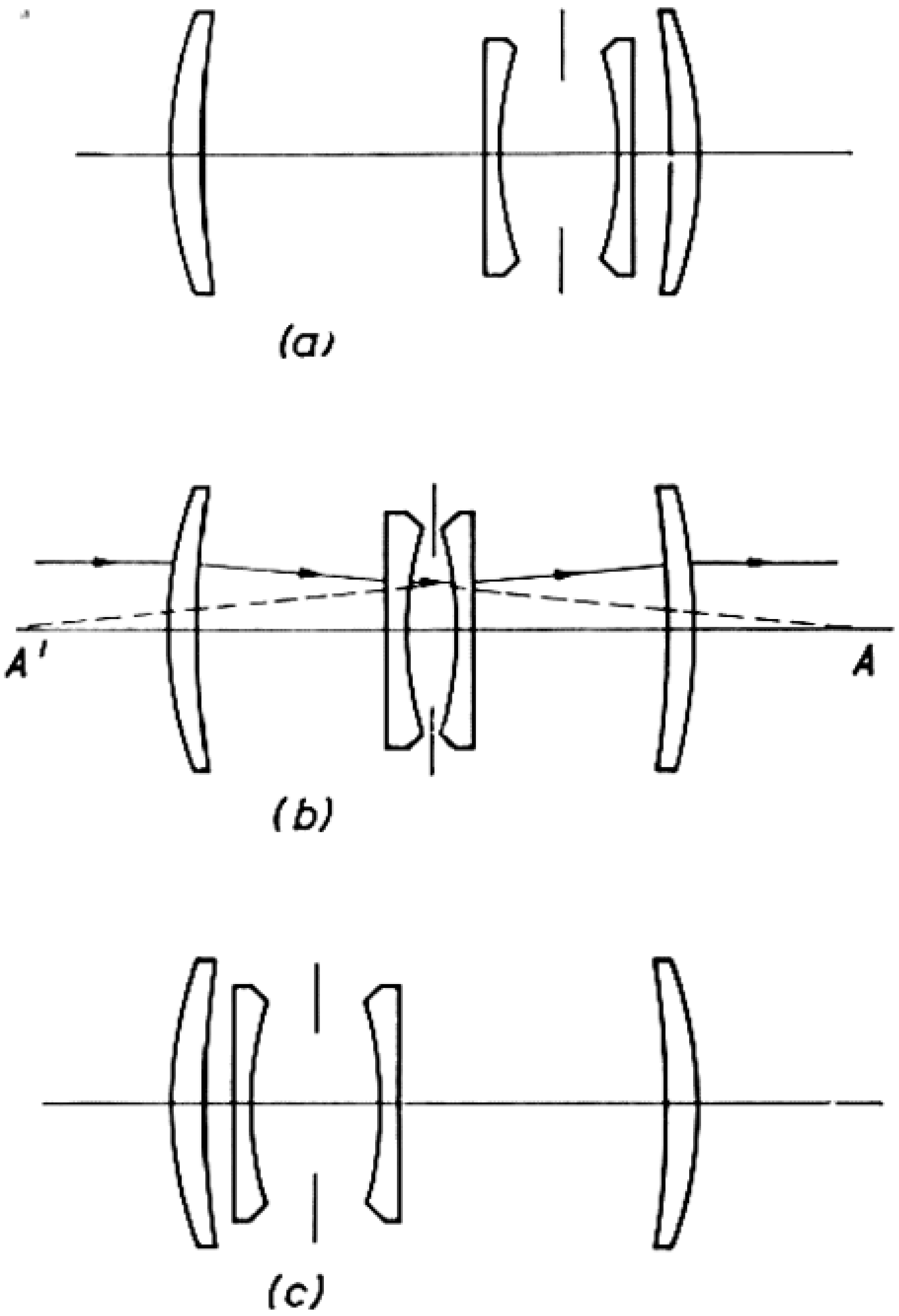


FIG. 33. (*a*) MAGNIFICATION m
(*b*) MAGNIFICATION 1
(*c*) MAGNIFICATION $1/m$

If the positions of the principal focal points of the outer lenses are as shown in Fig. 33 (*b*) at A and A′, respectively, the function of the inner negative combination is to project an image of A into the position A′, thus if the combination moves closer to A an increase in the (negative) power will be required, which is obtained by increasing the separation of the negative lenses. It is favourable if they have the same power, as will be seen. The combined power $\mathbf{F}$ is

$$\mathbf{F} = \mathbf{F}_a + \mathbf{F}_b - \mathbf{F}_a\mathbf{F}_b d$$

and the power changes most rapidly with d when the differential coefficient

$$\frac{d\mathbf{F}}{dd} = -\,\mathbf{F}_a\mathbf{F}_b$$

has its maximum value. Now when d is small $\mathbf{F}_a\mathbf{F}_b \approx \mathbf{F}_a(\mathbf{F} - \mathbf{F}_a)$; thus

$$\frac{\partial \mathbf{F}_a\mathbf{F}_b}{\partial \mathbf{F}_a} \approx \mathbf{F} - 2\mathbf{F}_a$$

so that when this differential coefficient is zero (for the maximum rate of change of $\mathbf{F}_a\mathbf{F}_b$ with $\mathbf{F}_a$) $\mathbf{F}_a = \mathbf{F}_b$.

Now the separation δ of the principal planes of a system of two thin lenses separated by a distance d is easily shown (from the expressions on p. 54, Vol. I) to be

$$\delta = -\frac{d^2\mathbf{F}_a\mathbf{F}_b}{\mathbf{F}} \quad . \quad . \quad . \quad . \quad (2.10)$$

When the middle lenses are of negative power, the increase of d increases the negative power of the combination, and also increases the value of δ which is then a positive quantity; but the object-image distance T is a negative quantity

$$\mathrm{T} = \delta + l' - l$$

Hence an increase of δ means a necessary increase in the absolute value of $l' - l$. The larger is the numerical value of $l' - l$, the smaller the increase of power necessary to effect the image formation with the conjugates l' and l. The reverse is the case if the middle lenses are the positive ones, and a greater relative increase of power is necessary to effect the image formation. (N.B. that the "first" principal point now lies to the right of the second as in Fig. 26, Ramsden eyepiece.)

If the system uses negative middle lenses, and the effective stop is outside (it will be situated in the actual image-forming lens at the rear of the afocal system) the aperture of the beam in the centre lenses will be smaller than in the positive ones. The reverse is the case if the negative lenses are outside, for they have already to take the full aperture of the beam and the centre position lenses must transmit a proportionately enlarged bundle of rays (see also "Zoom Lenses," p. 237).

The Field of View of a Telescope. In all telescopes we can distinguish a diaphragm which limits the angular divergence of the rays passing through the centre of the exit pupil. Referring to Fig. 12, the stop limiting the field is seen in the common focal plane. If it reaches the size when the full bundle of rays passing through a point in the margin cannot be transmitted by the eyepiece, the illumination of the boundary of the field will suffer. In most eyepieces intended for ordinary observation, the diameter and position of the stop is specified in the complete design. Take the case of the Huygenian eyepiece shown in Fig. 34, for which $f'_a = 2f'_b$ and $d = \frac{3f'_b}{2}$. The stop is situated in the first focal plane of lens b. If its radius is y, the apparent angular field is clearly $2\left(\tan^{-1}\frac{y}{f'_b}\right)$. The entrance pupil of the eyepiece is evidently formed at a distance l

from the field lens for which the conjugate $l' = \frac{f'_b}{2}$ and

$$\frac{1}{(f'_b/2)} - \frac{1}{l} = \frac{1}{2f'_b}$$

and thus $l = \frac{2f'_b}{3}$. The radius of the effective entrance pupil is therefore

$$\frac{yl}{l'} = \frac{4y}{3}$$

Hence the real field of view will be $2 \tan^{-1}\left(\frac{4y}{3f'_o}\right)$, where f'_o is the focal length of the objective. If f'_e is the focal length of the eyepiece

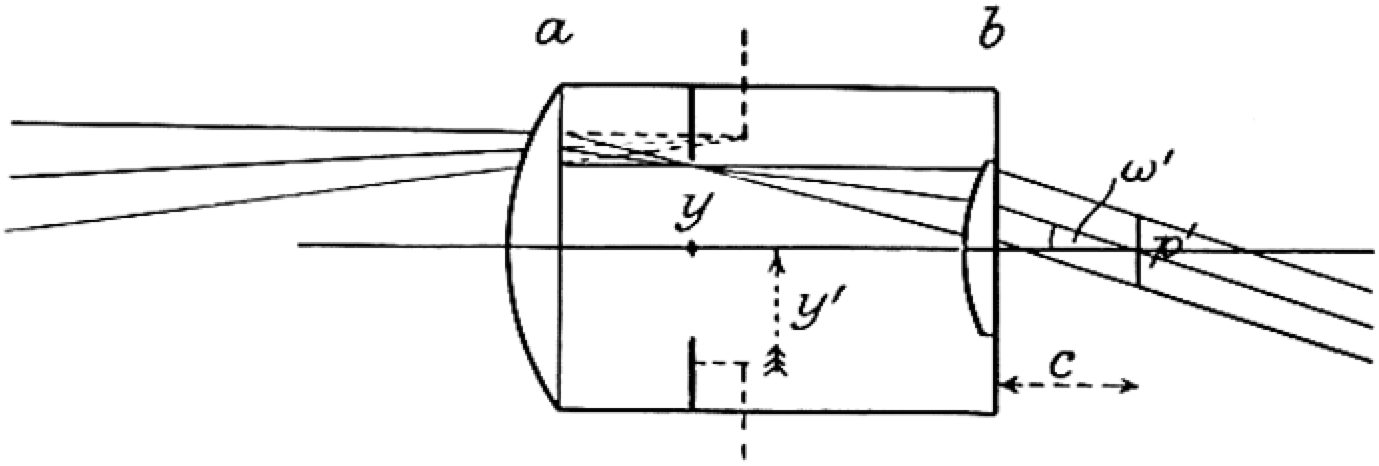


FIG. 34

as a whole, i.e. $4f'_b/3$, the apparent angular field is $2 \tan^{-1}\left(\frac{4y}{3f'_e}\right)$. If the angles are small, the ratio of apparent field to real field will therefore reduce to $\frac{f'_o}{f'_e}$, the ratio of the focal lengths of objective and eyepiece which is the angular magnification of the telescope. More generally, we calculate the radius y' of the effective stop situated in the first focal plane of the eyepiece system, and find the real field as $2\omega = 2 \tan^{-1}\left(\frac{y'}{f'_o}\right)$, and the apparent field as $2\omega' = \tan^{-1}\left(\frac{y'}{f'_e}\right)$. The distance of the exit pupil from the eye lens can be calculated. Let it be c, say. The radius p' of the exit pupil can also be calculated from $\frac{p}{m}$ where p is the radius of the entrance pupil and m the magnification of the telescope. Then the required radius of the eye lens will be $p' + c \tan \omega' = p' + \frac{c \cdot y'}{f'_e}$; this will allow the full oblique pencil to be transmitted.

Observation with Rolling Eye. In the case when the pupil of the eye is brought into coincidence with the exit pupil of the instrument, and the latter is smaller than the eye pupil, then the whole field of the instrument will be projected on the retina. In using hand instruments, small telescopes or binoculars, the head and instrument can easily be moved together in order to bring different images into the central region of distinct vision. If, on the other hand, the instrument is held on a stand and not easily moved, the eye can observe different parts of the field by turning the head so as to observe as it were through the small window of the exit pupil, or by holding the head stationary and moving the eye in its socket.

The eye, observing through the eye lens, sees the image of the stop (more or less well defined) which limits the field. This image forms the "exit window," and in the case of telescopes with positive eyepieces it is usually seen "at infinity" and sharply defined.

If the exit pupil coincides with the eye pupil, let the point of rotation be situated at a distance ρ behind the pupil, and let p_e be the radius of the pupil of the eye. When the eye is turned so as to bring the margin of the field to the fovea, the required radius of the exit pupil is seen from Fig. 35 (*a*) to be

$$p' = \rho \tan \omega' + p_e \sec \omega'$$

The eye will then lose no light on rotation.

It may be possible to design the instrument so that the exit pupil falls into the centre of rotation of the eye in ordinary conditions of vision. This case is illustrated in Fig. 35 (*b*), and it will be seen that the radius of the actual exit pupil need only be $p_e \sec \omega'$ in order that the eye pupil may be kept completely filled with light.

On the other hand, it will often be the case that the exit pupil of the instrument is smaller than the eye pupil, and the limit to which the eye can move without experiencing some curtailment of the light will be restricted. In Fig. 35 (*c*), PQ represents the exit pupil of the instrument, and RS the pupil of the eye. If the eye rotates, the iris will begin to cut off some light from the image when S reaches Q. No illumination of the image will be possible when S reaches P. Hence

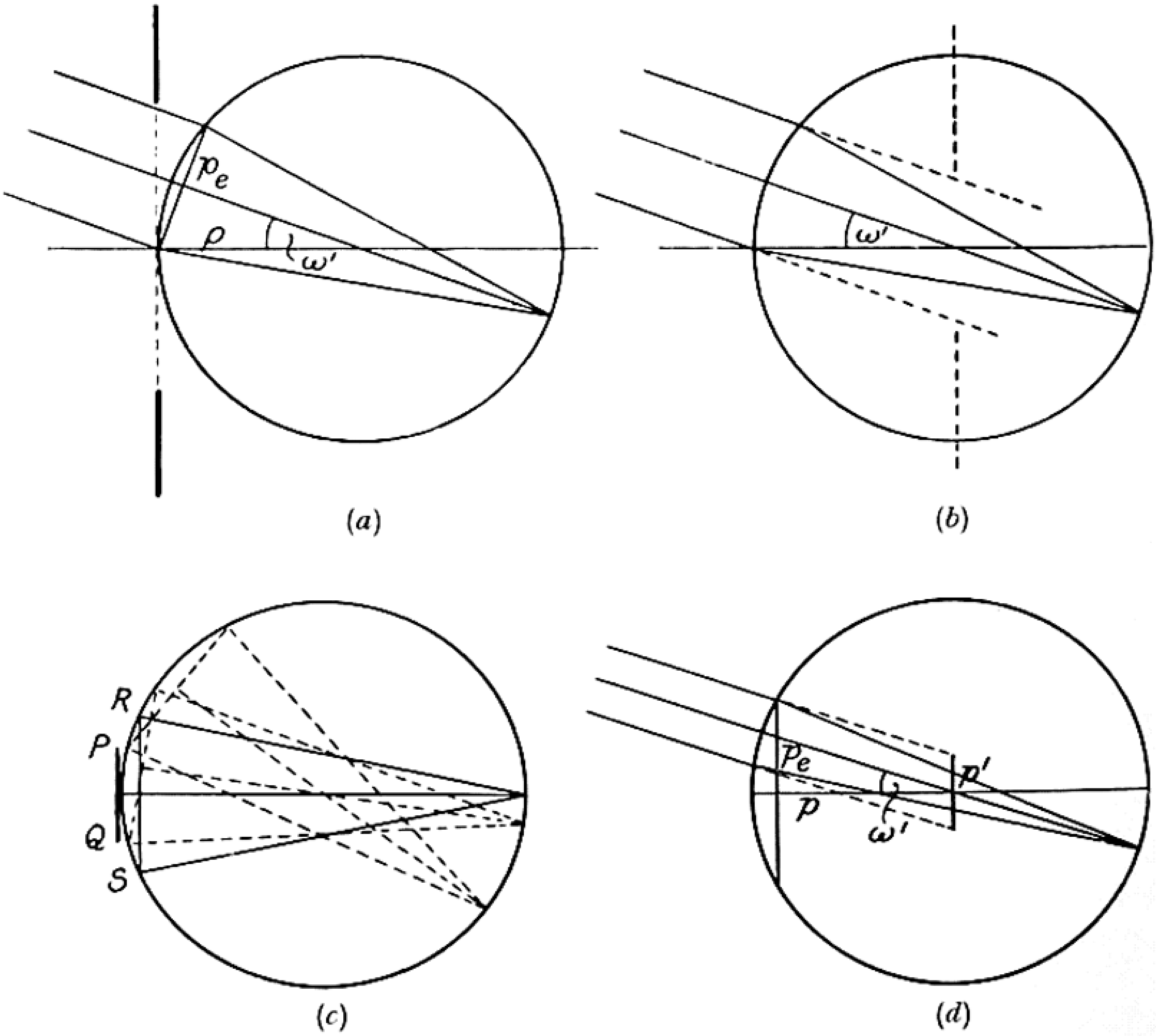
Total rotation possible while retaining full illumination

$$= \frac{p_e - p'}{\rho} \quad \text{(approx. angular measure)}$$

Total rotation possible while retaining partial illumination

$$= \frac{p_e + p'}{\rho}$$

When the instrument exit pupil coincides with the centre of rotation of the eye, no light will be cut off in this way for observation with the rolling eye until the margin of the field falls on the fovea, provided that the optical system gives full parallel bundles through the exit pupil up to the limit of the angular field. On the other hand, it will be noticed that the stationary eye at rest in the symmetrical position (Fig. 35 (*d*)) will not secure full illumination of the whole field simultaneously unless the radius of the eye pupil is at least



(*a*) (*b*)

(*c*) (*d*)

FIG. 35

(*a*) Eye pupil falls in a larger exit pupil
(*b*) Exit pupil formed in the centre of rotation of the eye
(*c*) Eye pupil coincides with a smaller exit pupil
(*d*) Observation with stationary eye

$$p_e = \rho \tan \omega' + p'$$

where ρ is the distance of the exit pupil behind the eye pupil.

If it should be smaller than the required amount, we can see that the limit of the angular field for full illumination will be

$$\tan^{-1}\left(\frac{p_e - p'}{\rho}\right)$$

while the limit of the angular field for partial illumination will be

$$\tan^{-1}\left(\frac{p_e + p'}{\rho}\right)$$

Visible Field in Relation to the Position of the Eye. In observing from the axial point of the exit pupil of an instrument, there will be some lens mount or diaphragm which limits the angular extent of the field of view; the image thus seen will be called the exit "window."

Let WW′ (Fig. 36) be the exit window and PP′ the exit pupil of a telescope. If the eye is placed anywhere within the quadrilateral APBP′ it will be clear that the limits of the visible field of view will be controlled by the margin of the "window" WW′. On the other hand, if the eye is placed outside the quadrilateral beyond B, as at E_2, the angular subtense of the visible field will be controlled by the exit pupil PP′ which now takes on the role of the exit window. The term "position of the eye" used above may refer either to the nodal point of a fixed eye, or to the centre of rotation of a rolling eye.

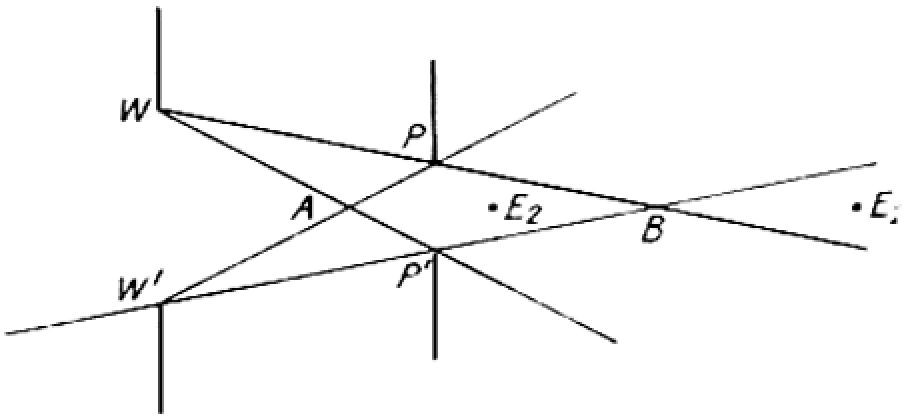


FIG. 36. EXIT PUPIL AND EXIT WINDOW

The case just mentioned, when the exit pupil limits the angular field, is typical of the Galilean telescope, and it is important enough to merit a closer discussion.

The Field of View of the Galilean Telescope. The "entrance pupil" of the Galilean telescope, Fig. 37, will be the diaphragm which limits the objective, of which the radius is p. The radius of the exit pupil is $\frac{p}{m}$ where m is the magnification of the telescope.

The distance of the entrance pupil from the first principal point of the eyepiece is

$$l = -(f'_a + f'_b)$$

Hence l', the distance of the exit pupil from the second principal point, is given by

$$\frac{1}{l'} + \frac{1}{f'_a + f'_b} = \frac{1}{f'_b}$$

Then
$$\frac{1}{l'} = \frac{1}{f'_b} - \frac{1}{f'_a + f'_b}$$
$$l' = \frac{f'_b\,(f'_a + f'_b)}{f'_a}$$
$$= f'_b\left(1 + \frac{f'_b}{f'_a}\right) = f'_b\left(1 - \frac{1}{m}\right)$$

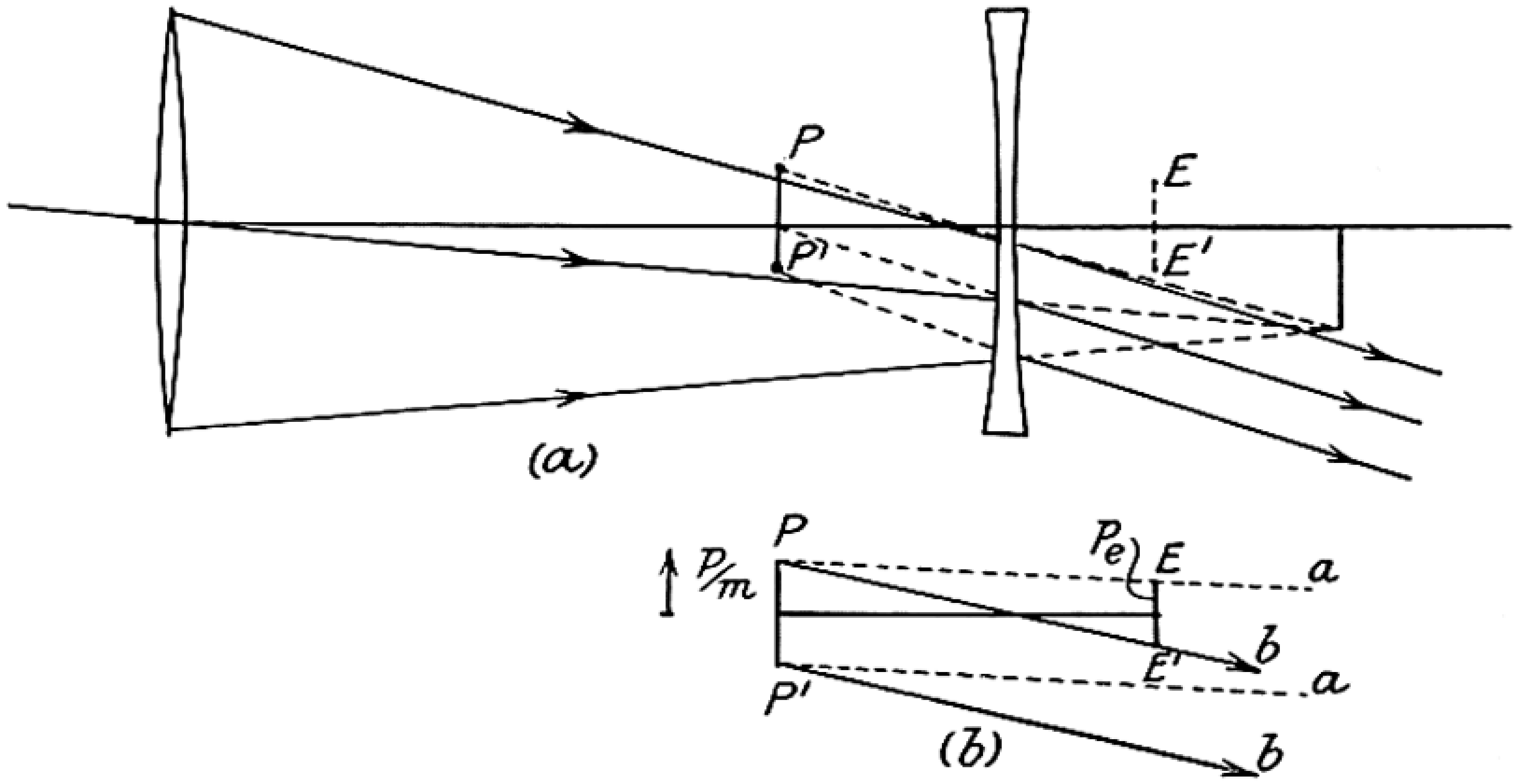


FIG. 37. FIELD OF VIEW OF A GALILEAN TELESCOPE

Let the distance of the pupil of the eye from the second principal point of the eyepiece be d, then its distance ρ from the exit pupil is given by
$$\rho = d - f'_b\left(1 - \frac{1}{m}\right)$$
Fig. 37 (a) shows the formation of the exit pupil PP′, and a bundle of parallel rays originally derived from an infinitely distant object point. Let the pupil of the eye be situated at EE′ (shown separately in lower figure). A bundle of parallel rays from the exit pupil will completely fill the eye pupil, provided that the inclination of the rays to the axis is not greater than that of the line (a) in the diagram; the angle with the axis is thus given by

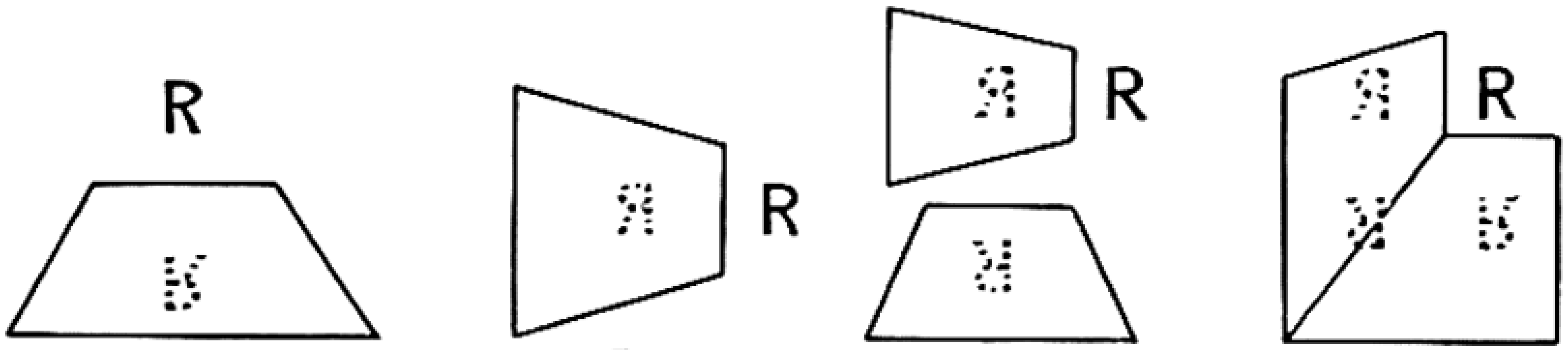
Maximum angle of full illumination
$$= \tan^{-1}\frac{\dfrac{p}{m} - p_e}{d - f'_b\left(1 - \dfrac{1}{m}\right)} = \tan^{-1}\left(\frac{p - mp_e}{md - f'_b\,(m - 1)}\right)$$

When the inclination of the bundle has increased to that of the ray (*b*) in the figure, no light can reach the retina.

Maximum angle of partial illumination

$$= \tan^{-1} \frac{\frac{p}{m} + p_e}{d - f'_b\left(1 - \frac{1}{m}\right)} = \tan^{-1}\left(\frac{p + mp_c}{md - f'_b\,(m - 1)}\right)$$

The visible field is therefore bounded by a ring of decreasing illumination which may be looked upon as the out-of-focus image of the exit pupil. We may clearly enlarge the field of view by increasing p, and diminishing d as far as possible. The quantity f'_b will be numerically negative so that by decreasing it (other things being equal) we shall increase the field of view, but this would call for a shorter focal length for the object glass to preserve the same magnification. The limit to progress in this connection is the difficulty of correcting the aberration of the negative eye lens when the focal length is reduced below 1 in. or thereabout. The objective radius p may be increased in a binocular until the mounts of the two objectives practically touch each other (if the interocular distance is to be adjustable, this condition must be reached at the lowest separation), and d is decreased as far as possible by bringing the eyes as close as possible to the eye lens.



(*a*) (*b*) (*c*) (*d*)

FIG. 38. MIRROR REFLECTIONS

The so-called "mean field" is given by

$$\tan^{-1} \frac{p}{md - f'_b\,(m - 1)}$$

and it will be seen that the effect of increasing m while other things remain the same will be to decrease the field. (The magnification would be increased by a longer focal length of the objective.)

Reflecting Erecting Systems. Perhaps the most important means of erecting the telescope image is now by the use of erecting prisms;

the suggestion was originally made by Porro, and put into practical use by Abbe.

The image of an object formed by reflection in a horizontal mirror is seen inverted (Fig. 38 (*a*)); the image formed by a vertical mirror is reversed, left to right (Fig. 38 (*b*)). The reversed image of the vertical mirror may be viewed in the horizontal mirror (Fig. 38 (*c*)). The result is a complete reversion and inversion such as is found in the real image formed by a telescope objective. If, therefore, the light from an objective suffers two such reflections between the objective and the image plane, it will be seen erect and correctly disposed when examined by the eyepiece. The light must not fall on the mirrors at too great an angle of obliquity, so that in general the direction of the principal ray will be more or less deviated after the two reflections.

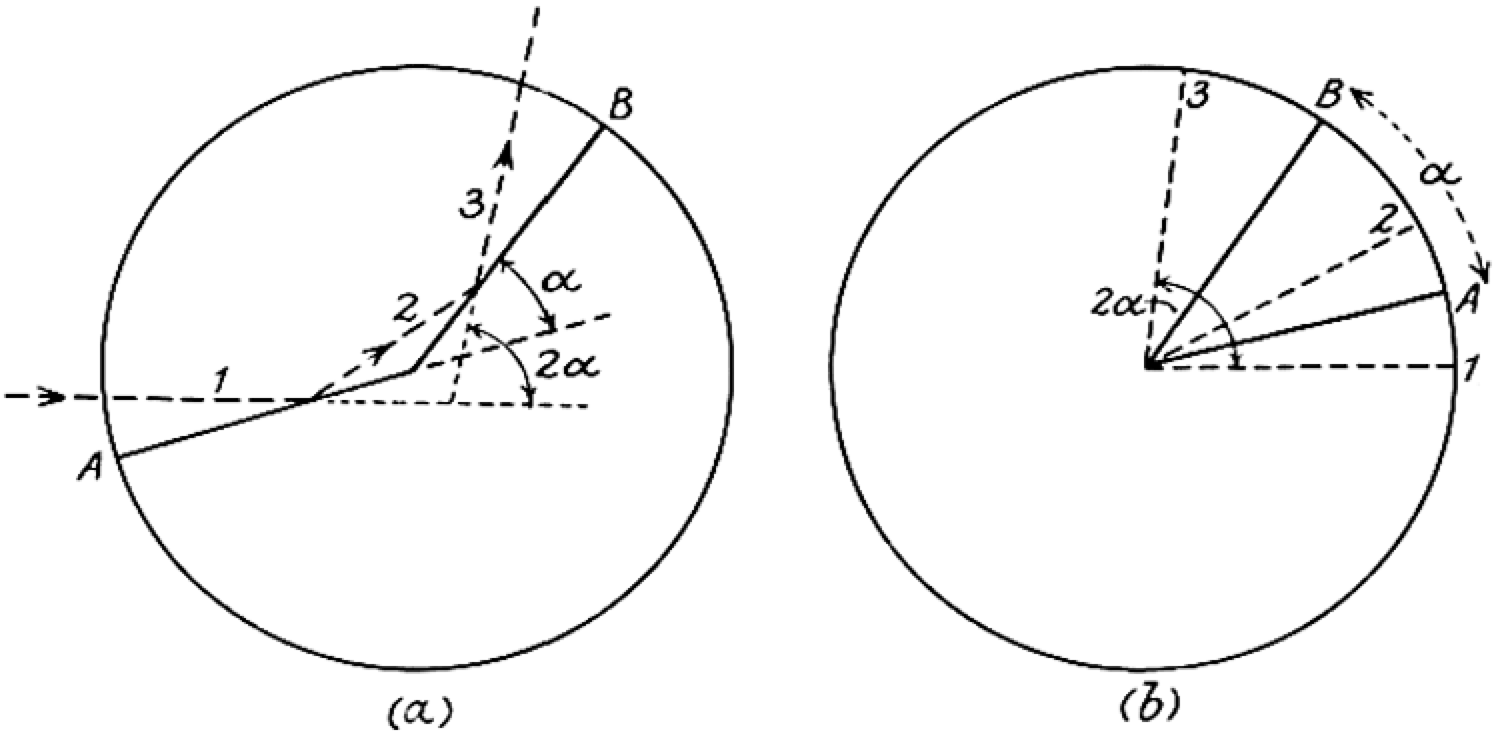


FIG. 39

Reflection at two inclined mirrors (*a*) and the corresponding polar diagram (*b*). In (*b*) the mirror direction is such that the reflecting side is met when going clockwise round the circle

If the two mirrors are placed together so that their planes intersect at right angles, reflections may take place at either one first, and in addition to the two images formed by one reflection in each mirror, there will be an inverted and reversed image to which reflection in both mirrors has contributed (Fig. 38 (*d*)).

In Fig. 39 (*a*), a ray shown in broken lines suffers reflection at two mirrors shown by full lines. It is easy to show that if the angle α between the mirrors is reckoned as shown in the diagram, then the deviation of the ray is 2α. Fig. 39 (*b*) is a "polar diagram" in which the directions and senses of the mirrors A and B are shown by the *outward* directions of the corresponding radii. The convention is that the reflecting surface represented in the polar diagram shall be met by going clockwise round the circle. In these and subsequent polar figures the ray directions are also shown by the outward directions of the corresponding radii. In Fig. 40 (*a*) the reflecting surfaces of the mirrors include an acute angle, and Fig. 40 (*b*) is the corresponding polar diagram. Let ray 1 pass through an object-point O between the reflecting surfaces; the path after reflection in mirror A is in a line passing through P, the mirror image of O. After reflection in B, the ray 3 appears to diverge from the point Q; it is easily proved that the angle subtended at the centre by the arc OBQ is equal to 2β, where β is the angle included between the reflecting surfaces of the mirrors. We might, however, draw a ray through O which strikes first the mirror B, then A. This ray would, after the double reflection, appear to originate from the point R, where the arc OAR also subtends an angle 2β at the centre. The points Q and R will evidently coincide only when $\beta = 90°$, and a single image will be seen. If β is not 90°, the angular separation of Q and R will be $360° - 4\beta$.

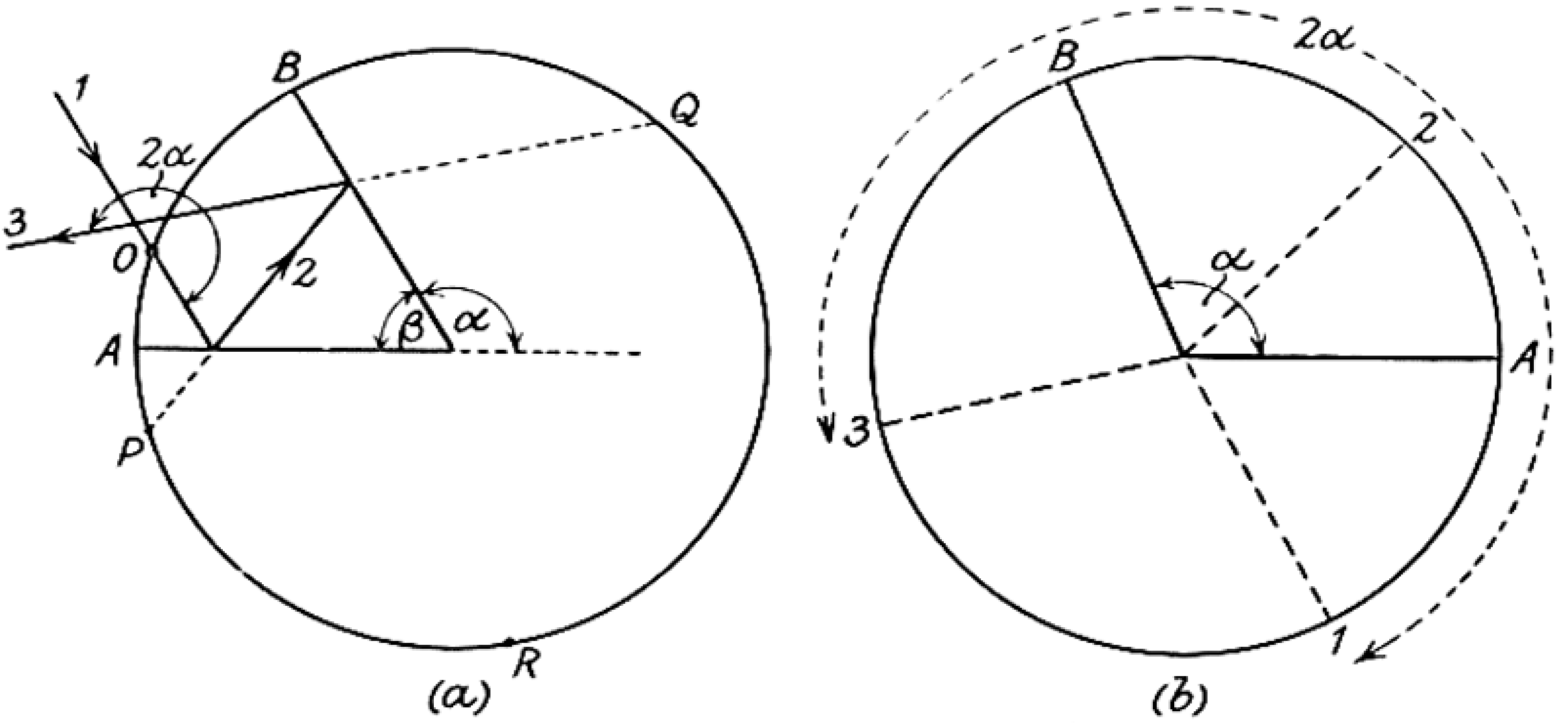


FIG. 40

(*a*) Reflection at two inclined mirrors (*b*) Polar diagram

A combination of two reflecting surfaces mounted so that reflection can first take place at either of them forms a "roof" reflector. If used in an instrument, the doubling of the image produced by any slight difference of β from 90° must not be perceptible to the eye. The tolerance will depend on the optical arrangements, but an accuracy within one second of arc may sometimes be called for. The eye would scarcely detect the doubling of the image if the separation of the components subtends an angle of less than 30 sec. in the visual field of view, but this tolerance is still somewhat large in regard to the full contour acuity of the normal eye. (Vol. I, Chapter V.)

Prism Faces as Reflectors. The isosceles inverting prism ABC (*Dove* prism) (Fig. 41) overcomes the difficulty, Fig. 38 (*a*), that the observer has to look into a new direction in order to see the image. The optical effect is conveniently studied by "developing" the prism and drawing the mirror image of the refracted ray paths, and we therefore find the effect as that of a thick inclined plate of glass in the path of the light. The effect of dispersion clearly separates a red and blue ray, but they emerge parallel. The lateral displacement has no effect if the object or image is at a distance so great that the confusion of the images is not perceptible, but it will be serious if the image is close to the prism. Parallel rays from a distant object will all be subjected to the same action by a parallel plate, so that no aberration can arise, but if the image is close to the prism surface, so that rays passing through any point of it to the prism surface are incident at various angles, then serious aberrations arise by refraction at the plane surfaces, which will clearly be especially objectionable if the axial ray to the central point of the field is obliquely incident. Hence, if any refracting prism surface must be placed in the path of a divergent or convergent beam, the surface should be normal to the axial ray in order that the central part of the field, where the corresponding angles of incidence have small obliquity, may be subject to the least possible disturbance. Fig. 42 clearly shows that if the base angles are not equal, the inverting prism will have a residual dispersive action which would (for example) draw the image of a star into a short spectrum.

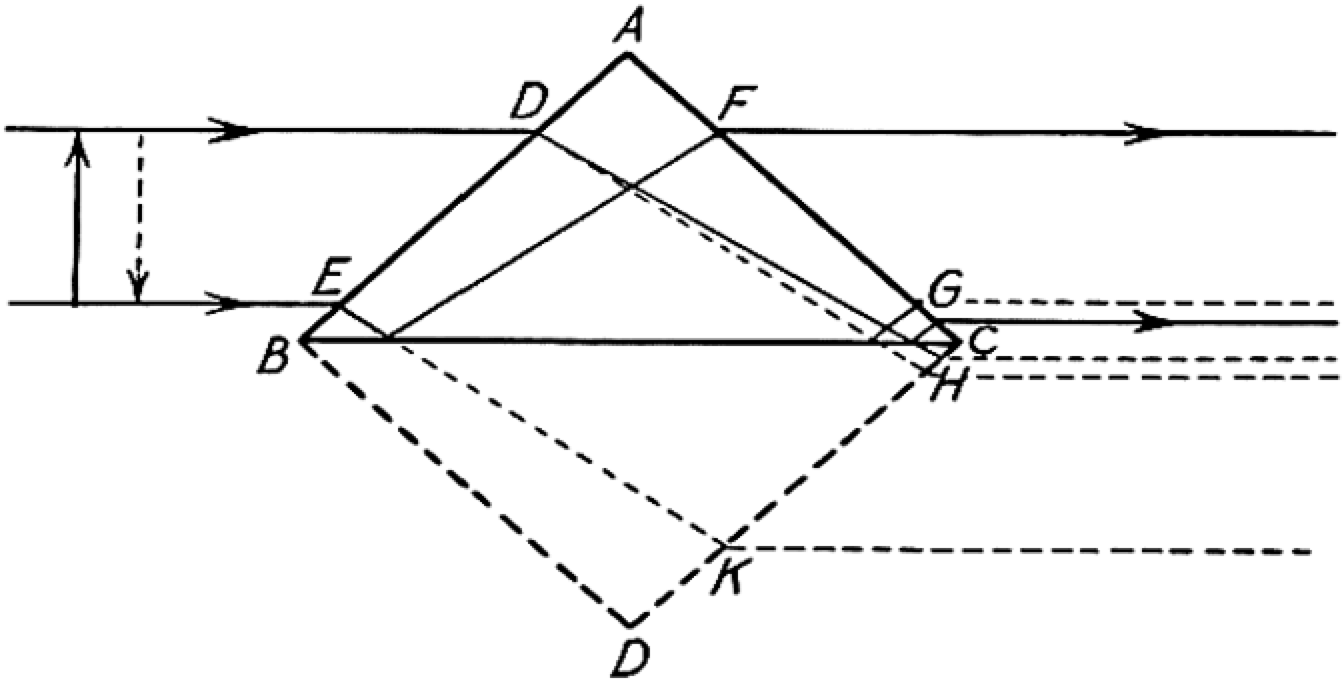


FIG. 41. EQUIVALENCE OF AN ERECTING PRISM AND A PARALLEL PLATE OF GLASS

Such a prism as that of Fig. 41 can, however, only produce inversion in the "up and down" sense while its base is horizontal. If it is placed in front of the objective of an inverting telescope the image will be erect, but reversed "left to right."

We can, however, obtain a prism which also gives reversion, by substituting the two faces of a roof reflector for the horizontal

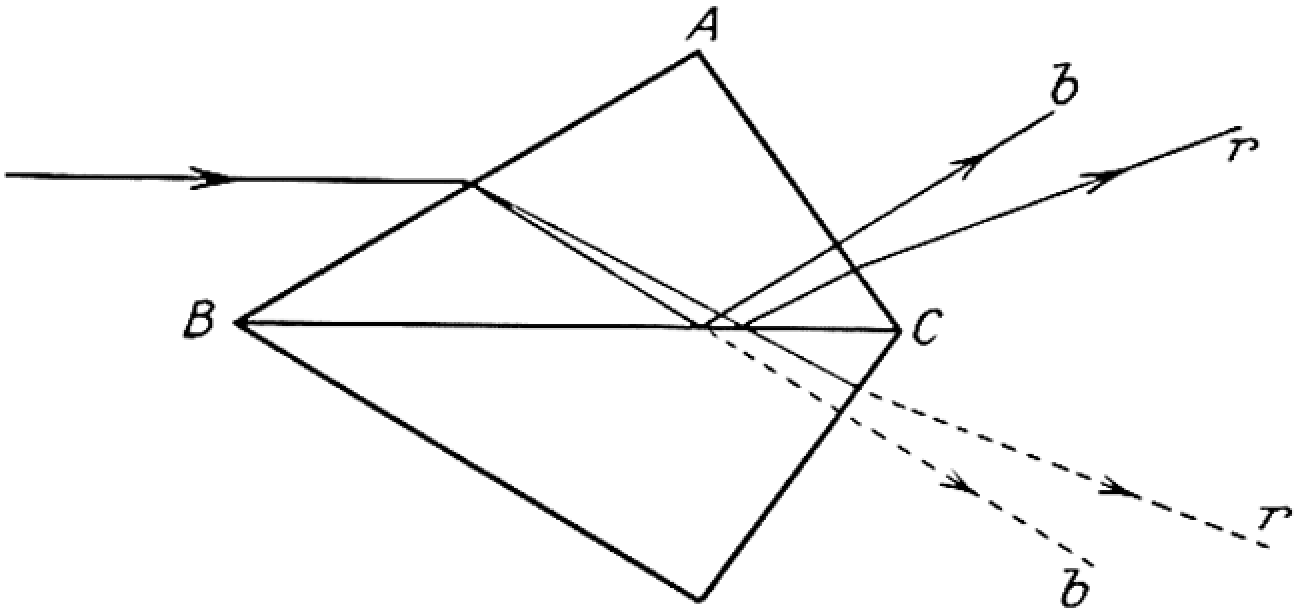


Fig. 42. Effect of Angle Error

reflecting face; each of the roof faces will be inclined at 45° to the horizontal plane, and the roof edge will be a horizontal line. Fig. 43 shows the end view and side elevation of such a prism. There is no need to retain the vertical sides of the simple reflecting prism (indicated in the end view). The top faces are, therefore, also cut at an

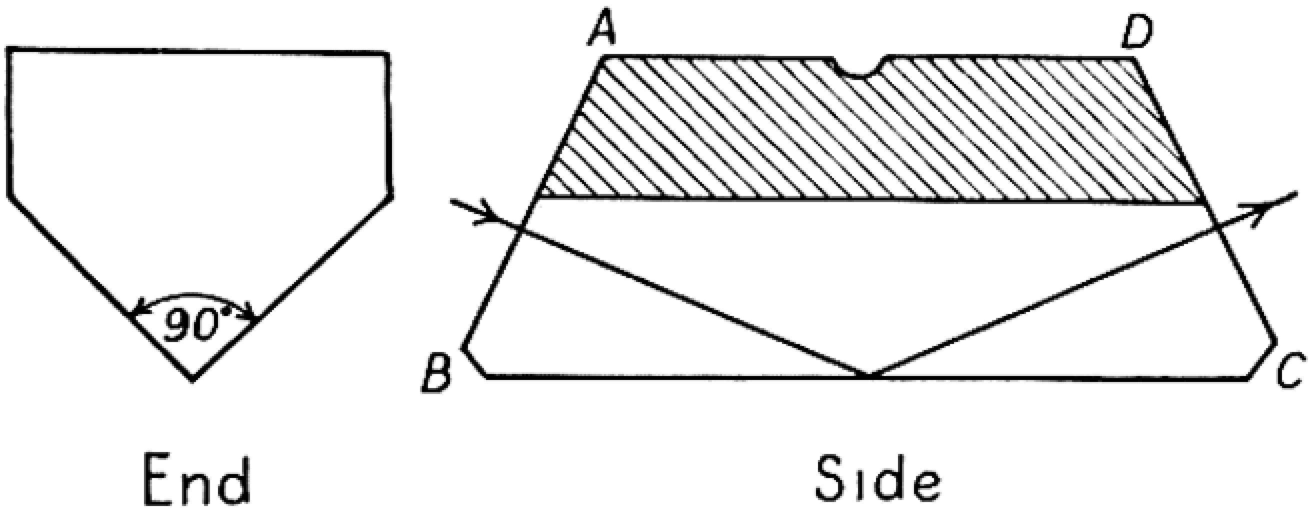


Fig. 43. Simple Roof Prism

angle of 45°, but only the end faces and the two roof faces are polished; the top faces are left rough-ground.

We can follow the action of such a prism by the aid of a three-dimensioned polar diagram as suggested by T. Y. Baker.[12] It must first be understood that when reflection takes place at two mirrors in turn (Fig. 44), *the image may be geometrically regarded as the result of swinging the object around the axis marked by the intersection of the reflecting surfaces.* If R is a real object lying in the plane of the diagram, images 2 and 2′ are formed after one reflection, 3 and 3′ after two reflections. The angle of swing is double the angle included between the reflecting surfaces. Fig. 44 (*b*) is a two-dimensional polar diagram of the same case, having significance only with regard to the orientation of the images and ray directions.

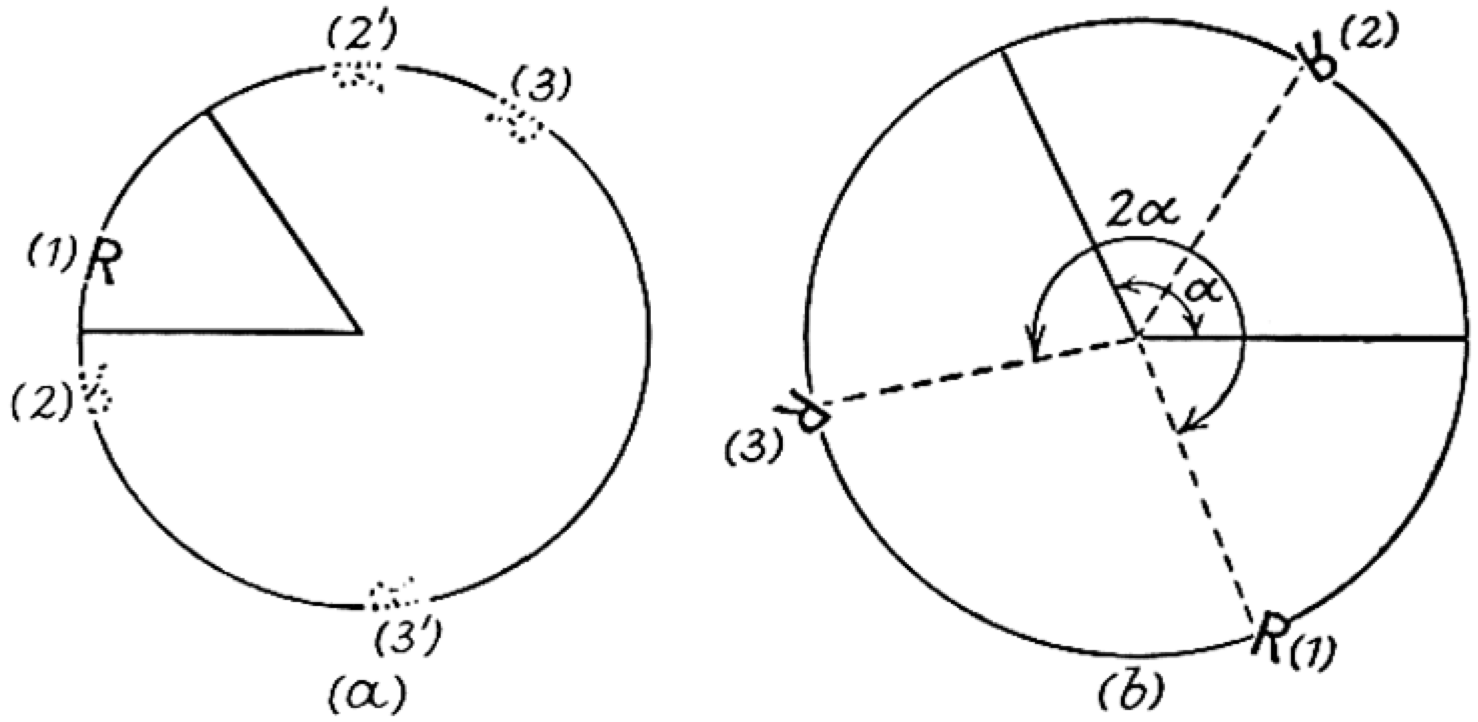


FIG. 44. ANGULAR SWING OF THE IMAGE

The letter R in positions 1, 2, and 3 is parallel to those in the left-hand diagram. The final image orientation at 3 is the result of a swing 2α round the axis, and is, of course, independent of the exact position of the reflecting surfaces, provided their line of intersection remains the same and that they include an angle α. Compare Figs. 44 and 40.

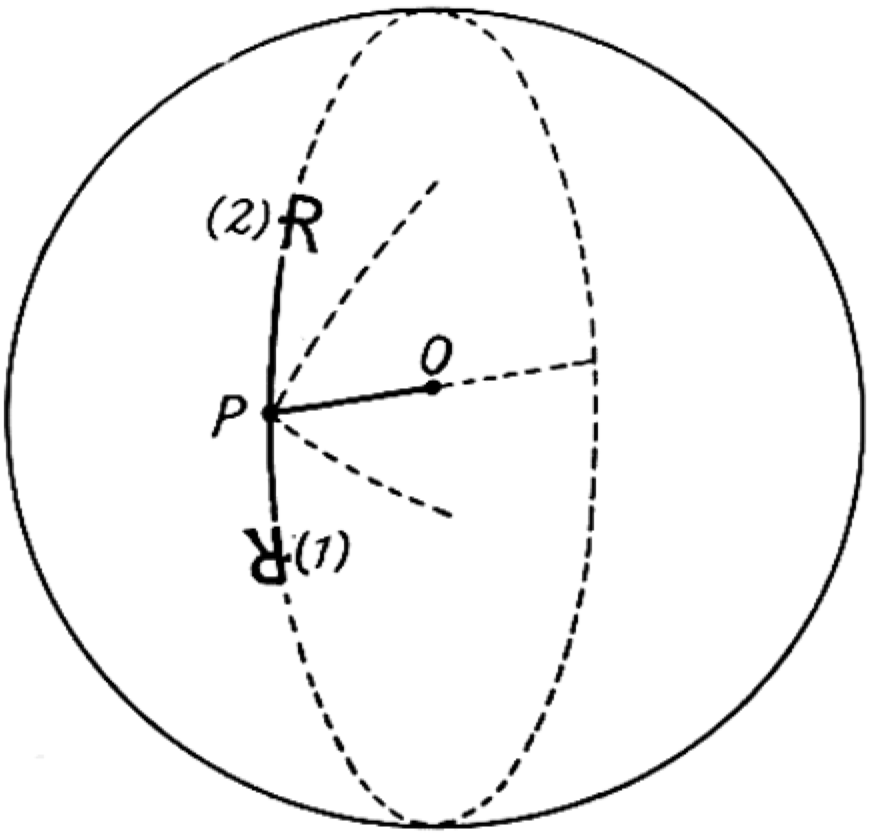


FIG. 45. POLAR DIAGRAM FOR "ROOF REFLECTOR"

We are now ready for the three-dimensional diagram. Fig. 45 represents a sphere with centre O, and the radius vector O1 to the point 1 on the circumference represents a downward directed ray corresponding to an inverted image suggested by the letter R. The point 1 is in the middle of the vertical stroke. Point 2 gives by the outward radial direction the required direction corresponding to an erect image. If both vertical strokes of the letters R are in a vertical great circle, it will be seen that rotation through 180° about a radius of the sphere, marked OP in the diagram, will be sufficient to effect the required change. This line OP, therefore, is the required direction of the intersection of the two reflecting surfaces required, and the angle between them must be 90°. Their directions can be shown (in the sense of the polar diagrams) by a pair of lines drawn on the surface at 90° through P. This represents the simple case discussed above when we use a roof prism with its roof edge horizontal. The roof prism is allowable since the rotation is to be exactly 180°. If some other angle, say 179°, of rotation were required, it would not be permissible to use a roof prism, because reflection must now take place first at *one* surface and then at the other for all rays if overlapping images are to be avoided.

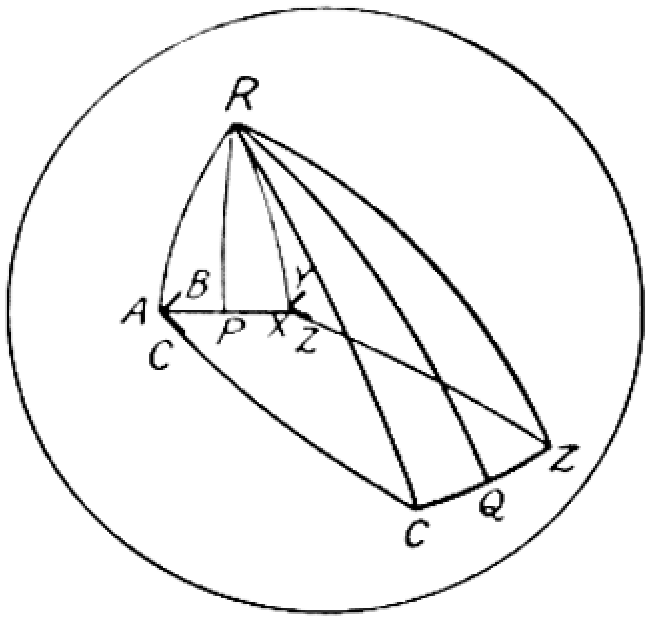


FIG. 46. GENERAL CASE

In general it can be shown that the displacement of a rigid body from one position to any other can be effected by (1) a movement of translation along a straight line, and (2) a movement of rotation about some axis.

An image can be given a shift of pure translation in any direction and of any amount by reflection in two suitably-disposed parallel mirrors (one reflection produces a reversal and a second is necessary to annul the reversal). Any required movement of rotation can be produced by successive reflection in two surfaces for which the roof edge represents the rotation axis. It follows that reflection from four surfaces will in general effect any required change in the position and disposition of an image, so as (for example) to bring it into the field of view of an eyepiece placed in any particular position. The vast majority of prism erecting systems employ four reflections.

The polar diagrams are helpful because they permit attention to be given in the first place to the required change of direction of the ray and orientation of the image. The lateral shift can be dealt with afterwards. General considerations are illustrated by Fig. 46. The outward radius to A represents the direction and sense of the axial ray before entering a prism system, while the point X represents the axial ray on leaving the system. The orientation of the object-space field is represented by the short perpendicular lines AB and AC which might represent the 12 o'clock and 3 o'clock directions respectively. The corresponding lines XY and XZ represent these respective directions in the image field.

The pole of the great circle containing AB is the point *c*, while the point *z* is the pole of the great circle containing XY. If we are to find some axis of swing, rotation about which will bring AB into the direction XY, this swing will bring the pole *c* to the point *z*. If we bisect AX at right angles by the line PR, the axis must lie on this line; equally, it must lie on the line QR which bisects *cz* at right angles. These bisectors intersect in R, the radius to which point marks the axis of swing. The angle of swing is clearly given

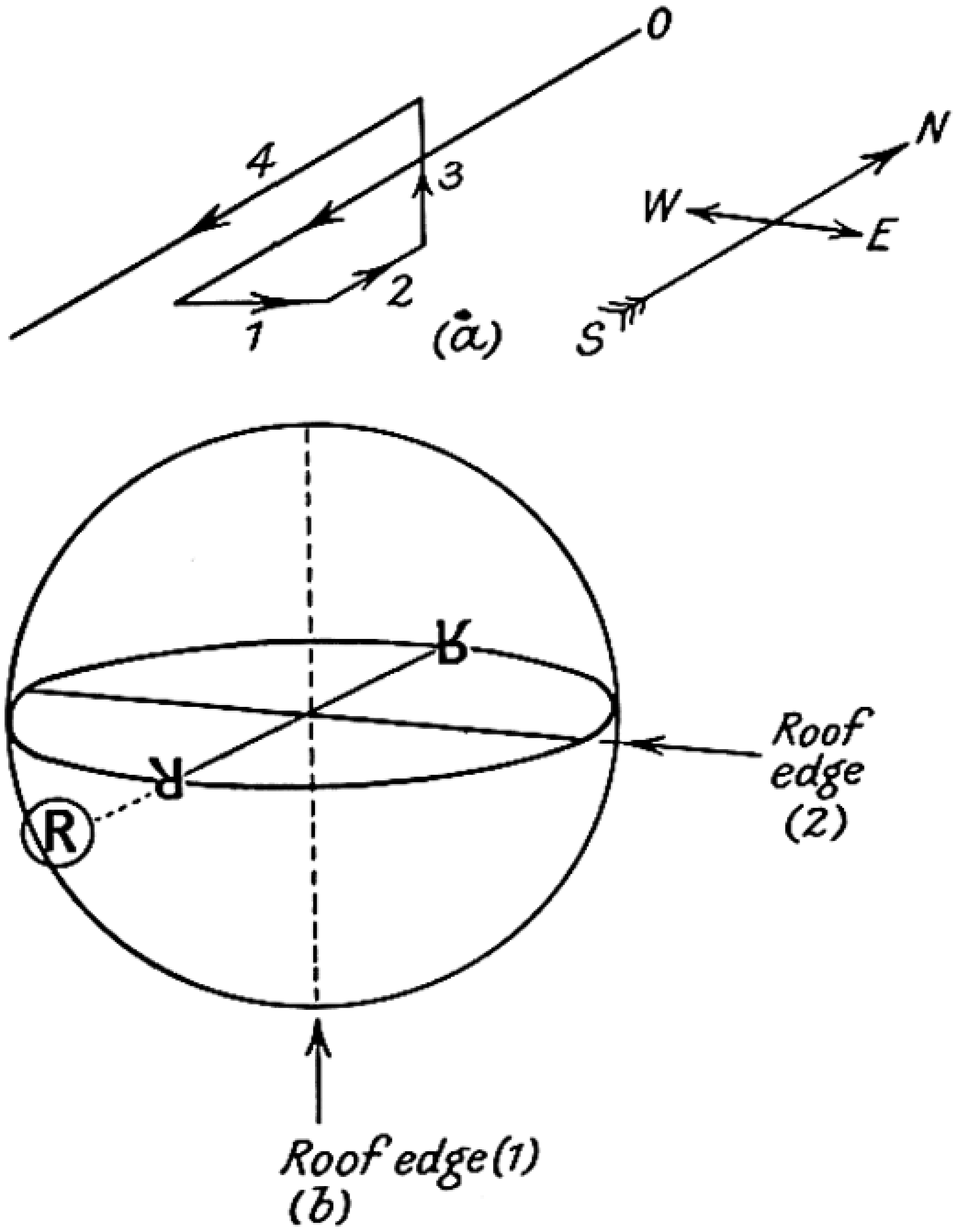


FIG. 47

(*a*) Path of ray in Porro prism system
(*b*) Polar diagram for Porro prism system

by $\widehat{ARX}$. The necessary angle between the reflecting surfaces, considered in the sense of the polar diagrams, will be $\widehat{ARX}$. Baker's original paper shows how to use the constructions in the graphical design of prism systems.

"Prism Binocular" System. In prism binoculars it is usually required for the final ray to be parallel to the direction of the incident ray. Fig. 47 (*a*) illustrates the case in which the incident ray O travels "south—east—north—vertical—south." This is accomplished by the aid of (1) a pair of reflectors at 90° with roof edge vertical, (2) a second pair with roof horizontal (see Fig. 48).

Referring to the polar diagram, in Fig. 47 (*b*), we see that the effect of (1) will be to swing the **ꓤ** round the vertical axis through 180°. The effect of (2) swings the image through 180° round the horizontal roof edge. The final result is **R**—distinguished with a ring in the diagram. Figs. 48 and 49 illustrate the arrangements of the prisms in practice. It will, of course, be clear that the system could be rotated as a whole round the incident ray (O) as an axis; the "erecting" effect would remain precisely the same, although the emergent ray would have moved with the system while remaining parallel to its original direction. The reflecting faces can be grouped in pairs for the purpose of the above theory, irrespective of the order in which they are encountered by the light.

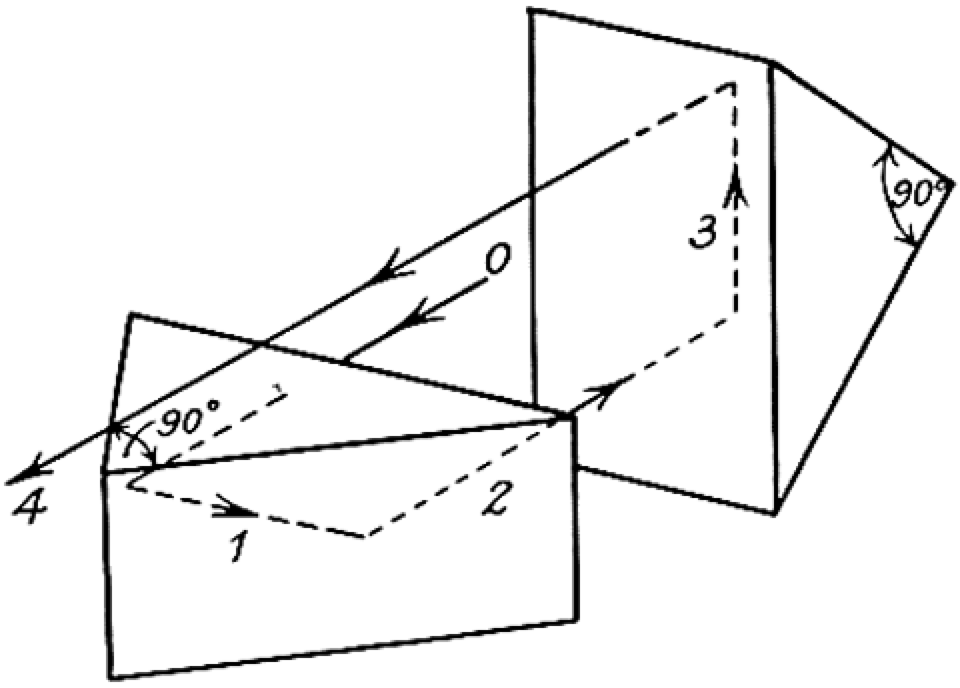
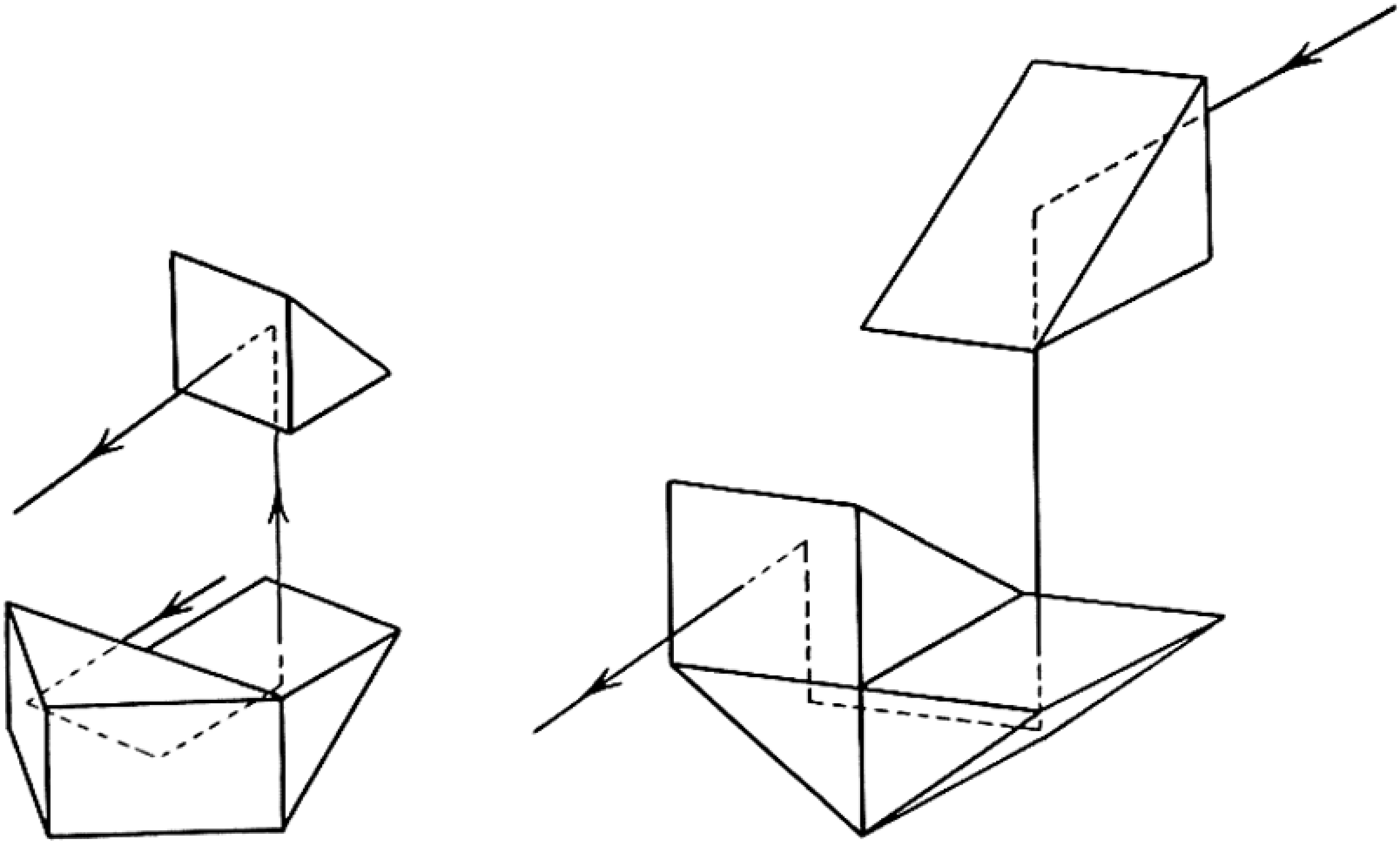


FIG. 48. PRISM BINOCULAR SYSTEM



(*a*) (*b*)

FIG. 49. VARIATIONS OF AN ERECTING SYSTEM

Such a system may, if necessary, be divided as shown in Fig. 49 (*a*), where part of the farther prism has been lifted vertically. In certain forms of periscopic telescope, the upper part of the system may be placed just over the objective, while the lower part of the

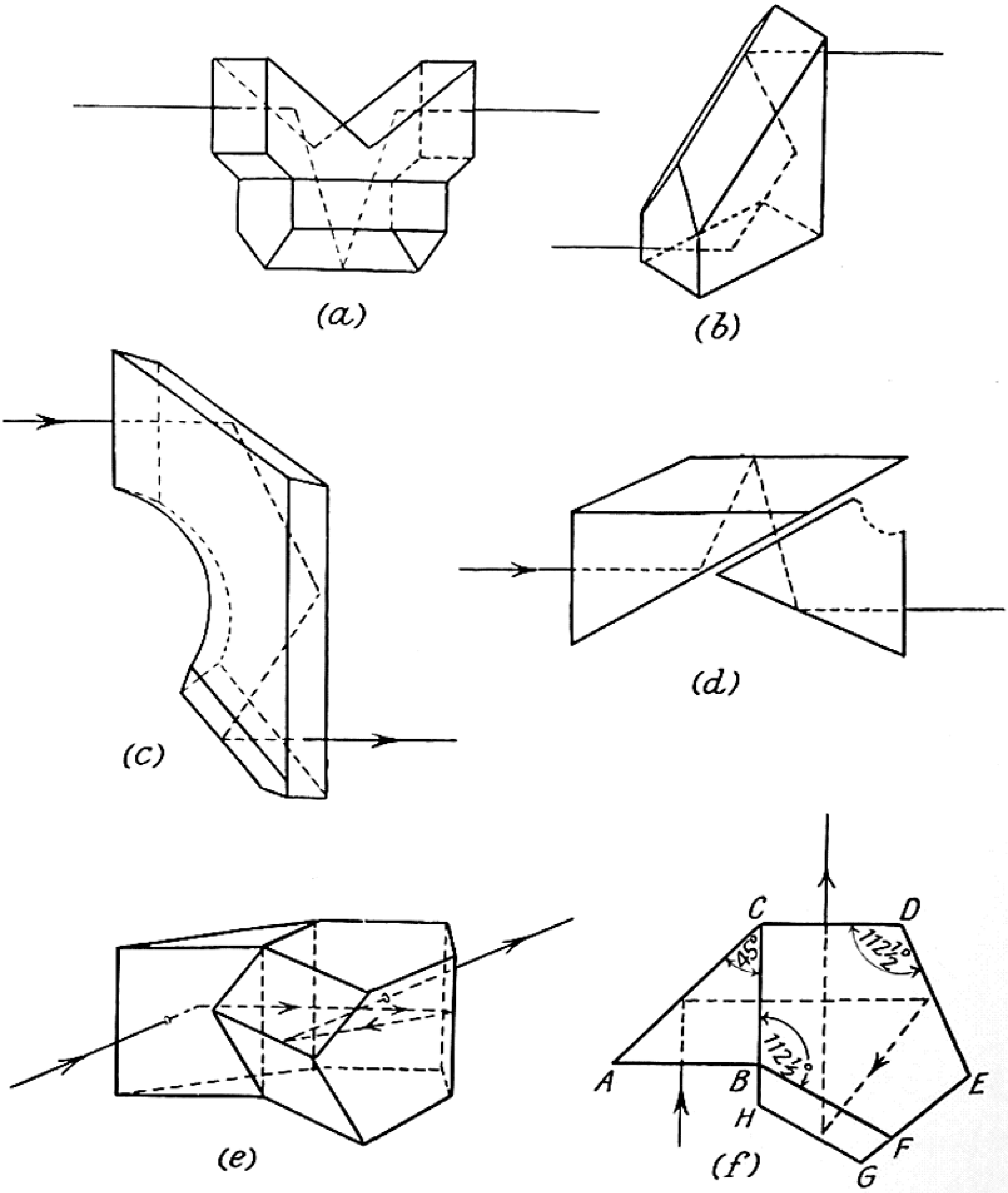


FIG. 50. A GROUP OF ERECTING PRISMS

(*a*) Abbe direct vision type
(*b*) Leman prism
(*c*) Modified Leman
(*d*) König prism
(*e*) Daubresse prism
(*f*) Daubresse prism (plan)

system is immediately followed by the eyepiece. Fig. 49 (*b*) shows a different arrangement; the reflections take place in a different order. All these cases involve a lateral displacement between the incident and emergent principal rays; reflection takes place at the various surfaces consecutively, and the extreme accuracy required for roof edge angles can be avoided. The 90° angles must be accurate to ± 3′ of arc, and the prisms should also be free from pyramidal error, i.e. non-parallelism of the 90° edge to the opposite face.

"Direct Vision Erecting Prisms." In many cases the loss of alignment between the axes of objective and eyepiece is undesirable, and the Abbe prism shown in Fig. 50 (*a*) illustrates an arrangement by which this is avoided. The disadvantage of this arrangement is the somewhat awkward size of the prism, which calls for a considerable enlargement of the telescope tube. An arrangement due to A. König, Fig. 50 (*d*), is somewhat similar, but allows of the use of a smaller prism; it will be seen that the division of the prism into two parts brings an air film into use, at which total reflection of the incident ray can take place at any point. When the ray returns from the roof edge it is incident at a very small angle, and passes through the air film with but slight loss.

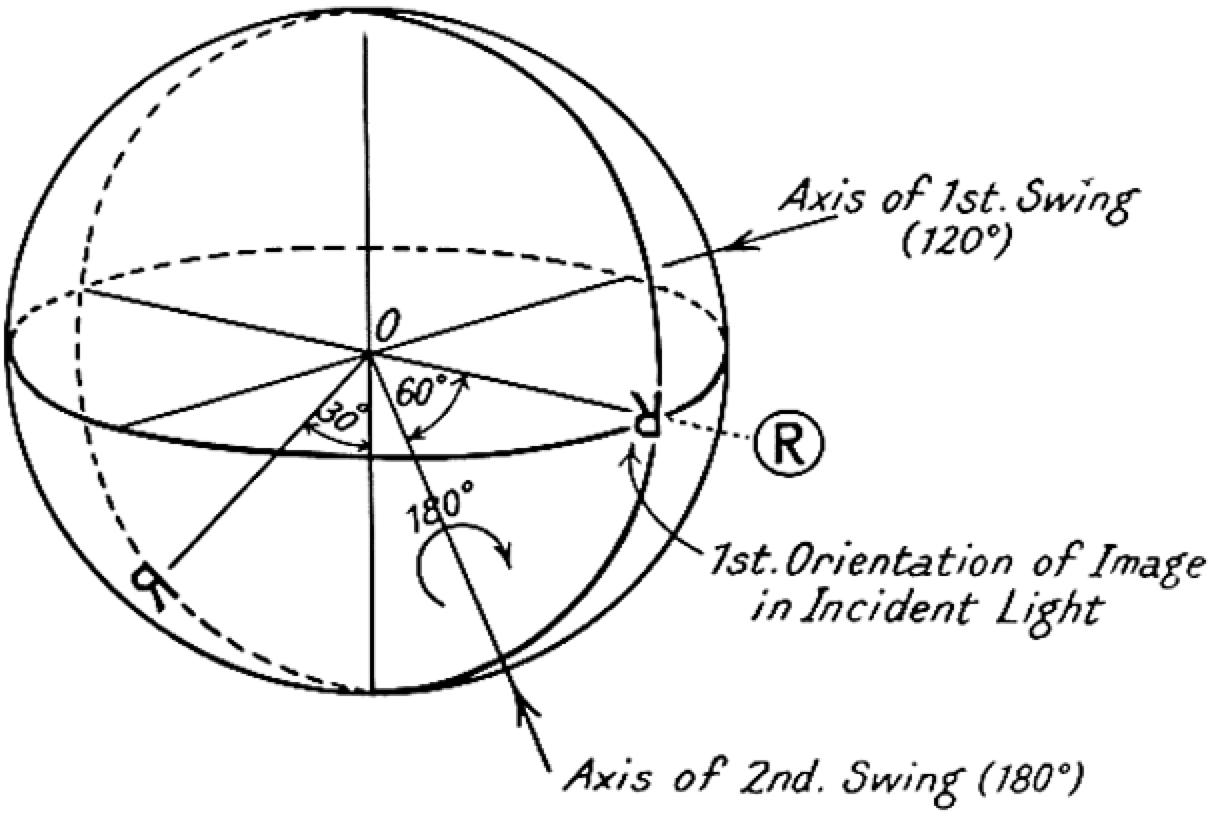


FIG. 51. ACTION OF MODIFIED LEMAN PRISM

A prism due to Leman is shown in Fig. 50 (*b*), and a modified form in Fig. 50 (*c*); the latter was widely used in an optical machine-gun sight in which the emergent ray is lowered by about 2 in. in order to secure useful protection to the eye of the gunner. The optical action of the system of Fig. 50 (*c*) is illustrated by the polar diagram of Fig. 51. The first axis of swing is horizontal, and the angle of swing is 120°. The next swing is one of 180° about an axis inclined at 60° to the horizontal (the lower roof edge). These two swings are clearly sufficient to erect the image.

The prism of Fig. 50 (*e*) is due to Daubresse, and can, perhaps, be understood more readily from the plan view (Fig. 50 (*f*)). It consists essentially of a right-angled prism ABC, together with one of the familiar prisms (the so-called "pentag") used in rangefinders (CDEFB). The "pentag" alone produces a deviation of 90° in the path of a beam, when it is important that the *deviation* shall not alter with any angular shift of the prism itself. Reflection in the "pentag" takes place at *silvered* surfaces corresponding to the lines DE and FB; the directions of these lines are at an angle of 45° with each other. Thus the "pentagonal prism" system consists essentially of two inclined mirrors *plus* a plane parallel block of glass. Rotation of such a system around the "mirror-intersection" axis produces no change in the angular deviation of a parallel beam doubly reflected from the mirrors.

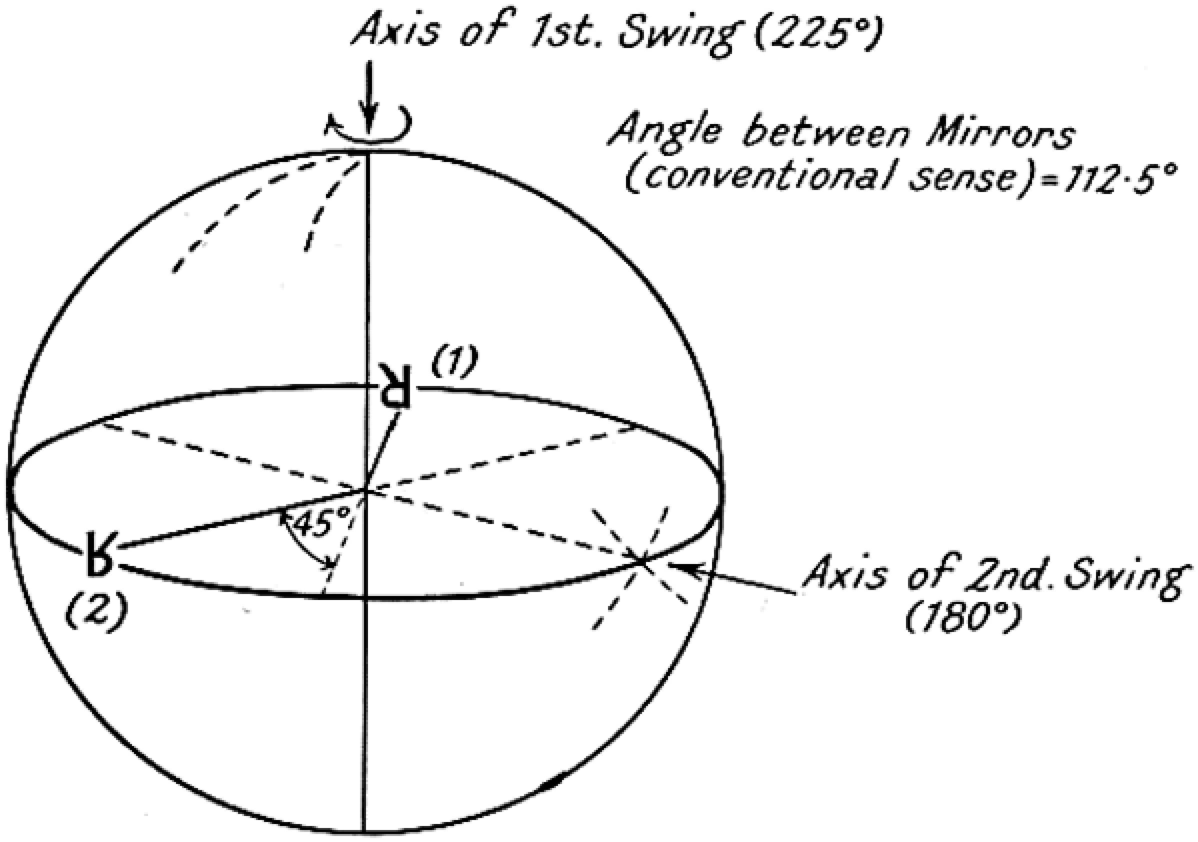

Axis of 1st. Swing (225°)
Angle between Mirrors (conventional sense)=112·5°
R
(1)
45°
R
(2)
Axis of 2nd. Swing (180°)

Fig. 52. Action of Daubresse Prism

In the Daubresse prism, the face FB is replaced by two roof faces which meet at 90° in a horizontal line. The action can now be understood from the polar diagram (Fig. 52). The inverted ᴚ at (1) marks the direction of the incident beam. The first axis of swing is vertical, and the swing is 225° produced by surfaces AC and ED inclined (in the sense of the polar diagram) at 112·5°. This brings the image to the position (2). The next swing must naturally be one of 180° about an axis symmetrically inclined to both (1) and (2), which makes an angle of 112·5° with the final direction 1. This is the direction of the roof edge HG in Fig. 50 (*f*). It will be clear that the prism ABC could be separated from the remainder of the system if it were desired to obtain a considerable separation of the incident and emergent rays, as, for example, in a periscope.

Periscopes and Other Specialized Telescopic Systems. Reference may be made to special articles (e.g. *The Dictionary of Applied Physics*, Vol. IV) for an extended treatment of this subject, of which only some salient features can be mentioned here. The essential optical problem involved is that of obtaining a reasonable field of view through a comparatively narrow tube.

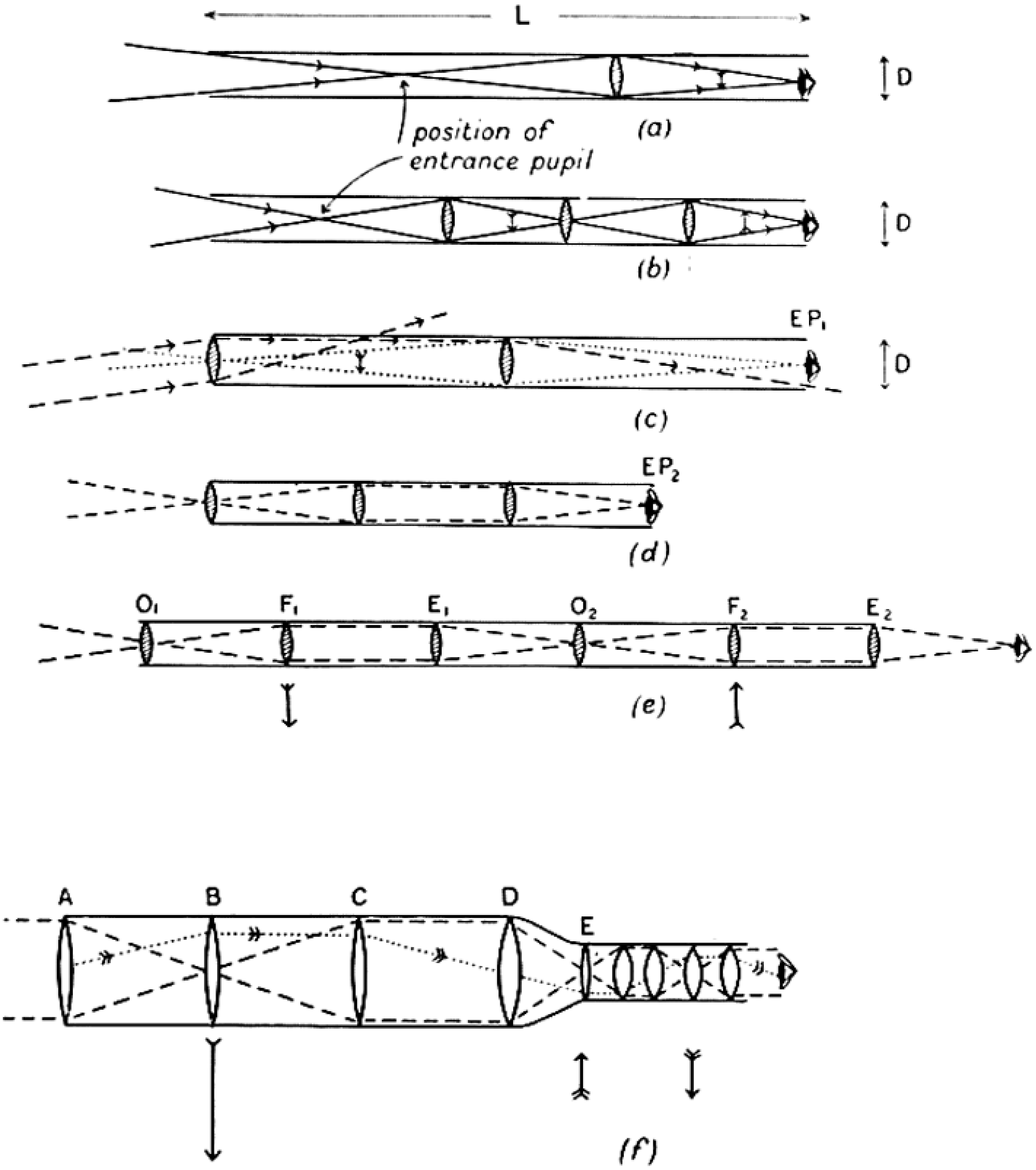


FIG. 53. PERISCOPE SYSTEMS

The optical system may be considered to project the image of the observer's pupil into a point near the entrance of the tube from which a field of view of adequate angle can be obtained. Thus if a lens of focal length $L/6$ is placed at a distance from the mouth of the tube equal to $2L/3$, where L is the length of the tube (Fig. 53 (*a*)), the tangent of the semi-angle of the field is $3D/2L$. The image must be seen by the accommodated eye; it will be observed (though inverted) under the same angle as is subtended by the object to the imaginary eye in the entrance pupil.

An increase of field is obtained by using more lenses. The case of Fig. 53 (*b*) gives an erect image, the semi-field being arc tan $5D/2L$; accommodation of the eye is again required.

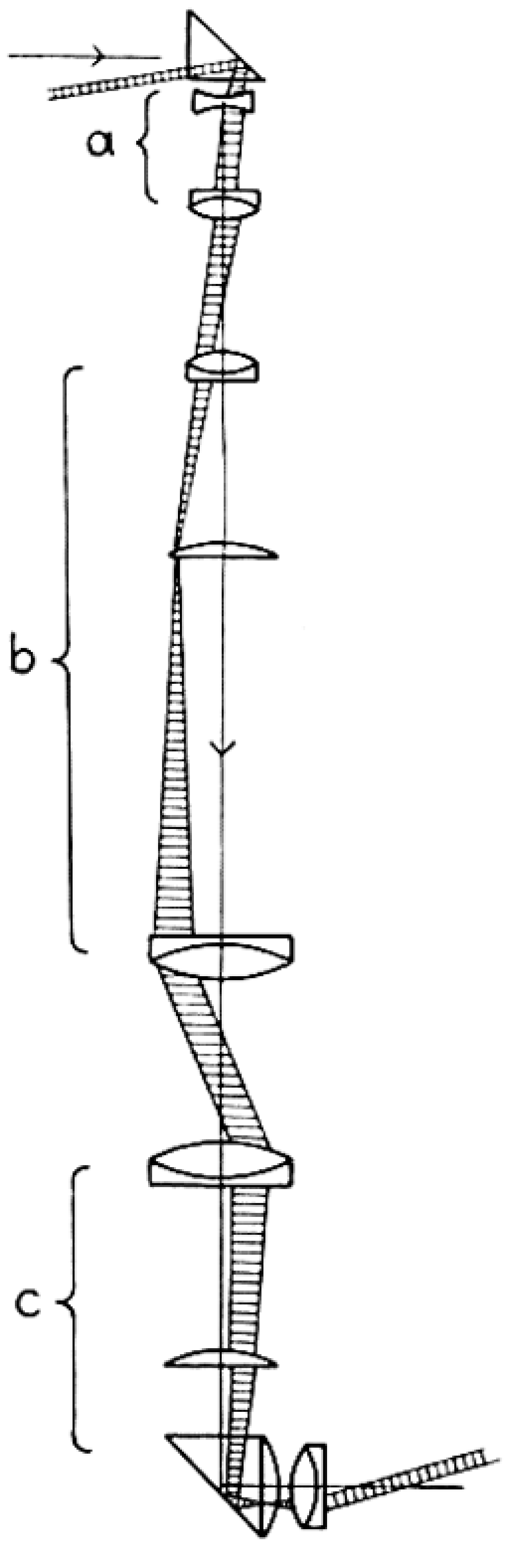


FIG. 54. PERISCOPE SYSTEM, BARR AND STROUD

This diagram is not to scale, and the ray paths as shown have exaggerated slopes. The optical system may be analysed into one erecting telescope (*a*), and two inverting telescopes (*b*) and (*c*)

The system of Fig. 53 (*c*) is a telescope of unit magnification; the lenses have equal focal lengths, i.e. $L/4$. The exit pupil of the instrument considered by itself is at EP_1. If the eye is placed here the semi-field of view is only D/L, but it is uniformly illuminated; if, however, a field lens is placed in the plane of the first image, having the same focal length as the others, the exit pupil is moved to EP_2 (Fig. 53 (*d*)) and the angular field will be doubled in size. If two such systems are used in series, we obtain the erecting system shown in Fig. 53 (*e*). The rays from any single object point are parallel between the lenses E_1 and O_2; consequently this separation may be varied without upsetting the focus of the instrument; but the position of the pupil will alter, and the full illumination of the field of view will be lost unless the eye is correspondingly moved.

Fig. 53 (*f*) suggests the means which must be adopted if the image is to be passed from a wider into a narrower tube. The lens D, which is placed in the exit pupil of the first telescope has (say) half the focal length of A, B, or C, and the image projected is therefore half the size of the image at B. The rays passing through the centre of the image in the plane of E converge at twice the angle of those through B, and would reach the walls of the half-diameter tube in a distance equal to $\frac{1}{4}$ CD. Hence, while D and E must have focal lengths equal to half those of A, B, or C, the subsequent lenses must have focal lengths of one-quarter of the same, and their mutual separation is one-quarter that found in the wider tube. In general, if the wide tube has *k* times the diameter of the narrower one, the relative frequency of the lens spacing in the latter is k^2.

Of course, the lenses in the telescopic systems forming such periscopes are not necessarily of the same diameter. Fig. 54 shows the optical system of a typical submarine periscope. An auxiliary Galilean telescope can sometimes be introduced (at will) into the system in order to vary the magnification. The use of auxiliary prisms in order to deviate the line of sight with any required direction is understood; they afford an additional means of erecting the image if required, i.e. with the aid of a roof reflector, etc.

The design of the lenses used in the "opposed telescopes" type of periscope follows the general lines of those in telescope systems. If non-achromatic field lenses are employed in systems of this type, the image definition may not appear to suffer much, but the chromatic aberration of the pupil position sometimes causes the field to show coloration. The curvature of field will often be additive in such systems, and the off-axis astigmatism can easily reach objectionable amounts.

Cystoscopes. Cystoscopes are instruments for the inspection of cavities in the body inaccessible to direct observation; they follow the usual optical design of periscopes, though the field of view required need not be very large, and everything depends on obtaining an extremely narrow tube. A minute electric lamp is usually added to the lower end.

"Flexible" Optical Systems. Considering a system such as that of Fig. 53 (*e*) it will be noted that whether we trace through the system a pair of rays which are parallel on entry, or a pair of principal rays which meet in the entrance pupil, they cross and recross from one side of the tube to another. Suppose now that a long tube contains a great many lenses thus arranged; if the tube is uniformly bent without changing its length along the axis it would be still possible to say that owing to this continued crossing-over the elongation of any path elements on the outside of a curve will be off-set by the contraction of an equal number of elements on the inside, and therefore the optical paths along such pairs of rays will still be equal in length.

Alternatively, we can consider that every telescope of unit magnification in such a system is an inverting one, and that owing to the bending of the tube, the rays received from the eyepiece of one system enter the second obliquely, but then (through the inversion) they are transmitted so as to conform generally to the change of direction of the containing tube.

Multiple systems of this character invented by G. Loeck are mounted in flexible tubes, and can be used for examination of interiors which can only be reached by flexible connections—such as the human stomach, etc. There are, however, numerous difficulties in making such systems work satisfactorily, and they have not (within the knowledge of the writer) yet been developed to the point of successful use in clinical practice (see Fig. 55).

Fibre Optics. If light is admitted to a flat-ended dielectric cylinder immersed in a medium of lower refractive index all interior rays incident on the outer surface at angles greater than the critical angle are totally reflected. Hence a thin narrow cylinder (glass fibre in air, for example) can serve as a conductor of the energy of the light, and this takes place even if the fibre is bent. A bundle of such fibres bound closely together can therefore conduct the energy from the elements of an optical image falling on a common plane of flat fibre-ends, and transfer it to a new plane. The chief optical troubles are (*a*) absorption of light in the material, and (*b*) interference with total reflection owing to the touching of the fibres.

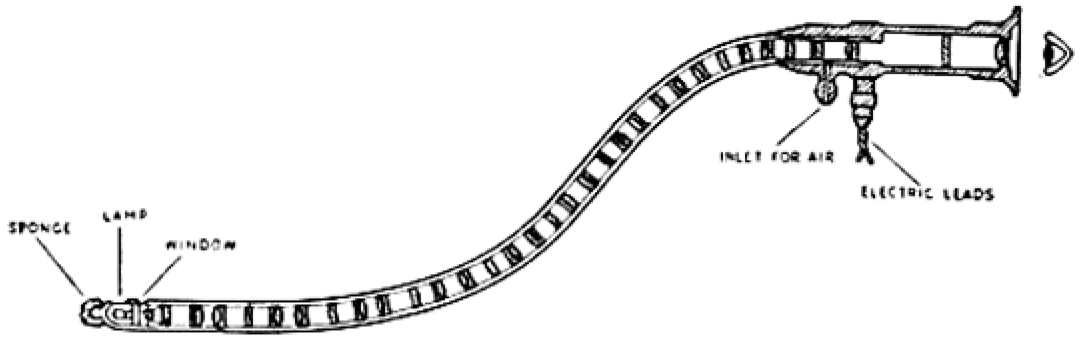


FIG. 55. FLEXIBLE CYSTOSCOPE

Related problems are discussed by N. S. Kapany[13] [14] and his collaborators. While the system has an obvious promise, it has apparently not yet been possible to develop it to the point of providing a satisfactory flexible cystoscope or other device.

Sighting Telescopes. The use of spider-web lines in the common focal plane of the objective and eyepiece of a telescope so as to form an optical sight or pointer was the invention of William Gascoigne, who also employed micrometers in the same situation for refined angular measurements. Spider webs are now replaced in many instruments by lines engraved, etched, or photographed on the surface of a plate, or *graticule*. Thus a scale of degrees, etc., is often carried by the graticule in a binocular. The objective and eyepiece of instruments using such devices should (in order to obtain the best results with them) be independently corrected as far as possible.

Precision of optical pointing requires (*a*) an aperture in the system adequate to secure the necessary resolution of the objects; (*b*) webs or lines, etc., suitable to the object; (*c*) adequate magnification so that some required liminal error of pointing becomes easily recognizable by the eye; and (*d*) adequate illumination of the field. If the magnification is too high the exit pupil of the instrument becomes too small; the definition of the visual image becomes poor, and the visual acuity further diminishes through lack of light (see p. 286).

Optical sights are of all degrees of complexity, from simple erecting telescope of unit magnification to very complex systems, one of which, an articulated sighting telescope (designed by Heyde), is illustrated in Fig. 56.

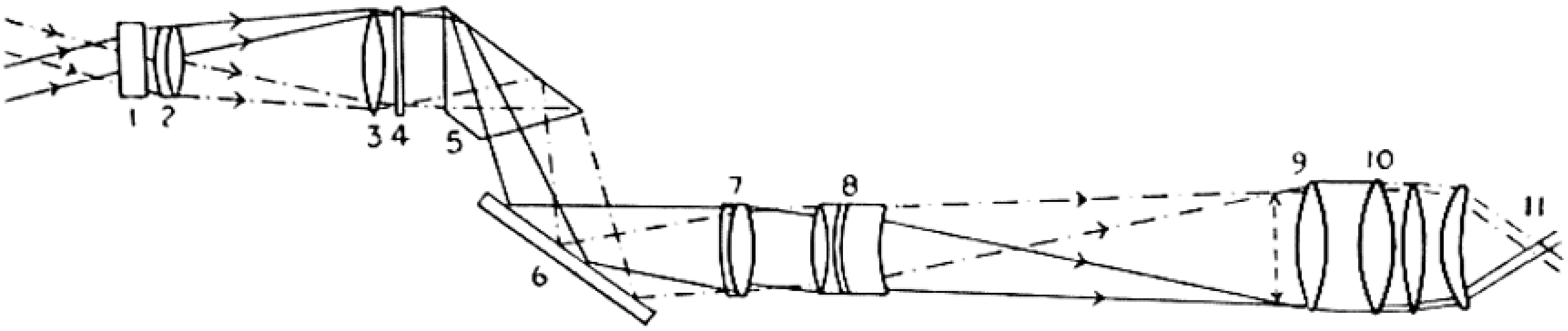


FIG. 56. ARTICULATED SIGHTING TELESCOPE
(*Heyde*)

(1) Window
(2) Objective
(3) Field lens
(4) Graticule (and 1st image)
(5) Isosceles prism
(6) Front surface metallized mirror
(7) 1st erector lens (renders beams parallel)
(8) 2nd erector lens
(9) Plane of erected image
(10) Eyepiece system
(11) Exit pupil

The front assembly, which can be moved to follow the elevation of a target, consists of the parts numbered 1–5. The reflecting surfaces of the prism and mirror are parallel when the axes of the objective tube and main tube are parallel. Hence when elevation takes place, the mirror 6 rotates in the same direction as the prism, but through half the angle, so that it always reflects the beams into the fixed main tube.

Reflecting Telescopes. In 1639 *Mersenne* proposed to make a telescope from spherical reflecting surfaces* by substituting a concave reflector for the objective, and a convex reflector for the negative eyepiece of a Galilean telescope. While it is true that reflecting astronomical telescopes are of great importance, the eyepiece in modern instruments is always of the refracting form.

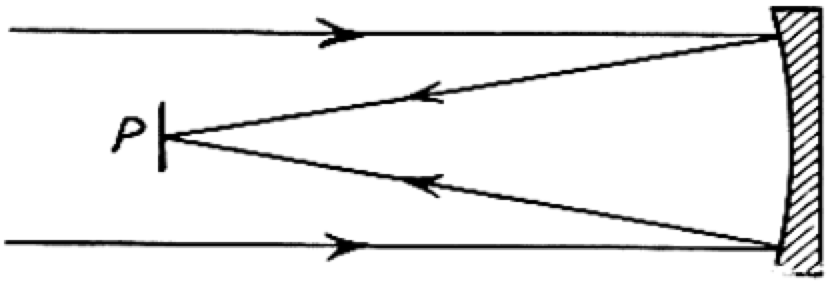


FIG. 57. PLATE IN FOCUS OF ASTRONOMICAL REFLECTOR

The "parabolic" mirror (of which the surface is a paraboloid of revolution) brings a bundle of rays parallel to the axis to a focus without spherical aberration. In the first method of using the mirror (Fig. 57), a photographic plate is mounted in the focal plane to register the image, which is not observed visually. The plate obscures a certain proportion of the mirror aperture; the concentration of light in the image "star discs" is slightly diminished; a circular mount will be desirable for the plate in order to avoid radial irregularities in the concentration.

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* Reflecting systems are sometimes spoken of as "catoptric" systems, in distinction from "dioptric" systems of lenses. Combinations of mirrors and lenses may be called "catadioptric" systems.

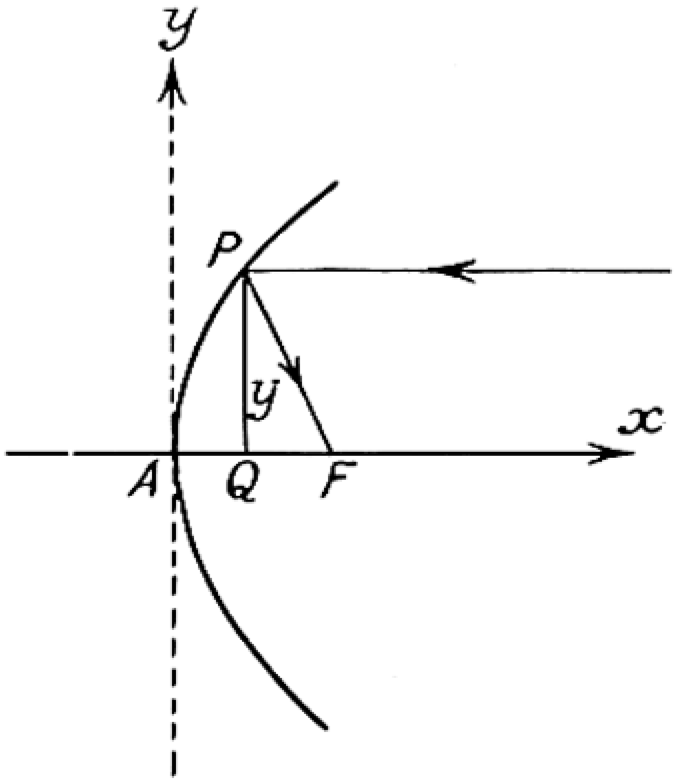

FIG. 58. REFLECTION AT A PARABOLOIDAL SURFACE

Aberrations of the Image. Consider the image formation at the focus of a parabola AP (Fig. 58); while it is free from spherical aberration, it is afflicted with coma. It was shown above that the optical sine condition for freedom from coma is

$$\frac{y_m}{\sin \alpha'} = \text{const.}$$

In our case $\frac{y_m}{\sin \alpha'}$ is represented by the length FP for each zone, and is clearly not a constant; let P be the point x,y, A being the origin of co-ordinates, and AF the axis of x. Let AF $= f$, the "focal length."

$$\begin{aligned} \text{FP}^2 &= \text{FQ}^2 + \text{PQ}^2 \\ &= (f - x)^2 + y^2 \end{aligned}$$

The equation to the parabola is $y^2 = 4fx$, so that

$$\text{FP}^2 = \left(f - \frac{y^2}{4f}\right)^2 + y^2$$

i.e.

$$\text{FP} = f\left(1 + \frac{y^2}{4f^2}\right) \quad . \quad . \quad . \quad (2.11)$$

In the absence of spherical aberration, the magnification, m_m, for a marginal zone may be written, using the sine theorem and putting $\sin \alpha = \frac{y}{l}$, where y is the incidence height on the mirror and l is the object distance (assumed to be very great),

$$m_m = \left(\frac{h'}{h}\right)_m = \left(\frac{n}{n'}\right)\frac{\sin \alpha}{\sin \alpha'} = \frac{n}{n'}\frac{y}{l \sin \alpha'} = \frac{n}{n'}\frac{\text{FP}}{l}$$

For the paraxial zone the magnification is

$$m_p = \frac{n}{n'}\frac{\text{AF}}{l}$$

Hence the offence against the sine condition is

$$\frac{m_m}{m_p} - 1 = \frac{\text{FP}}{\text{AF}} - 1$$

$$= \frac{y^2}{4f^2} \quad . \quad . \quad . \quad . \quad (2.12)$$

For the common value $\frac{y}{f} = \frac{1}{20}$ for a mirror, the above amount comes to $\frac{1}{1600}$. We mentioned above that general experience shows that for ordinary aperture ratios, the offence against the sine condition should not be allowed to exceed one part in 400; hence for parabolic mirrors of ordinary aperture, the coma is well within the tolerance and compares well with that characteristic of well-designed refractors; the amount speedily grows, however, when relatively large aperture ratios are used.

It may be shown (*Applied Optics;* Conrady, p. 393), that the physical tolerance for offence against the sine condition, in cases where primary coma alone need be considered, is expressed by $0{\cdot}5\lambda/h' \sin \alpha'_m$, where h' is the image height and α'_m is the angle of inclination of the marginal ray. If we put this equal to $y^2/4f^2$ we have

$$h' = 2\frac{f^2}{y^2}\left(\frac{\lambda}{\sin \alpha'_m}\right).$$

The factor in brackets is, approximately, the diameter of the Airy disc, and in the case where $f/y = 20$ the formula indicates that the radius of the field set by the physical coma tolerance is 800 Airy discs, a result of significance for astrographic work.

The astigmatism of the oblique pencils will be encountered in precisely the same way and for the same cause as that in a refracting telescope objective passed centrally by the principal ray, i.e. the slower increase of power for the sagittal than the tangential rays; the amount will be approximately the same as that calculated on page 38 of Vol. I.

The curvature of the field may be calculated from the ordinary expression for that characteristic of a refracting surface (Vol. I, page 137). We had

$$\frac{1}{n'\text{R}'_t} - \frac{1}{n\text{R}_t} = -\left(\frac{n'-n}{nn'r}\right)$$

Putting $n' = -n$, the field curvature for an infinite radius in the object field comes out to $\frac{2}{r}$; the radius of the Petzval surface is equal to the focal length.

Tangential and Sagittal Fields for a Concave Mirror. When the ordinary equations for the conjugate distances in narrow tangential and sagittal fans refracted at a spherical surface (Vol. I, page 34) are adapted to the case of reflection by putting $n' = -n$, we obtain

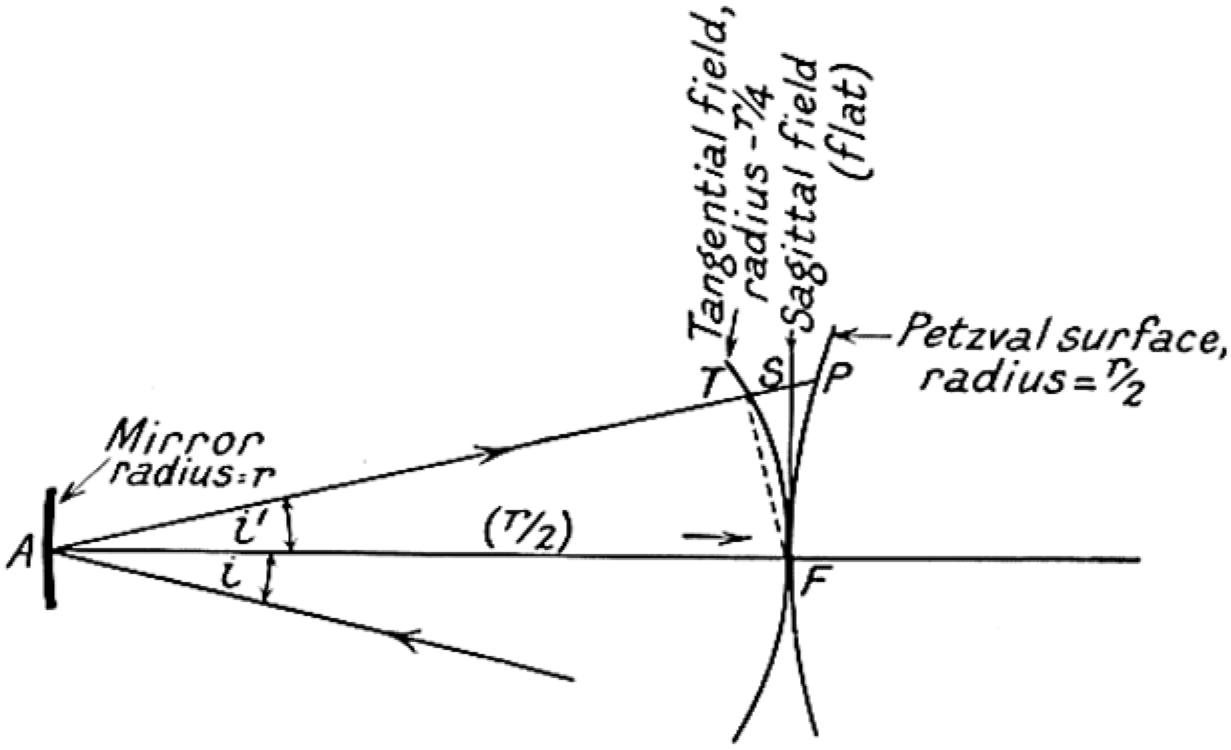

FIG. 59. IMAGE FIELDS WHEN OBJECT FIELD IS AT INFINITY

Formula for tangential fan: $\frac{1}{t'} + \frac{1}{t} = \frac{2}{r \cos i}$

Formula for sagittal fan: $\frac{1}{s'} + \frac{1}{s} = \frac{2 \cos i}{r}$

Referring to Fig. 59 we see that the image distance when t is infinite is given by $t' = (r/2) \cos i$, and is therefore found by drawing a perpendicular FT from the focus F to the principal ray AP. The tangential field near the axis is therefore spherical and has a radius $r/4$, since from elementary geometry: "the angle in a semicircle is a right angle."

Again, $s' = r/(2 \cos i)$. The sagittal field is evidently flat. If FP is the Petzval surface,

the interval TS $= \frac{h^2}{2\,(r/4)}$, approx., where $h =$ FS,

and SP $= \frac{h^2}{2\,(r/2)}$

so that TS = 2 SP, and, as we expect from theory, the tangential focal line is three times as far as the sagittal line from the Petzval surface. The above results apply strictly, of course, to very narrow apertures, but the system compares favourably with a refracting telescope.

The most important advantage of the reflector is, however, that the chromatic aberrations, both axial and radial, are absent.

The main difficulty in securing a good performance with a reflector is due to the necessary extreme accuracy of figure of the surface.

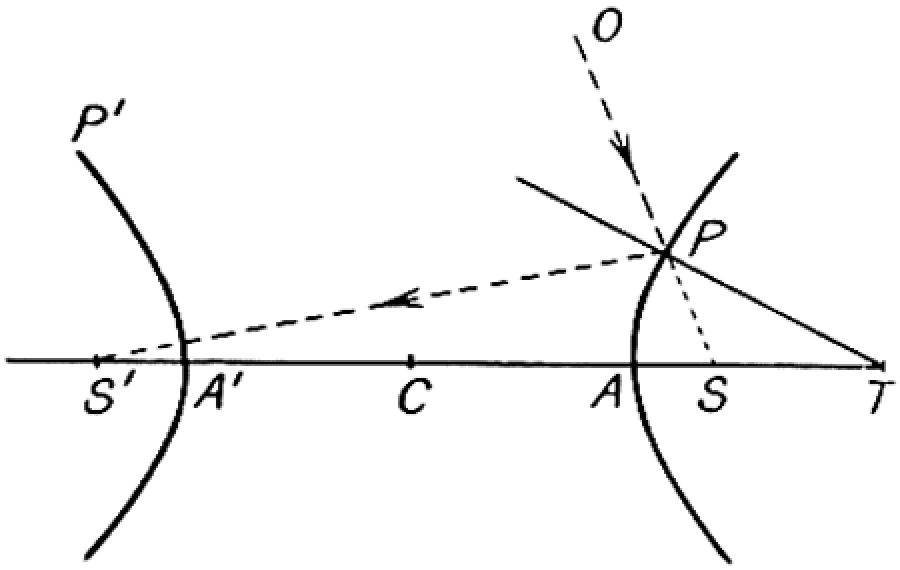

FIG. 60. REFLECTION AT A HYPERBOLOIDAL SURFACE

In order that the disturbance from the object point may meet in the image within the usual tolerance of $\frac{1}{4}\lambda$ path difference, the figure of the surface must be accurate to within $\frac{1}{8}\lambda$. For methods of testing see p. 301.

Cassegrain Telescope. In order to appreciate the action of the Cassegrain and Gregorian telescopes, we may introduce a simple theorem in analytical geometry. If AP, A′P′ are two branches of a hyperbola (Fig. 60),

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

the normal at the point P $(x'y')$ is

$$\frac{x - x'}{x'/a^2} = \frac{y - y'}{-y'/b^2}$$

which intersects the x axis at the point T given by putting $y = 0$, and obtaining

$$x = x'\left(1 + \frac{b^2}{a^2}\right)$$

But if e is the eccentricity of the hyperbola,

$$\frac{b^2}{a^2} = -(1 - e^2)$$

so that
$$x = x'e^2$$
and the distances of the intersection point from the two foci are
$$\text{ST} = \text{CT} - \text{CS} = x'e^2 - ae$$
$$\text{S}'\text{T} = \text{CT} + \text{CS}' = x'e^2 + ae$$
Now we have
$$(\text{S}'\text{P})^2 = (ae + x')^2 + y'^2$$
and
$$y^2 = (a^2 - x^2)(1 - e^2)$$
so that
$$(\text{S}'\text{P})^2 = (xe + a)^2$$
and similarly
$$(\text{SP})^2 = (xe - a)^2$$
Hence
$$\frac{\text{SP}}{\text{S}'\text{P}} = \frac{xe - a}{xe + a} = \frac{\text{ST}}{\text{S}'\text{T}} \text{ (from above)}$$

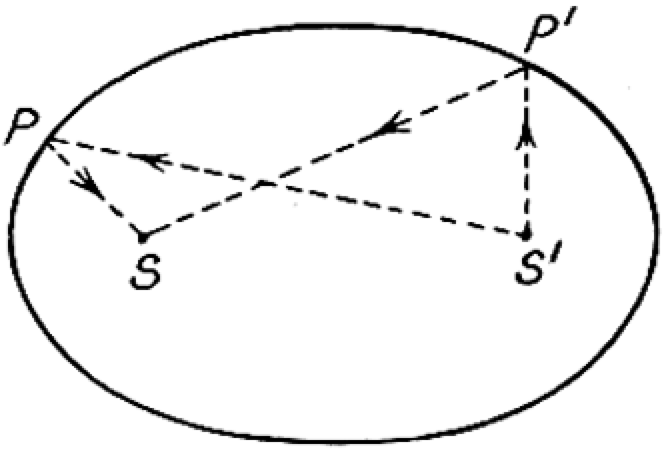

FIG. 61. REFLECTION AT AN ELLIPSOIDAL SURFACE

By Euclid, VI, A: "If, in any triangle the segments of the base produced have to one another the same ratio as the remaining sides of the triangle, the straight line drawn from the vertex to the point of section bisects the external angle."

Hence a ray OP proceeding to one focus S of the hyperbola would, if reflected at P, proceed towards the other focus S′. The action is exactly analogous to that of the conjugate foci of an elliptical reflector (Fig. 61).

In the Cassegrain reflector (Fig. 62), the main mirror is pierced by a central aperture. Instead of the Newtonian plain mirror, a convex mirror of hyperbolic form is placed so that its two foci are situated at S′ and S (the main mirror focus and the region of the aperture respectively). In this case an image free from spherical aberration will be formed at S. Sampson recommends an eccentricity of about 3. This determines the position of the auxiliary reflector.

It will be noticed that the use of such a reflector is like that of a negative lens in a telephoto system; it increases the magnification, approximately in the ratio of the distances of the surface from the two foci of the hyperbola. The diameter of the small mirror and that of the aperture in the large one are about equal in the usual case. If the diameter of the small mirror is $\frac{1}{p}$ that of the large one, then the resultant focal length can easily be shown to be $(p - 1)f'$ approximately, where f' is that of the main mirror.

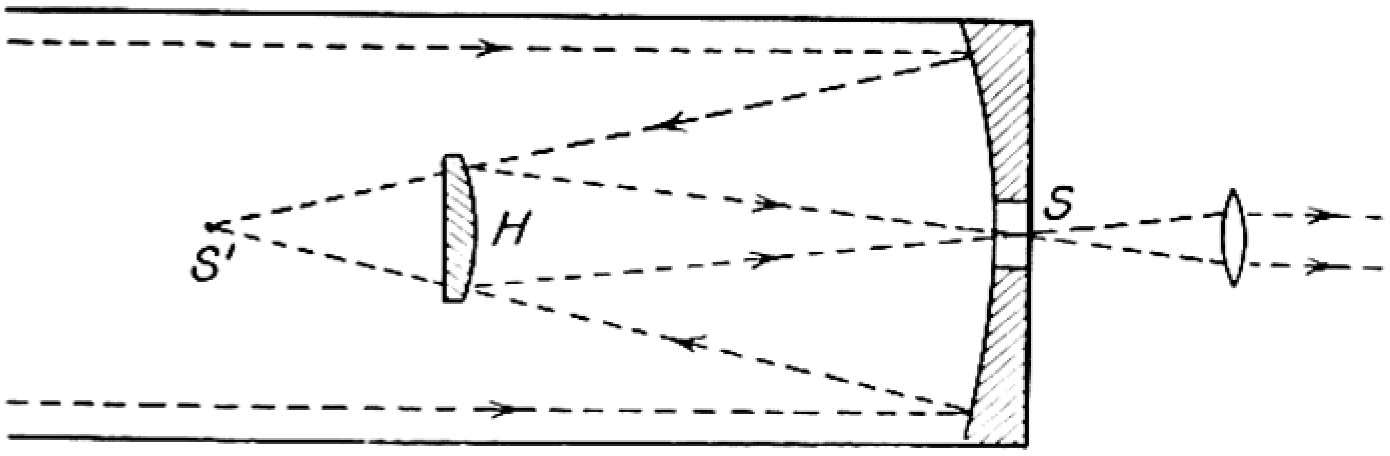

FIG. 62. CASSEGRAIN REFLECTOR

The Gregorian Reflector. In this case the auxiliary reflector is placed beyond the focus of the main mirror, and should therefore be given an elliptical section. The Gregorian form is less common than the Cassegrain.

Herschel's Telescope. Herschel's method of using the reflector is not widely used now; it is shown diagrammatically in Fig. 63. While it possesses some advantages in simplicity, and consequent conservation of light, over the arrangements described below, it is clear that the minimum angular tilt will be about $y/2f$, or y/r. Herschel's mirrors had an approximate aperture ratio of 1/10, i.e. $y/2f = 1/40$. Consequently, the inclination of the principal ray to the axis will be about 1·5°. Therefore the difference of the sagittal and tangential focusing distances will be $f \tan^2 (1{\cdot}5^\circ)$. It can be shown that, for a mirror of 1·22 metres diameter and 12·2 metres focal length, the intercept between the astigmatic foci is therefore 4·2 mm., and if an eyepiece is used which will make the telescope

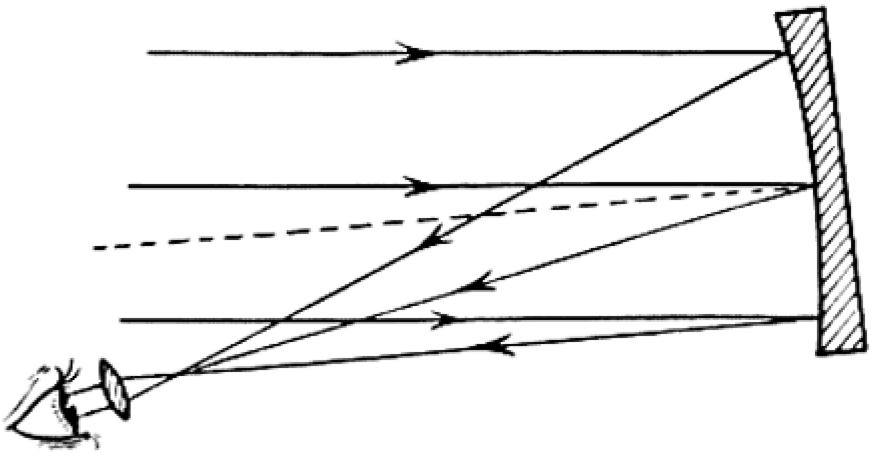

FIG. 63. HERSCHEL'S REFLECTING TELESCOPE
(Diagrammatic)

magnification 400, the intercept requires a 4-dioptre difference of accommodation for the two foci. The removal of this defect calls for special figuring of the mirror. In addition to the astigmatic defect, the effects of coma also become objectionable at such a comparatively large distance from the axis.

Newton's Telescope. It will be remembered that Newton saw no hope of making achromatic lenses, and therefore turned his attention to reflecting telescopes (1672). In the Newtonian form (Fig. 64), a small auxiliary mirror, at 45° to the principal ray, deflects the convergent beam through 90°, so that the focus is formed approximately in the locus of the surface of the telescope tube where it may be conveniently examined by an eyepiece. If D is the diameter of the objective, the distance of the mirror from the focus is D/2. The shape of the mirror must clearly be an elliptical conic section, the minor axis being $D^2/2f$, and the major axis approximately $D^2/f\sqrt{2}$. By this arrangement, the centre of the field lies on the axis. and unsymmetrical aberration is avoided.

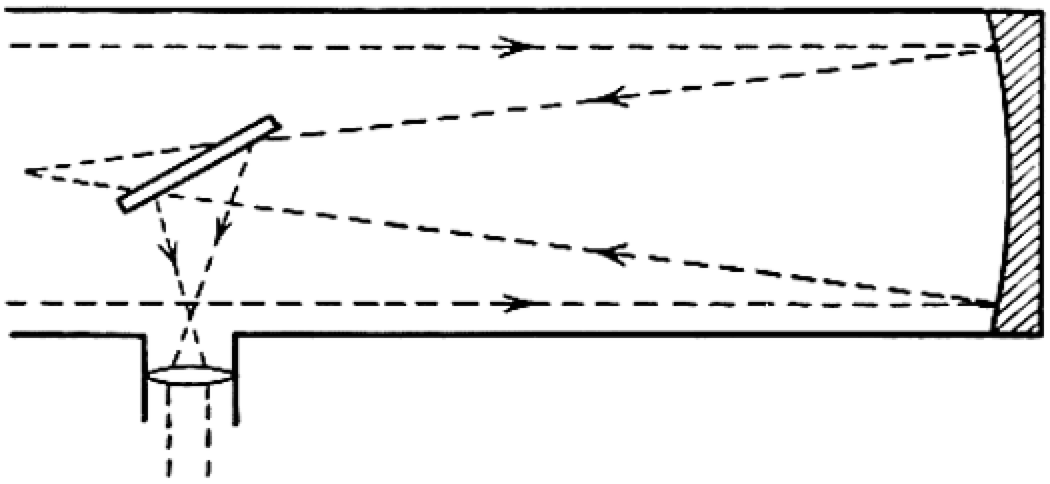

FIG. 64. NEWTON'S REFLECTING TELESCOPE
(Diagrammatic)

General Remarks on Reflectors. It is sometimes the practice to equip a reflecting telescope with alternative methods of securing the image. Thus in the Mount Wilson photographic reflectors, the image can be secured either with a Newtonian reflector, or with a combination of Cassegrain reflector and auxiliary Newtonian reflector, the latter being placed close to the mirror. Practically all modern mirrors are silvered on glass; the use of speculum metal has been very largely discontinued. Many difficulties are encountered when very large mirrors are required, since thick glass discs are necessary which are difficult to keep free from the disturbing effects of internal strain and temperature variations; these may distort the surfaces. For this reason it has been proposed to grind the discs in fused quartz, which has a much lower coefficient of expansion than glass. Pyrex glass has been employed for the 200 in. Mount

Palomar reflector. Recent developments in reflecting telescopes using aspheric correcting plates, etc., are described in Chapter VIII.

Reflecting telescopes have hitherto been little used for purposes other than astronomy. However, the relative ease in perfecting chromatic correction has been exploited in designs due to Wild (and described by Lotmar[15]) for a high-precision theodolite, and to Bouwers[16] for compact general-purpose systems.

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CHAPTER III

THE MICROSCOPE

It is possible that compound microscopes were constructed by Giambattista della Porta, but the actual introduction of the instrument must be credited to Zacharias Jansen, and Lippershey, of Middelburg, about the year 1610. The principle of the telescope would naturally suggest that of the microscope, and it is known that Galileo also developed both these instruments.

The practice of microscopy was developed by Hooke, whose *Micrographia* was published in 1665. The researches of Leeuwenhoek (born 1632) with the simple microscope, and Bonnani with his compound instrument (1697), helped on the development, but the early objectives were necessarily made of very small aperture in order to avoid overwhelming chromatic aberration. The discovery (1733–1758) of the achromatic lens finally enabled a great advance to be made in this instrument, but progress was slow until the beginning of the nineteenth century when Marzoli (1808–1811) constructed plano-convex cemented objective lenses, used with the plane side turned towards the object. He was followed in this step by Chevalier (1825). About this time J. J. Lister, in England, and G. B. Amici, in Italy, began their work. In 1830 Lister published his discovery of the two pairs of spherical-aberration-free conjugate points for plano-convex doublets constructed from suitable glasses. Principles of compensation were used in Amici's objectives of 1827. Lister was one of the earliest to realize the value of a wide aperture in the object glass.

The theory of the "Lister" points will be understood from Fig. 65. The "thin lens" approximate discussion of Chapter II indicates a shape somewhat similar to that of the diagram for the aplanatic crown-flint achromatic combination used for a telescope object glass. If the object point B is brought nearer the lens it is still possible to find a lens corrected for spherical aberration, and approximately for coma, which has a form not greatly different. Reference to the spherical aberration equation (p. 38) shows that having once calculated such a form we could derive an equation giving the aberration of the crown lens in terms of $\boldsymbol{L}_1$, the vergence $\left(\frac{1}{l_1}\right)$ to the object. The flint lens would furnish another equation in $\boldsymbol{L}_{1b}$, which would be transformed by putting

$$\boldsymbol{L}_{1b} = \boldsymbol{L}_{1a} + \boldsymbol{F}_a$$

We should therefore obtain a quadratic equation in $\mathbf{L}_1$, for the spherical aberration of the complete lens, and the solution would furnish *two* roots which represent *two* pairs of conjugate points free from spherical aberration for small apertures of the lens. It is found that the residual coma has an opposite sign in the two cases so represented.

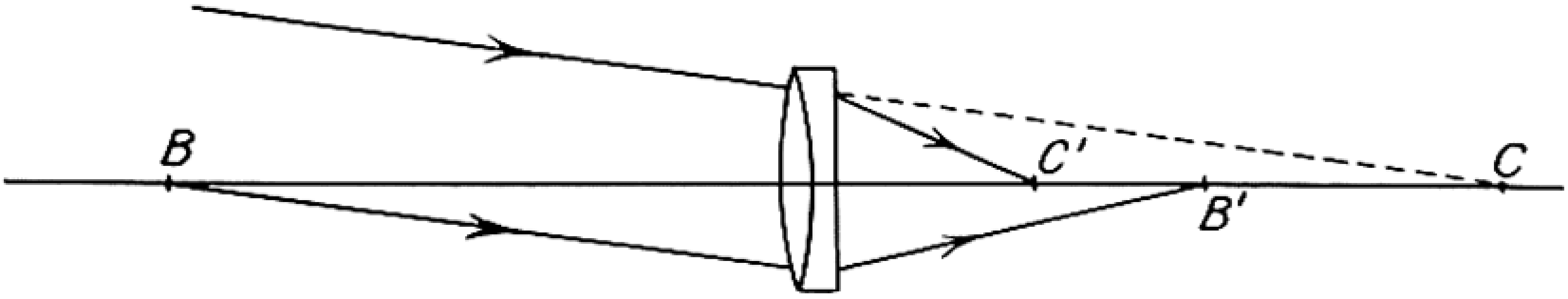


FIG. 65. THE APLANATIC POINTS OF A DOUBLET LENS

Two such pairs are shown in Fig. 65. One pair, B and B′, are real object and image points, but of the other points, C and C′, the point C represents a virtual object.

Fig. 66 shows an aplanatic combination of two cemented doublets. The first lens (*a*) works on the conjugate pair represented by B and B′, Fig. 65; but lens (*b*) works on the conjugate pair C and C′. Reverse the direction of the light, treating C′ as the object point, and we have an aplanatic objective of the Lister type. The advantage thus gained is the production of a system of short focal length in which the aberrations are not excessive. If it is sought to attain the same short focal length by using one doublet lens and increasing the curvatures of both components, then the aberrations of the system become much more troublesome, for even if spherical aberration is corrected for one zone, there are very serious amounts of spherical aberration for other zones (zonal aberration), for the restriction of which it is necessary to diminish the *aperture* of the lens very severely.

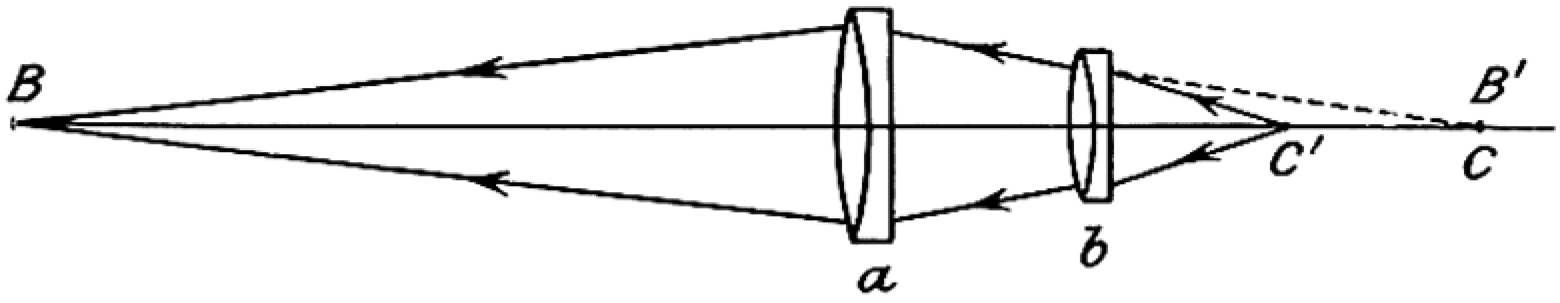


FIG. 66. LISTER OBJECTIVE

The importance and exact significance of the "aperture" of the lenses will be dealt with before discussing the further development of the microscope objective.

Magnification. The microscope consists of two positive systems, "a" and "b" (Fig. 67), of which the adjacent principal foci are separated by a distance g; the systems are the objective and eyepiece respectively.

The object of height h has an image of height h' which is projected by the objective into the first principal focus of the eyepiece.

$$h' = -h\frac{g}{f'_a}$$

The image is viewed under the angle ω' where

$$\omega' = \frac{h'}{f'_b} = -\frac{hg}{f'_a f'_b}$$

An object viewed by an unaided eye will naturally be placed in the nearest point of accommodation, which is at a distance β, say, from the first principal point of the eye. Thus,

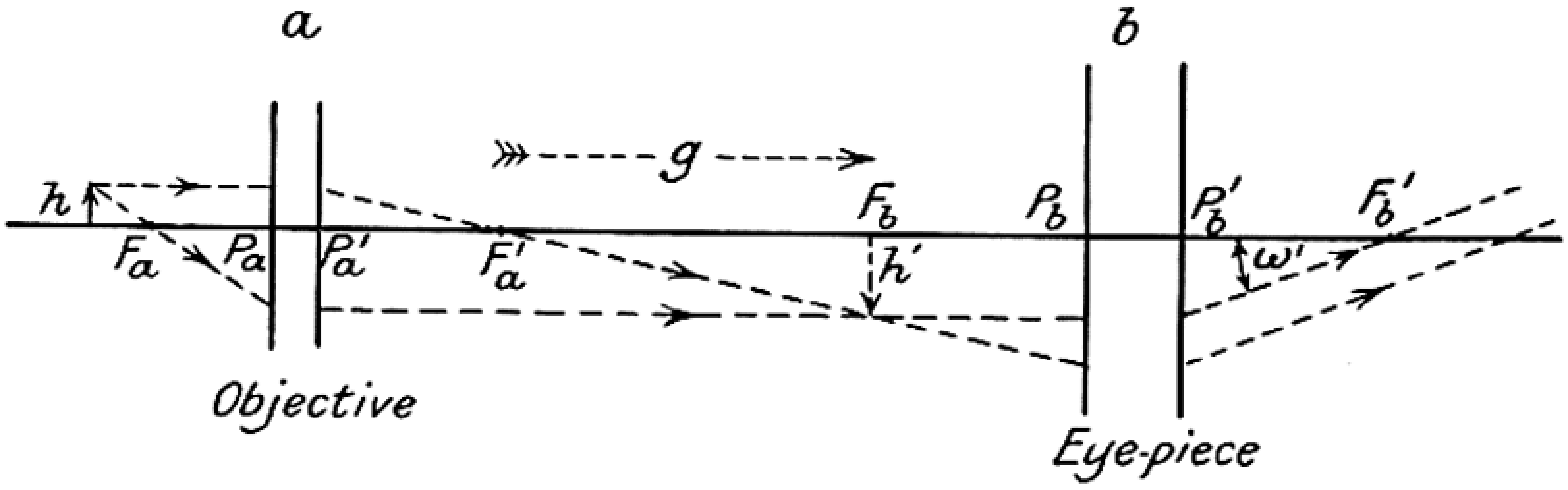


FIG. 67. A GAUSSIAN DIAGRAM OF THE MICROSCOPE

$$\text{Angular subtense of object to unaided eye} = -\frac{h}{\beta}$$

Hence,

$$\text{Magnification} = \frac{\text{Angular subtense of image to eye}}{\text{Angular subtense of object when at near point}}$$

$$= \frac{\beta g}{f'_a f'_b}$$

The expression can be written $\left(\frac{g}{f'_a}\right)\left(\frac{\beta}{f'_b}\right)$, and it will be seen from above that the first bracket represents the "first magnification" of the objective, i.e. ratio of linear size of image to object measured transversely to the axis. The second bracket $\left(\frac{\beta}{f'_b}\right)$ is clearly the "magnification" of the eyepiece used as a simple magnifier.

Modern models of the microscope often have a length such that the normal value of g, the "optical tube length" is 160 mm. (sometimes 170 mm.). The older English tube had an optical tube length of 10 in.

Amongst English makers it is usual to specify the eyepieces by the magnification, i.e. (approximately)

$$\frac{10 \text{ in.}}{\text{Focal length of eyepiece in inches}}$$

or

$$\frac{250}{\text{Focal length of eyepiece in millimetres}}$$

taking 10 in. or 250 mm. as measures of the least distance of distinct vision. In the Continental catalogues, however, the objective magnification is listed as

$$\text{Objective magnification (Continental)} = \frac{250}{\text{Focal length of objective in millimetres}}$$

and

$$\text{Eyepiece magnification (Continental)} = \frac{160}{\text{Focal length of eyepiece in millimeters}}$$

This is equivalent to writing the microscope magnification as

$$\text{Magnification of microscope} = \left(\frac{\beta}{f'_a}\right)\left(\frac{g}{f'_b}\right)$$

which arose from Abbe's special method of deriving the magnification formula, which need not be reproduced here. We may, however, note that when the image formed by an instrument is projected "to infinity," then

$$\text{Magnification} = -\frac{\beta}{f'}$$

where f' is the focal length of the combined system, given by

$$f' = -\frac{f'_a f'_b}{g}$$

Thus

$$\text{Magnification} = \frac{\beta g}{f'_a f'_b} \quad . \quad . \quad . \quad . \quad . \quad (3.01)$$

Resolving Limit of the Microscope System. It will be assumed in the first instance that the object field consists of an assembly of discrete self-luminous points, although such a condition rarely obtains in the actual use of the microscope, and it will be necessary later to review the conclusions reached by such an assumption.

In Fig. 68 let R and R′ be the entrance and exit pupils respectively of the whole image-forming system, including the eye if visual observation takes place. Let α and α' be the extreme angles with the axis, made by rays passing through the axial points of object and image respectively. Then

$$\text{Radius of Airy disc in the image} = h' = \frac{0{\cdot}61\lambda'}{\sin\alpha'}$$

where λ' is the wave-length of light in the image space.

In order to find the dimension h in the object plane, which has an image of size h' (the radius of the Airy disc) in the image field,

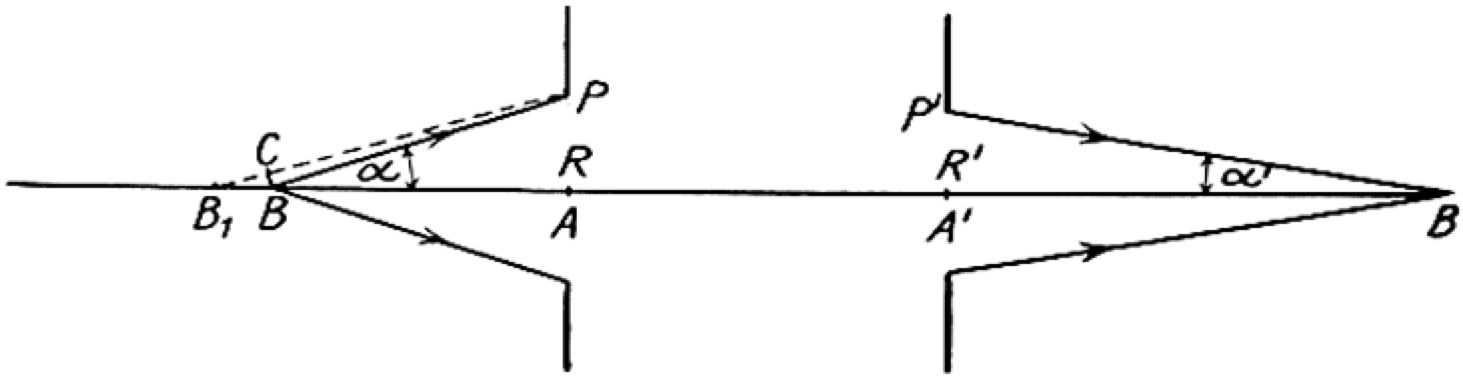


FIG. 68

(assuming the system to fulfil the "sine condition") we use the sine relation

$$nh\sin\alpha = n'h'\sin\alpha'$$

whence

$$h = \frac{n'h'\sin\alpha'}{n\sin\alpha} = \frac{0{\cdot}61\, n'\,\lambda'}{n\sin\alpha} = \frac{0{\cdot}61\,\lambda_o}{n\sin\alpha}$$

where λ_o is the wave-length of light in the air.

It was explained in Vol. I, p. 100, that the closest possible approach for two elementary star images which allows of recognizable resolution is approximately this radius of the Airy disc. Thus the distance of closest approach for two *object* points is inversely proportional to the product of the refractive index of the object space, and the sine of the semi-aperture of the cone of rays entering the entrance pupil of the system from the object point. Abbe called this product the "Numerical Aperture" of the system, and we shall write it NA, thus

$$h = \frac{0{\cdot}61\lambda_o}{NA}$$

It is remarkable that the only necessary assumption about the construction of the optical system is its ability to produce images free from aberrations of optical path.

Visual Resolution. As in the case of the telescope, the visual resolution of the images requires that they shall be presented to the eye under a sufficient angular magnitude. If ω is the angle subtended by the image at the eye, then

$$\begin{aligned}\omega &= (\text{Angle subtended by the object at the near point}) \\ &\qquad \times (\text{magnification}) \\ &= \left(\frac{h}{\beta}\right) m_a, \text{ numerically, where } m_a \text{ is the magnification,} \\ &= \frac{0{\cdot}61\lambda\, m_a}{\beta NA}\end{aligned}$$

if h is the separation necessary to secure that the images are just resolved physically. In order to present the image separation under the very minimum angle for visual resolution for high-contrast object details (say one minute of arc, which is 0·00029 in angular measure), we first get from the formula

$$m_a = \frac{\omega NA\beta}{0{\cdot}61\lambda}$$

then by taking $\lambda = 0{\cdot}00058$ mm. and $\beta = 250$ mm.

m_a (the minimum essential magnification) $= 200\ NA$ approx.

It will be found, however, that the critical image distance should, in actual practice, be presented under a much larger angle than one minute. Four or five minutes will not be too large, and even more will be required if the image-contrast is low. The formula required is

$$m_a = 200\ NA \ (\omega \text{ in minutes})$$

so that if we are working with $NA = 1{\cdot}2$, and we desire to make $\omega = 3'$ in order to have comfortable observation, a magnification of 720 will be called for. The required eyepiece can then easily be calculated from the formula

$$f'_b = \frac{250 \times 160}{f'_a \times m_a}$$

Take, for example, a case in which $m_a = 750$ and $f'_a = 2$ mm.

Then $$f'_b = \frac{250 \times 160}{2 \times 750} = 27 \text{ mm. approx.}$$

In the English system this represents a magnifying power for the eyepiece of $\frac{250}{27} = \times\ 9$ approx. The "Continental magnifying power" is $\frac{160}{27} = \times\ 6$ approx. It is quite usual to work with magnifying powers considerably higher than these when employing a

2 mm. immersion lens. In fact, the ordinary microscope system can usefully employ a great deal of magnification, in contrast to the case of small telescopes and prism binoculars, where the magnification is usually not nearly high enough to use the full resolving power of the system. Note, however, that if the object detail is not resolved by the objective, no amount of magnification by the eyepiece will render such detail visible to the eye.

Size of Exit Pupil and Measurement of Magnification. Let R and R′ (Fig. 69) be the axial points of the entrance and exit pupils of the optical system of a microscope, while B and B′ are the axial points of object and image respectively. The extreme ray from B passes through the margin of the exit pupil after refraction by the system, since entrance and exit pupils are conjugates, and it must be directed from B′. If h and h' are the sizes of object and image, then

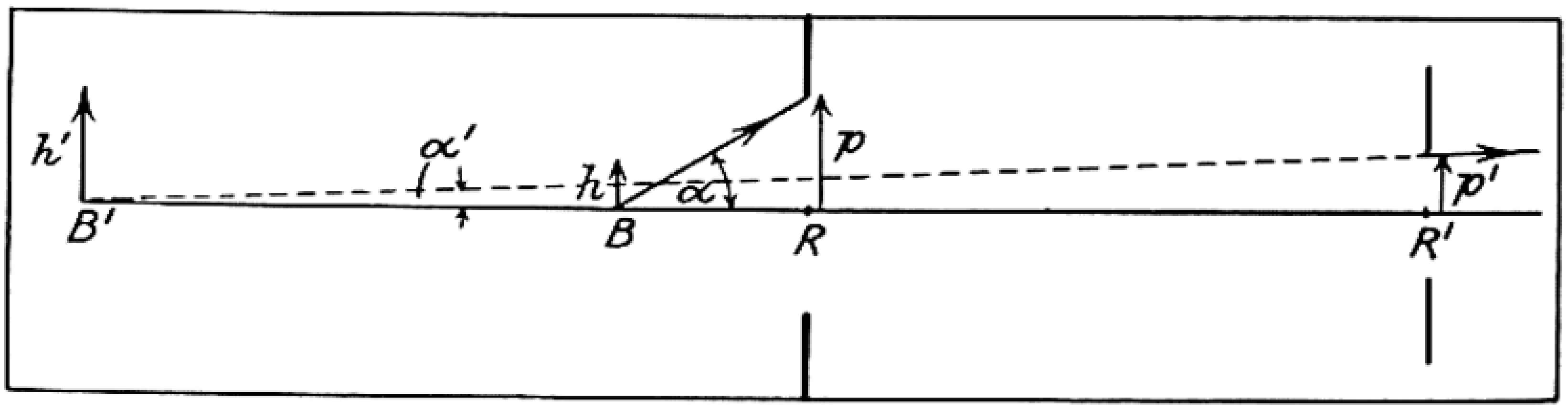


FIG. 69

$$nh \sin \alpha = n'h' \sin \alpha'$$

But $n \sin \alpha = NA$, and $\sin \alpha' = \dfrac{p'}{\mathrm{R'B'}}$, within allowable approximation if R′B′ is large. The sine relation becomes

$$hNA = \frac{n'h'p'}{\mathrm{R'B'}}$$

$$= (n'p') \times \text{(angular subtense of image taken at centre of exit pupil)*}$$

Hence,
$$\frac{h}{\beta} \cdot NA = \left(\frac{n'p'}{\beta}\right) \times \text{(angular subtense of image)}$$

But $\dfrac{h}{\beta}$ is the numerical value of the angular subtense of the object held at the near point. Hence since $n' = 1$,

$$\frac{NA \,.\, \beta}{p'} = \frac{\text{Angular subtense of image}}{\text{Angular subtense of object}} = \text{Magnification.}$$

---

* We are only concerned with numerical relations, and shall not need to consider the signs of the angles in this section.

This gives a useful method of finding the magnification of the microscope.

Since we found about a minimum value for the necessary magnification

$$m_a = 200\ NA$$

$$p' = \frac{\beta}{200}$$

The conventional value for β is 250 mm., so that p' should then be 1·25 mm. The diameter of the exit pupil of the microscope ought to be *at most* 2·5 mm. if all the resolvable detail is to be visible to the eye. It will be an advantage, as shown above, to employ magnification four or five times as great, so that the exit pupil diameter may well be reduced to 0·5 mm.

Depth of Field in the Microscope. Visual Factors. Unlike the case of the telescope, the magnification in the microscope is often so great that the Airy disc corresponding to a point object would subtend an angle, in the visual field of the instrument, comparable with the visual limit of resolution. In consequence of this there will be grounds for supposing that three main factors will be concerned in determining the total depth of field, i.e. the "physical" depth, the depth dependent on the visual discrimination limit, and the depth associated with variations of the accommodation of the eye. There is, however, no straightforward way of assigning theoretical weights to these factors, which may vary in some special way with the system used, and the magnification. A simple addition would give (see pp. 16, 18 and 19)

$$\Delta = \frac{n\lambda}{(NA)^2} + \frac{nc'}{(NA)m} + \frac{n}{n'}\left(\frac{b^2}{m^2}\right)\left(\frac{1}{k} - \frac{1}{b}\right) \quad . \quad (3.02)$$

A fairly comprehensive practical investigation was carried out by M. Berek,[1] using a wide range of objectives and low-power and high-power eyepieces, of which the principal results were as follows: He found no evidence that possible changes of accommodation play any significant part in increasing the depth of field in the ordinary use of high-power microscopes. No dependable evidence of a change in the mean error of focusing with a change of wave-length was found with weak eyepieces, but a definite change (well above the apparent limits of experimental error and in accord with his adopted formula, given below) was found with the high-power eyepieces.

If c' in the last formula is put equal to $b\sigma$ where b is the visual distance of distinct vision and σ is the limiting angle, the theoretical formula might thus be (omitting the "accommodation" term)

$$\Delta = \frac{n\lambda}{(NA)^2} + \frac{nb}{(NA)m}\sigma$$

However, b/m is the focal length of the microscope system, so an alternative formula becomes

$$\Delta = \frac{n\lambda}{(NA)^2} + \frac{nf'}{(NA)}\sigma \quad . \quad . \quad . \quad (3.03)$$

If we take $b = 250$ mm. and $\sigma = 0{\cdot}00058$ (or 2 minutes in radians), $b\sigma \approx 1/7$, so a trial theoretical formula is

$$\Delta = \frac{n\lambda}{(NA)^2} + \frac{n}{7m(NA)} \quad . \quad . \quad . \quad (3.04)$$

Taking the total "experimental" depth of focus T to be on theoretical grounds five times the standard deviation of his focusing results, an *empirical* formula which gave good agreement with Berek's experiments was

$$\mathrm{T} = \tfrac{1}{2}\frac{n\lambda}{(NA)^2} + \frac{nf'}{(NA)}(0{\cdot}00136) \quad . \quad . \quad (3.05)$$

The theoretical formula (3.04) fits the results fairly well for the high-power and immersion lenses but gives results considerably too small for the lower-power lenses. Berek remarks that the results seem to be independent of the type of object used in the tests.

The connection between theory and experiment here seems unsatisfactory. The comparatively simple two-term expression is probably inadequate to represent the interaction of the physical and visual factors concerned. The following short table will suffice to show the kind of relation between the depth of field T, calculated by Berek's empirical expression (these figures correspond well with five times the standard deviation in his experimental focus-settings) and the depth of field Δ, calculated by the more usual formula. The latter appears to give results which are too low for the smaller magnifications.

The depth of focus in the *image* depends on the aperture ratio of the convergent beam. With average objectives it may be of the order of one or two millimetres. When a projection eyepiece is used to form the final image for photomicrography the depth of focus in the final image will be very considerably greater still, owing to the very small angular apertures of the convergent pencils;

Depth of Field (Microns) Calculated by Different Expressions

Objective	Numerical aperture	Eyepiece	Total magnification	T	$\frac{n\lambda}{(NA)^2}$	$\frac{n}{7m(NA)}$	Δ
1	0·12	× 4	12·8	241	38·2	93·1	131·3
1	0·12	× 20	64	63·6	38·2	18·6	56·8
3	0·25	× 20	200	11·2	8·8	2·9	11·7
6*a*	0·85	× 20	840	0·86	0·76	0·21	0·97
1/12*a*	1·32	× 20	1900	0·45	0·48	0·09	0·57

it may be several inches or more. Note that this does *not* refer to focus changes produced in other ways than by moving the final receiving screen.

Aplanatic Surfaces. It has been shown that the resolving power of the objective is dependent on the numerical aperture which can be attained, and, hence, the means for increasing the numerical aperture to the widest possible limits must receive attention.

In Fig. 70 let P be a point on the circumference of a solid refracting sphere of radius r. The ray DP lies in the plane containing the radial line CB′B, and is directed towards the point B. The points B′ and B lie on the circumference of spheres struck with radii rn/n' and rn'/n respectively, where n' is the refractive index of the sphere, and n that of the medium surrounding it.

Then it may be shown that any ray DP directed towards the point B will be refracted so as to pass through B′; to this end it will be sufficient to show that the path DPB′ obeys the law of refraction. Thus from the above radii we write

$$\frac{\mathrm{CB'}}{\mathrm{CP}} = \frac{n}{n'} = \frac{\mathrm{CP}}{\mathrm{CB}}$$

Hence, since the triangles BCP and PCB′ have a common angle, they are similar in all respects.

Now $\widehat{\mathrm{BPC}}$ = angle of incidence = $\widehat{\mathrm{PB'C}}$, and $\widehat{\mathrm{CPB'}}$ = angle of refraction, but

$$\frac{\sin \widehat{\mathrm{PB'C}}}{\sin \widehat{\mathrm{CPB'}}} = \frac{\mathrm{CP}}{\mathrm{CB'}} = \frac{n'}{n}$$

Hence, the law of refraction is fulfilled for *any* ray directed towards B and refracted towards B′ after a single refraction at the spherical surface, for by rotating the diagram about the axis CB we may imagine a complete cone of rays directed towards B and refracted towards B′.

If n' is greater than n, these "aplanatic" surfaces on which B and B′ lie are situated as shown in Fig. 70, but if n is greater than n' (refraction, say, from glass into air) the point B must then lie within the sphere while B′ lies outside of it.

From the above work, $\widehat{CBP} = \widehat{CPB'}$. Hence

$$\frac{\sin \widehat{CB'P}}{\sin \widehat{CBP}} = \frac{\sin \widehat{CB'P}}{\sin \widehat{CPB'}} = \frac{CP}{CB'} = \frac{n'}{n} \text{ (constant)}$$

This shows that the refraction fulfils the "sine condition"; it is free from both spherical aberration and coma. This is implied in the usual use of the term "aplanatic."

Use of Aplanatic Refraction in Microscope Objectives. Referring to Fig. 71 (*a*), the sphere S of radius r has a refractive index n', and is immersed in a medium of refractive index n. An object point B within the sphere lies on a concentric circle drawn with radius rn/n'; all rays from this point suffer aplanatic refraction at the surface of the sphere. The corresponding virtual image B′ is situated on a corresponding circle of radius rn'/n, and is free from spherical aberration and coma. Thus the wide divergence of the fan of rays from B is very considerably reduced.

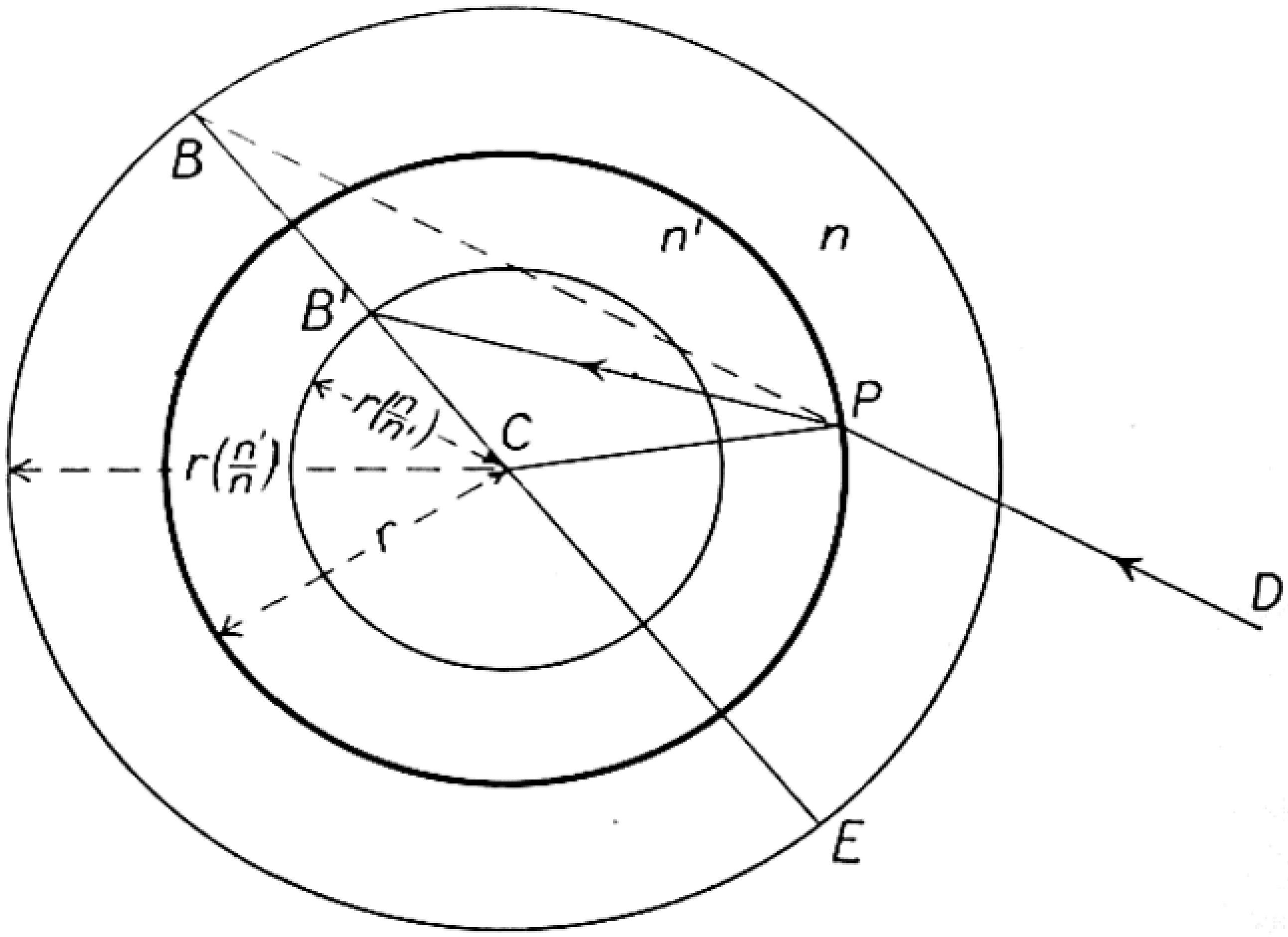
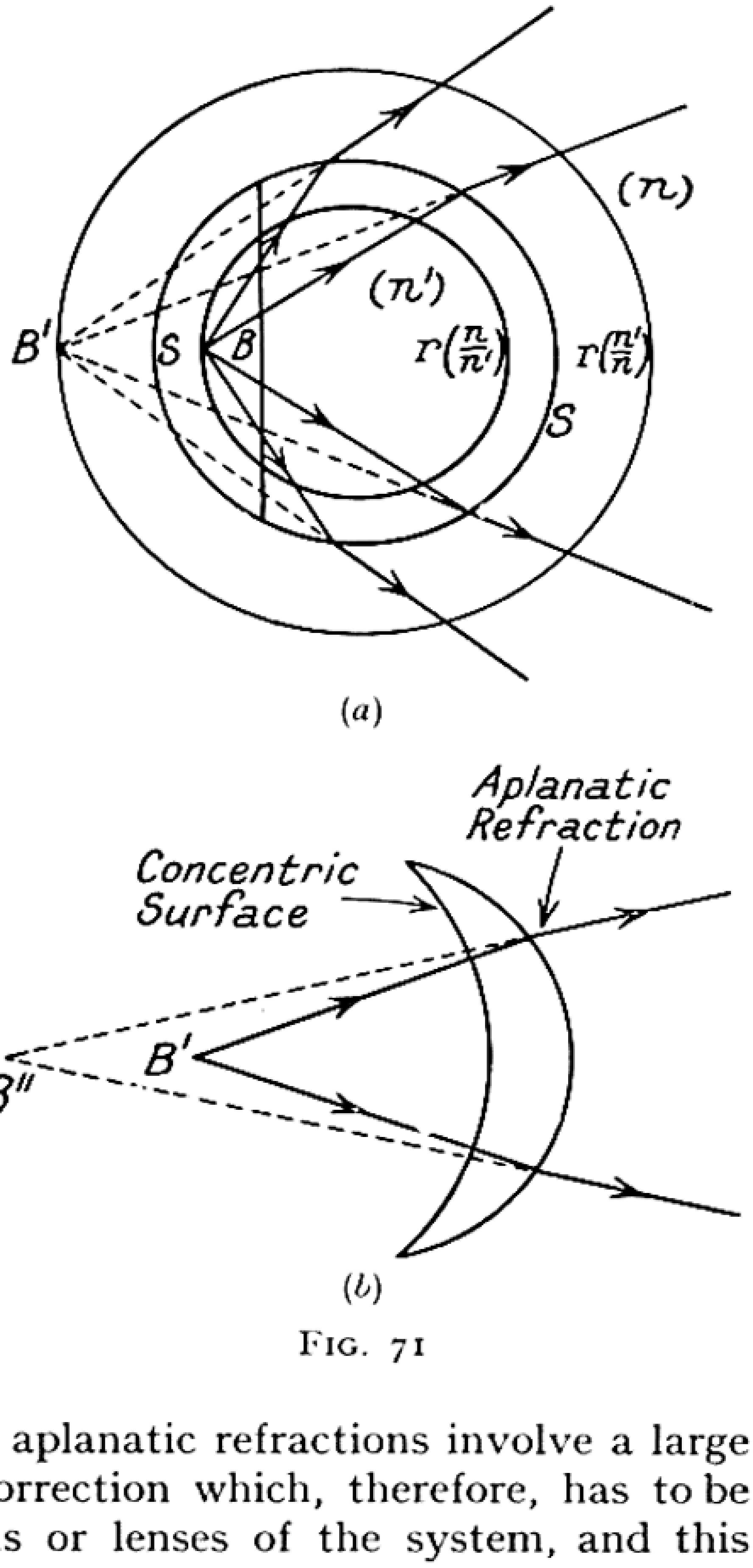


FIG. 70. APLANATIC SURFACES OF A SPHERE

Such a fan, with reduced divergence, may now encounter a spherical refracting surface which has its centre coincident with the common divergence point B′ of all the rays, which thus suffer on deviation and no aberration. This condition is illustrated in Fig. 71(*b*), where the rays enter the first surface of the meniscus. It is clear, however, that the back surface of this lens can also be made to produce an aplanatic refraction if B′ is one of the aplanatic points, and the divergence of the rays can be still further reduced. In this stage they may be made to encounter a "Lister" combination or other arrangement by which the divergence from a virtual image is changed into convergence towards a real image, as suggested Fig. 72 (*a*).

The object point cannot be placed, of course, within a spherical lens; hence this front lens is cut off by a plane face, as suggested in Fig. 71 (*a*), and the continuity of refractive index of the media is ensured by placing a film of liquid, usually cedar oil of refractive index 1·517, between the lens and the cover glass ($n = 1{\cdot}515$, about) which is practically in contact with the object. There is then little change of refractive index between the front lens and the farther side of the cover glass, as the front lens is usually of glass of fairly low refractive index, not above 1·54. The above principles represent the main basis of the control of spherical aberration and coma in high power immersion lenses.



(*a*)

(*b*)

FIG. 71

On the other hand, the two aplanatic refractions involve a large amount of chromatic under-correction which, therefore, has to be compensated by the back lens or lenses of the system, and this is usually only attained by the use of fairly deep curves in the contact faces of these lenses. The result is that although spherical aberration can be compensated for one zone without very great difficulty, it is somewhat difficult to obtain it for all zones, i.e. zonal aberration is likely to be present. The successful balancing of the aberrations calls for considerable skill and resource on the part of the designer, and successful selection of the materials.

Apochromatic Objectives. Efforts to reduce the secondary spectrum characteristic of the "achromatic" systems made of glasses of the older types were made by Abbe when the new Jena glasses were introduced, but the success of his "apochromatic" lenses was largely due to the use of "fluorite" as a component of some ot the lenses. The "optical constants" of fluorite as compared with some other glasses are given in the Appendix to Vol. I.

It will be recalled in the discussion of achromatism (Vol. I, Chapter VII) that the "powers" of opposing lenses in achromatic combinations are lowest when there is a large difference of V values; when the powers of the lenses are relatively low, the curves are relatively shallow, and zonal aberration is less likely to arise. It was also shown that the residual secondary spectrum was reduced by a large difference of V values; on the other hand, the relative partial dispersions of fluorite are quite comparable with those of ordinary crown glasses, and also come fairly close to those of an extra light flint or "telescope flint." Hence with combinations using fluorite in place of the "crown" lens, and "extra light flint" in place of the ordinary flint lens of an achromatic combination, it is possible not only to reduce the secondary spectrum, but also largely to reduce the zonal aberrations. Fig. 72 (*b*) shows an "apochromat." Low power apochromats (dry) are also made.

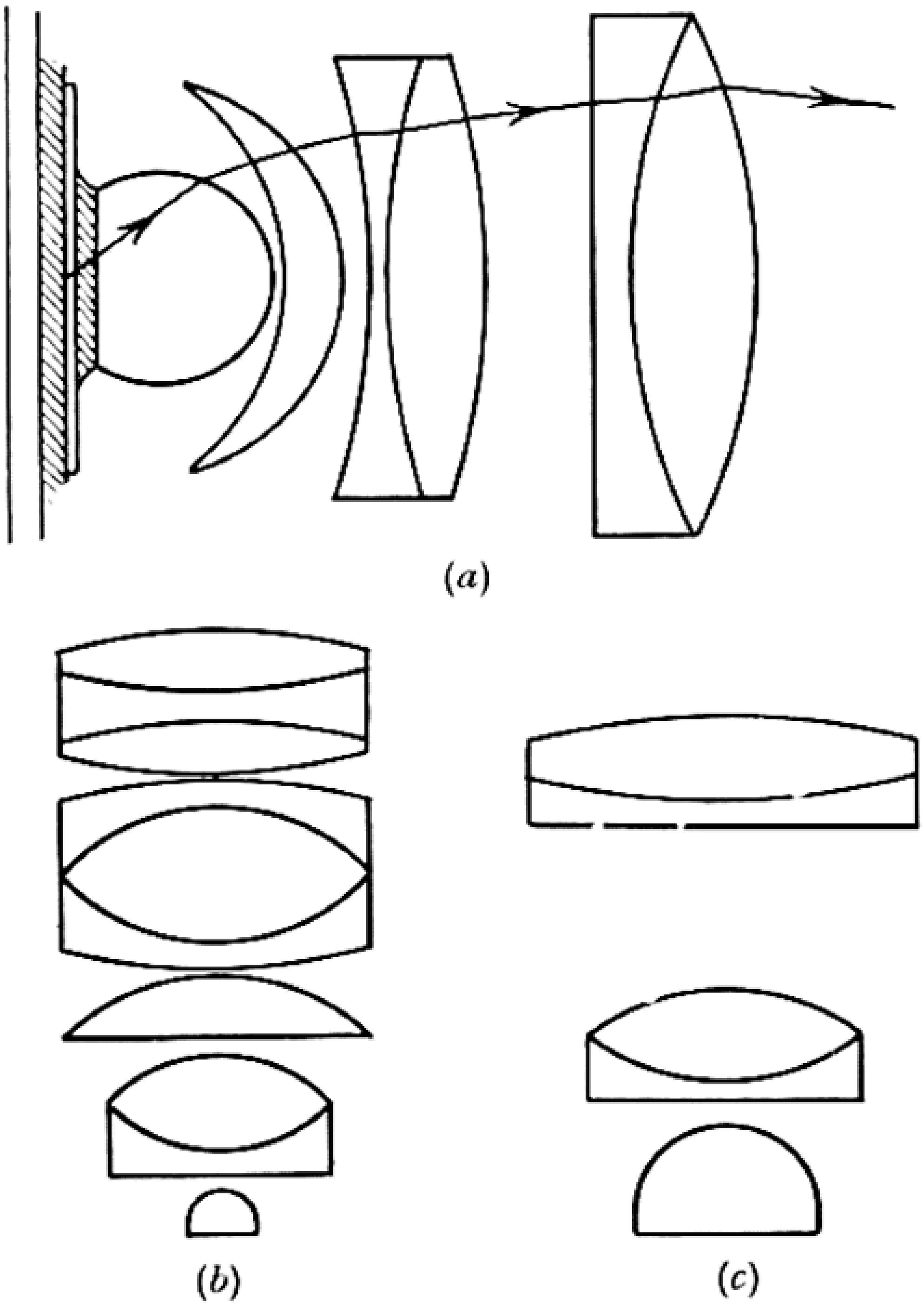


FIG. 72

(*a*) Section of Immersion Achromat (2 mm., $NA = 1{\cdot}25$)

(*b*) Section of Typical Apochromatic Objective (2 mm., $NA = 1{\cdot}40$)

(*c*) Section of Typical Dry Lens (8 mm., $NA = 0{\cdot}65$)

Apochromatic objectives are found usually to possess noticeable chromatic difference of magnification, which is best compensated by the use of special eyepieces. (See p. 104.)

High Power "Dry" Lenses. Dry lenses are made with numerical apertures up to 0·95, and, therefore, the extreme rays in the object space may have to make an angle of $\sin^{-1} 0{\cdot}95 = 72°$ with the axis. The arrangement of the majority of dry lenses of intermediate power is similar to that shown in Fig. 72 (*c*), which might be regarded as a Lister pair combined with a hemispherical front lens. In this case, however, considerable spherical and chromatic aberration arises at the front face of the "hemispherical" lens, and this has to be corrected by the rear components; not only this, but the aberration arising at the cover glass surface must be considered. If the front lens and the cover glass are of equal refractive index, the cover glass may be considered as an addition to the thickness of the front lens in the course of design, but this means that the system will usually be very sensitive to the thickness of cover glass with which it is used. Many objectives are corrected for use with a cover-glass thickness of 0·17 mm. Refraction at a plane surface transforms a spherical wave-front into an ellipsoidal wave-front, and, if a decrease of refractive index is involved, the *major* axis of the ellipse is the normal to the surface, so that the disturbances travelling along oblique paths suffer a relative *retardation*. Since an infinitely thin cover glass could produce no effect, it will be clear that increasing thickness of cover glass produces more "over-correction" in the spherical aberration.

In practice, slight over-correction arising through the use of a too thick cover glass can be compensated by using the objective with a shorter tube length and, consequently, a greater working distance between the front lens and the object point; this produces an opposing tendency to under-correction, mainly in the front lens.

Eyepieces. Little need be added to the discussion of eyepieces given in the preceding chapter. The Huygenian type is perhaps the most widely employed, the design varying with the required magnification; the focal length of the field lens may not exceed 1·5 times that of the eye-lens at the lowest magnifications. Rings are fitted to a set of eyepieces so that they are parfocal, i.e. they can be interchanged without large changes of focus; the optical tube-length is thus kept constant. Various eyepieces of the "orthoscopic" and other types are in favour with some microscopists who prefer a very wide angular field.

Ordinary achromatic objectives have little chromatic difference of magnification, and are best used in conjunction with ordinary Huygenian eyepieces. With the $\frac{1}{12}$ in. oil immersion objectives, and with the apochromatic objectives, there is a slightly greater magnification for the "blue" or shorter wave-lengths, as compared with the red, and this is best overcome by the use of so-called "compensating" eyepieces which are designed to give a correspondingly greater magnification to the red than to the blue. If held to the light the border of the field stop appears tinged with orange in a compensating eyepiece, and blue in an ordinary Huygenian. Abbe designed a series of apochromatic objectives for Zeiss in which all had the same chromatic difference of magnification, and thus could be used to advantage with the same compensating eyepieces; and various other makers have special arrangements in this connection.

The "projection eyepiece" usually has a single field lens with a triple projecting lens. The exit pupil is limited by a small stop in order to cut out stray light.

Curvature of Field. Owing to the importance of keeping up a maximum numerical aperture, it is not usual to provide any diaphragm in ordinary microscope objectives, other than one behind the system to cut off stray light from the edges of the lenses. More or less vignetting (see below, p. 192) is likely to occur for points off the axis, and the principal ray can be regarded as one which is derived from the centre of the apparent pupil. Thus in a Lister objective the principal rays will pass through an axial point between the lenses. As will be explained in the chapter on the Photographic Lens, it is possible to modify the primary astigmatism of a lens passed eccentrically by a principal ray, provided that it has residual spherical aberration or coma. For this reason it is usual to depart from the strict aplanatism of the individual parts of the Lister objective in order to obtain some over-corrected astigmatism; though the whole lens can still be aplanatic. In this way an approach to a flattened field can be achieved, which is a great advantage in photo-micrography.

Similar steps are taken in regard to designs more complex than the Lister type and it is in regard to such points, as well as the fine correction of the zonal aberrations, that the chief problems in microscope objective design are found. As in other systems, much depends on the successful choice of glasses.

As mentioned above, the ordinary forms of eyepiece have a negative curvature of field according to the usual formula—

$$\frac{1}{R} = \Sigma\left(\frac{-1}{nf'}\right)$$

where n is the refractive index and f' the focal length of a constituent lens of the system reckoned as for an infinitely thin lens with faces of the same curvature. In 1918, Conrady proposed to introduce as the front lens of an eyepiece system, a carefully computed achromatic lens of negative power which would be sufficient to give a positive curvature to the field of a telescope objective. This might then be followed by a suitable eyepiece.

An anastigmatic flat field telescope on these lines was later independently designed and produced by H. Dennis Taylor.

In microscope objectives, the curvature of field of the primary image is so much greater that an achromatic negative combination can hardly be expected to do more than flatten the field of the primary image without compensating the curvature due to an observing eyepiece.

In 1922, H. Boegehold and A. Köhler, of Messrs. Carl Zeiss, brought out the "Homal," a projection lens of negative power, which flattens the field of the microscope objective for which it is designed. As in the case of the Galilean telescope, the exit pupil lies within the lens; the field of view to an eye placed behind it is very small indeed, and the system is useless for visual work, but is quite satisfactory for photography.

References to a number of devices using systems of negative Petzval curvature on the image side of the objective are given in a paper by Blaschke.[2]

Moreover, it is possible to design objectives having powerful negative refracting surfaces close to the object as in the planachromatic series of Zeiss. Blaschke's paper describes such a system in which the front lens is a meniscus, and the second is also a thick meniscus with a strongly curved concave front surface close to the posterior surface of the front lens; these two surfaces work nearly aplanatically, and have the duty of correcting the astigmatism. This design is embodied in the Cooke Flatfield Objective of focal length 6·0 mm. and numerical aperture 0·8; it is intended for use with uncovered specimens (dry) as in metallography.

Numerical Aperture: General Remarks. Since "numerical aperture" is the product of refractive index and the sine of the angle of obliquity of the extreme ray, a medium of refractive index n cannot transmit a ray of numerical aperture greater than n. This has to be remembered in connection with the condenser as well as the objective. The law of refraction, $n \sin i = n' \sin i'$, shows that the "numerical aperture" characteristic of a ray is independent of any refractions at plane surfaces normal to the axis, such as those of the cover glass, etc.; therefore, if the original object is in air, the objective cannot work at a numerical aperture greater than unity. If the medium is water (refractive index 1·333), the limit of possible numerical aperture is equal to or under 1·333 provided that all the media have at least this refractive index. To take full advantage of an objective having any high *NA* such as 1·6 (such could be constructed), it would be necessary to have the mounting medium, cover glass, and immersion medium of this refractive index at least. The action of dark-ground illuminating systems has to be considered with this point in mind. (See below, p. 138.)

Historical Note. As remarked above, Lister's paper on the aplanatic points of an achromatic doublet was published by the Royal Society in 1830. About the same time, G. B. Amici, in Italy, was working somewhat on the same lines, but attempting to find how the spherical aberration of higher orders and the coma could be eliminated by the mutual action of doublet lenses, none of which were necessarily working strictly aplanatically. In the "eighteen-fifties," Amici introduced the strongly curved plano-convex lens as a front lens, which later developed into the hemispherical and hyper-hemispherical form of modern high power objectives. This allowed of a great increase of numerical aperture, the importance of which in resolution of fine structures had been realized by various workers, and also noticed by Lister in 1830.

Immersion systems were first introduced as a means of overcoming the effects of the thickness of the cover glass, and also its possible variations and irregularities; the liquid employed was almost invariably water. Such systems were employed by Amici and others. The great advantages of immersion systems in high power work were only slowly realized, although advances had been made by Tolles in America, who used glycerine and balsam immersion lenses. Abbe published his diffraction theory in 1877, and explained on this basis the theory of the effect of homogenous immersion in increasing numerical aperture and securing increased resolving power. He also introduced the apochromatic lenses. The formula given in the text on p. 16 shows that for a given numerical aperture the depth of field is increased by the use of "immersion" lenses.

The Illumination of the Object. The assumption made above, that the object in the microscope consists of an assembly of discrete self-luminous points, is seldom realized in practice. The majority of biological specimens are viewed by transmitted light, while opaque objects, such as metallurgical specimens, are illuminated by light which they reflect into the instrument.

Material objects which are non-self-luminous may obstruct light completely or partially. They may also reflect, refract, diffract, or scatter light. By these actions they become "visible," and the interpretation of the "image" formed on the retina usually leads us to a more or less correct knowledge of the physical character of the corresponding object, but often mainly through collected experience. The simplest case is that of a "silhouette" pattern in which the geometrical outline pattern of a flat object is seen by the obstruction of light, partial or complete. If the object manifests selective absorption of light, the geometrical boundaries may be indicated by the fields of various colours.

If such an object is to be viewed away from a microscope, we may hold it in front of a uniformly luminous white surface to obtain the best view, and the "true" colours. In the microscope we *may* project the image of a uniformly illuminated or luminous white surface approximately into the plane of the object by the aid of the substage "condenser" system (Fig. 73), and this affords a close approximation to the best condition for obtaining an image pattern similar to the object itself in a geometrical sense.

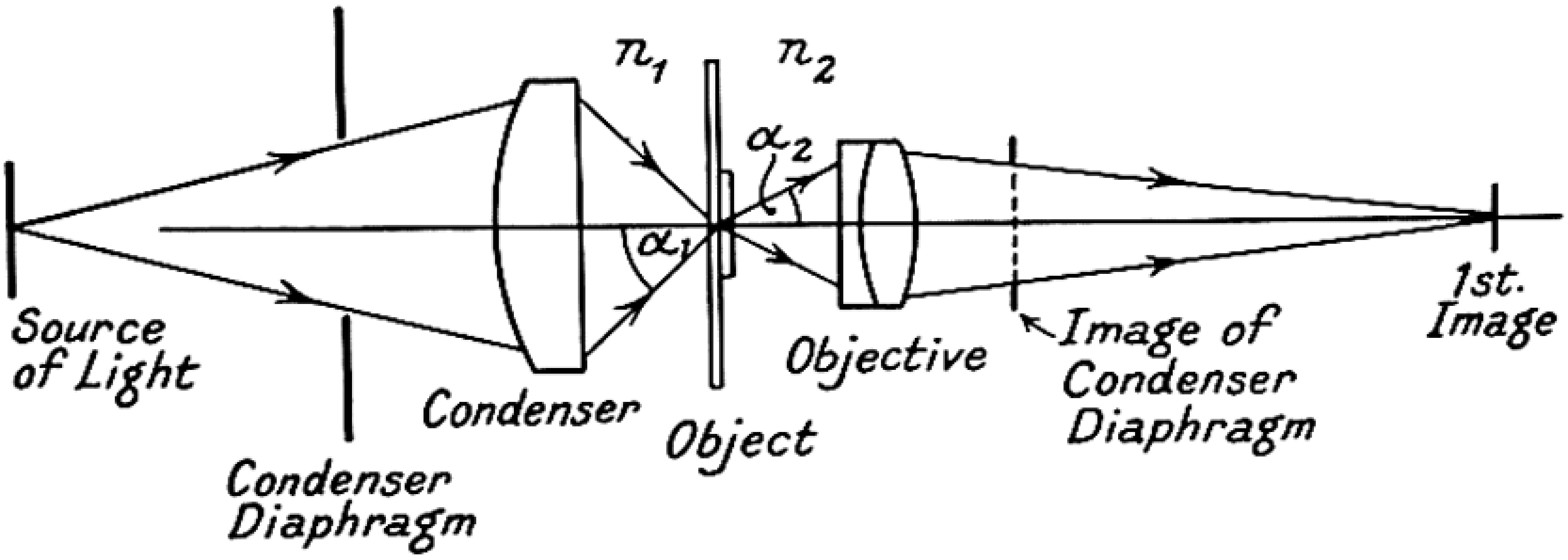


FIG. 73. USE OF THE SUBSTAGE CONDENSER
(Diagrammatic)

The above conditions may not, however, be at all suitable for yielding an image from which the physical characteristics of certain other classes of object can be inferred. As an experiment, take a piece of glass with a pattern moulded in its surface, such as is employed for doorways and the like. Where such a glass is held close to a broad, evenly illuminated source of light, the "pattern" practically disappears. In order to obtain a strongly marked, easily visible appearance of some kind, the glass is held between the eye and a small source of light which is thus giving a more or less directed beam. The "pattern" stands out strongly because certain parts of the glass refract or reflect light into the eye, and other parts do not. The effects of cumulative experience again largely enter into the interpretation, which may or may not be true. Hence, we need to guard against the easy assumption that there is always a possibility of obtaining a "true" picture of the object. The successful microscopist will adapt his illumination to the end in view. He will regard any image pattern not so much as a picture of the object, as a piece of physical evidence from which the physical characteristics of the object can possibly be established. The different methods of illumination possible with the modern substages should be regarded as additional weapons in the armoury. An important theoretical distinction can thus be made between two classes of objects, i.e. those which are visible by reason of varying opacity, and those which manifest variations of optical path, and thus cause corresponding variations of phase in the transmitted light. They may be called "amplitude objects" and "phase objects" respectively if a brief theoretical terminology is desired. Ordinary biological object slides often contain both these types of object detail.

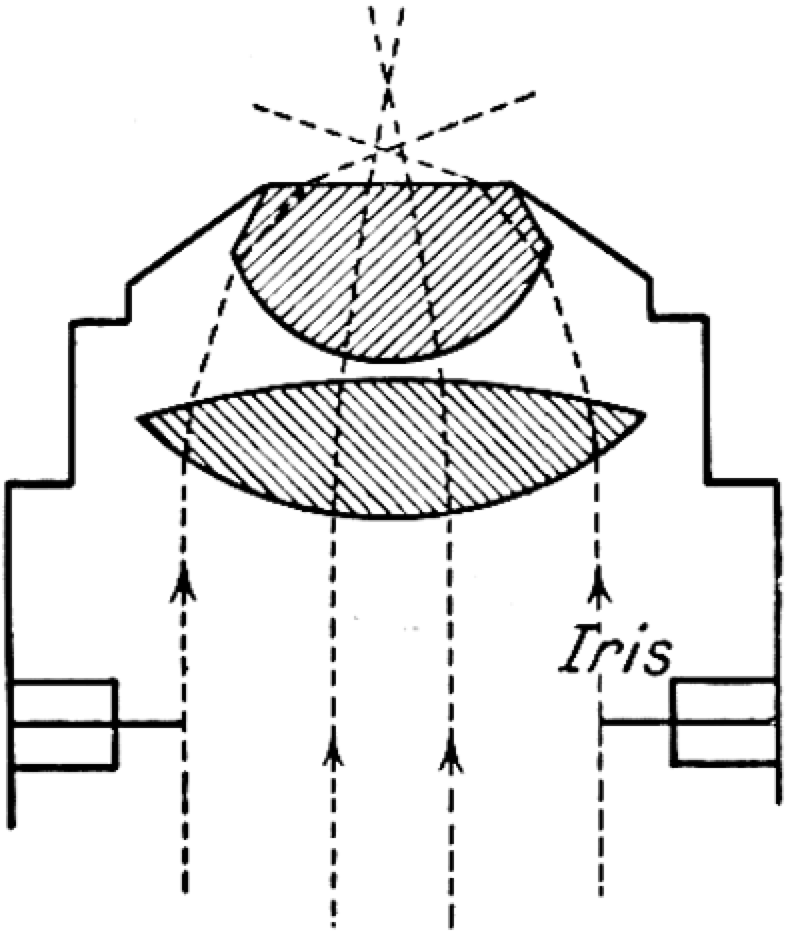


FIG. 74. ABBE TWO-LENS CONDENSER ($NA = 1{\cdot}2$)

Substage Condensers. Theoretical considerations (to be developed more fully below) show that in order to obtain the optimum resolving power, i.e. the distinction of the finest possible detail in many classes of object, (but not necessarily every case), it should be possible, if required, to pass light through the object in such a way that the rays diverging one from any point of it spread out and fill, uniformly, the whole aperture of the objective. The case is illustrated in Fig. 73. If the refractive indices of the media on the two sides of the object slide are n_1 and n_2, and the angular divergence of the extreme ray which can enter the objective from an axial point of the object plane is α_2, then we must have

$$n_1 \sin \alpha_1 \nless n_2 \sin \alpha_2$$

i.e. the numerical aperture of the condenser system must at least be equal to that of the objective if the above condition is to be fulfilled.

In the simplest case, there is no lens system in the substage. An image of a fairly broad source of light, i.e. a lamp flame or an opal glass electric lamp, is thrown approximately into the object plane by a concave mirror. The divergence of the light from the object plane is sufficient to fill an objective of small numerical aperture—up to about 0·25. It is then advisable to use a small stop immediately below the object to limit the illumination to the part of the object required. Sometimes when a broad source of light is available, the plane mirror may be used to reflect the light through the object. The angular divergence of the extreme rays passing through any point of the object plane is then determined by the angular subtense of the source of light.

For larger apertures a lens system in the substage is employed. Fig. 74 shows a type frequently employed, due to Abbe, with which a numerical aperture of 1·2 can be obtained. There is, however, considerable spherical aberration, so that the rays do not all pass through one point, and there is no correction for chromatic aberration. The disadvantage of this aberration will be understood from Fig. 75, in which the condenser S brings the marginal parallel rays from a small distant source to an axial focus at C—intermediate rays to B, and paraxial rays to A. If an object is placed in the plane of the point C, any point of it can only be illuminated by marginal and paraxial rays. A pinhole at C would block the rays from an intermediate zone. In practice, the difficulty is usually overcome by using a broad source of light, and rays from another part of the source may then pass through the intermediate zone and through the axial point of the object. This could be studied by drawing a figure similar to Fig. 75, and bringing a bundle of rays into the condenser at a suitable angle with the axis. The use of a broad source of light is, however, likely to be accompanied by difficulty in controlling the light and restricting the presence of "stray" and unwanted light.

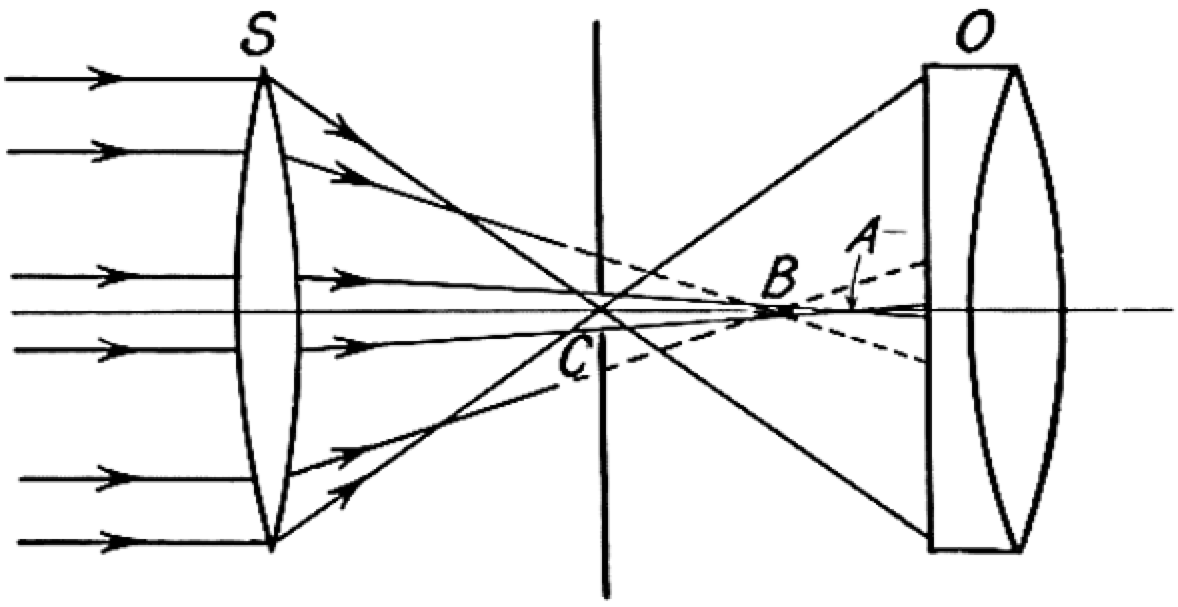


FIG. 75. EFFECT OF SPHERICAL ABERRATION IN A CONDENSER

This uniform distribution of light into the objective is most important in practice. It should always be checked by removing the eyepiece of the focused microscope, and observing the image (formed by the objective) of the diaphragm below the condenser, or of whatever stop limits the numerical aperture of the extreme rays passing through the system. If light diverges uniformly to the objective from any point of the object, this image (which is seen just above the objective) will appear as a uniform disc of light. If a small source is employed, and the condenser is subject to spherical aberration, then the disc at the back of the objective will probably be illuminated either at the margin and the centre only, or in an intermediate zone. This must be remedied by the use of a larger source of light, or by the use of an auxiliary condensing lens as described below.

If three or four lenses are employed in the construction of the substage condenser, as shown in Fig. 76, it may be made free from spherical aberration and given an NA up to 1·40. It is sometimes useful thus to be able to use quite a small source, such as a glowing thorium pastille heated by a small gas flame, or the glowing tungsten ball of a small Pointolite lamp,* as the illumination of the object can then be limited to one small part under examination. This is most helpful in avoiding stray light. In order to make perfectly sure of the above test, it is advisable to make the observation through a small pinhole in a card placed at the end of the tube. This will save any mistake due to glare.

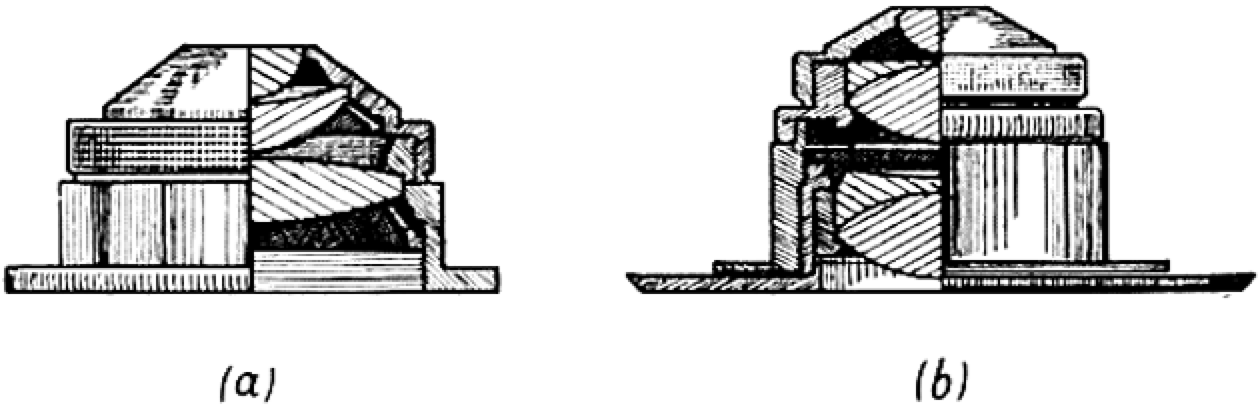


FIG. 76. TYPICAL CONDENSERS

(*a*) Three-lens aplanatic condenser (*b*) Achromatic condenser $NA = 1{\cdot}0$

Chromatic aberration is a little troublesome in cases where it is desired to use such small sources, and the best substage condensers are of the achromatic type. They are made on practically the same lines as objectives of equivalent numerical aperture, but the degree of precision required in their manufacture is naturally less, as the sole requirement is that they shall secure this uniform distribution of light into the objective, and any small deficiencies are not noticed because a source of light of finite size is invariably employed. The main advantage they confer is the possibility of obtaining the proper conditions with the minimum of trouble when it is desired to limit the illuminated area of the object. A diagram of such an achromatic condenser is shown in Fig. 76 (*b*). In certain cases it may be useful to use an objective (reversed) as a condenser, especially when using immersion systems of high numerical aperture; in such a case the object will be mounted between two cover glasses.

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* Manufactured by Messrs. Ediswan Electric Co., Ltd. If a Pointolite lamp is used for visual work the intensity of light will have to be reduced by a suitable light filter or screen. Other small sources such as zirconium arcs or high-pressure mercury arcs must also be employed with suitable screens and filters.

The diaphragm which limits the aperture of the condenser is an important adjunct. It is usually of the "iris" variety, so that its aperture can be reduced from the full lens diameter down to a few millimetres, and it is preferably mounted so that it can be displaced perpendicular to the axis of the system in order that a narrow cone of very oblique light may be projected through the object. This is useful in cases where evidence of the very finest structures is sought for.

Condensers required to give a greater numerical aperture than 1·0 *must* be used as immersion systems, a spot of cedar-wood oil being placed between the surface of the objective and the slide, otherwise $n \sin \alpha$ can never rise above unity, even if the extreme rays emerge from the condenser into the air practically at right angles with the axis. Even if the condenser is not designed for an *NA* greater than 1·0, the use of oil immersion saves much loss of light by reflection at very oblique angles. If the objective in use has an *NA* of 1·2 or more, it is only very seldom, however, that the allowable *NA* of the condenser (as actually stopped down for use in observation) may exceed 1·0, and some observers, if pressed for time, will omit the use of the oil for the condenser on this account. Naturally, an immersion objective must *always* be oiled on.

The working distance of the condenser should be so arranged that if the source of light is placed at about 10 to 12 inches from the mirror, an image of the source will be projected into the plane of the object as mounted on a slide of normal thickness, say, 1·0 mm. The distance between the front surface of the condenser and the slide should then be about 0·2 mm., so that a drop of oil may be squeezed out between the surfaces. The condenser will always be held in a focusing mount, and a good rack and pinion gives ample accuracy for the adjustment. It saves anxiety if a suitable stop prevents the surface of the condenser from actually rising above the plane of the surface of the stage on which the slide has to rest. There can then never be any danger of breaking the slide in that way.

In practice, a number of difficulties may be encountered in securing correct illumination which is evenly distributed over the required region of the object, amongst which are the following—

1. *The Image of the Source is Too Small to Illuminate the Required Area of the Object.* In this case an auxiliary condenser must be used, as shown in Fig. 77. The lens L_1 projects an image of the source S into the lens L_2, which represents the microscope condenser. If L_1 is placed at the working distance for L_2, and further, if L_1 projects the image without spherical aberration, it (L_1) will act as a uniform source, projecting light into the image of S. Clearly, this image should be large enough to fill the aperture of L_2 sufficiently to attain the required *NA* of the illuminating beam.

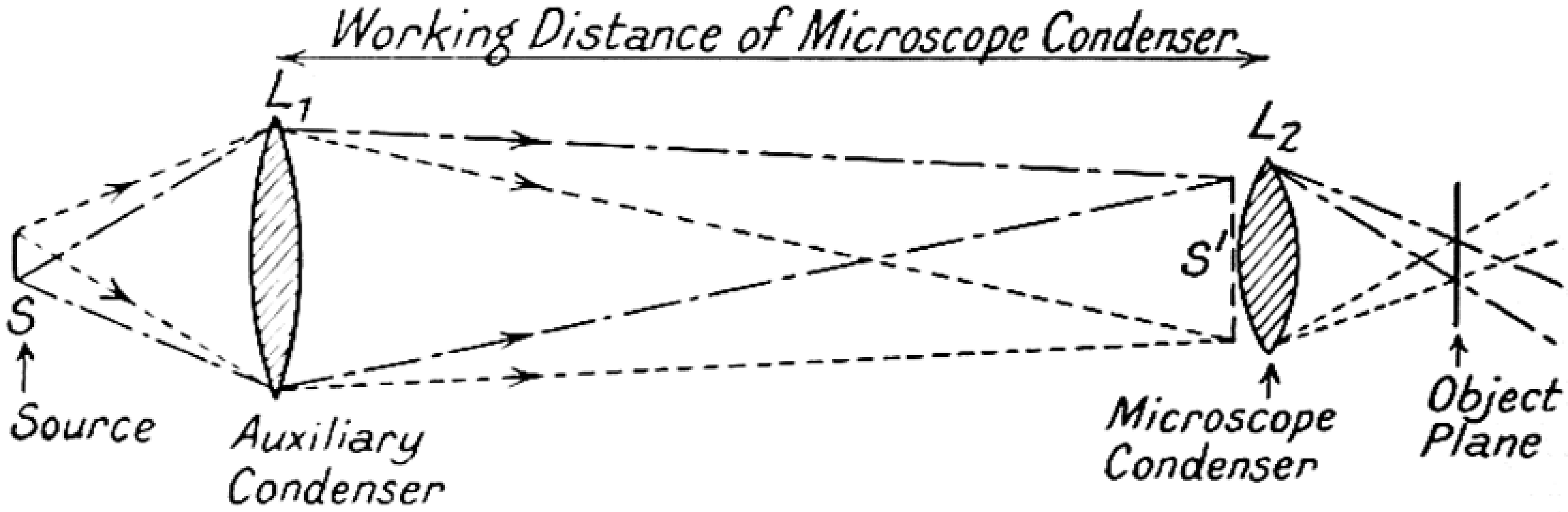


FIG. 77. USE OF AUXILIARY CONDENSER (KÖHLER ILLUMINATION)

Special "aplanatic" auxiliary condensers can be obtained for the above purpose. If only a plano-convex bulls-eye condenser is available, it should be employed with the plane side towards the shorter conjugate distance to minimize the spherical aberration as far as possible.

If the above arrangement proves impossible for any reason, owing, perhaps, to the difficulty of filling the microscope condenser aperture, use may be made of the double condenser system shown in Fig. 78. The first auxiliarly lens L_1 projects an image S' of the source S into the lens L_2. The lens L_2 now projects an image of L_1 into the aperture of the microscope condenser, and if L_1 is reasonably free from spherical aberration, the illumination of the condenser will be uniform. The

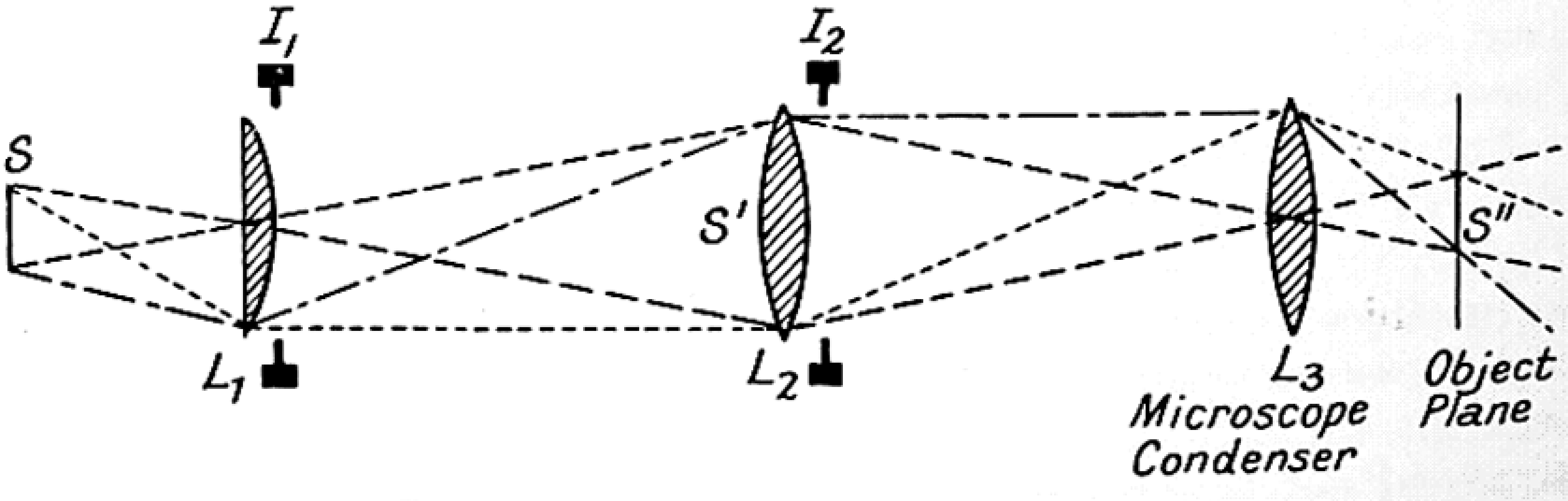


FIG. 78. DOUBLE CONDENSER SYSTEM

microscope condenser projects an image of L_2 into the plane of the object, so that the original source of light is finally imaged there also.

If the lenses L_1 and L_2 are fitted with iris diaphragms, shown in the diagram by I_1 and I_2, the use of I_1 clearly controls the aperture of the microscope condenser which is illuminated and, hence, the numerical aperture of the illumination, while I_2 clearly controls the area of the object plane which is illuminated.

2. *The Working Distance of the Condenser is Too Small or Too Great.* If a very thick slide is used, it may be found that the condenser will not give a satisfactory image of the source or effective source when used at the ordinary working distance, even when the condenser is racked up as far as possible. If this results in uneven distribution of the light into the objective, the source must be brought much nearer to the condenser, or an image of it projected into such a position as may be required.

If, on the other hand, the working distance is too great, the condenser may, of course, be racked down to obtain the required focus, but difficulty may then be found in retaining a film of immersion oil between the condenser and the slide. The space may, however, be partly filled by one or two cover glasses, oiled on both sides, and this usually overcomes the difficulty.

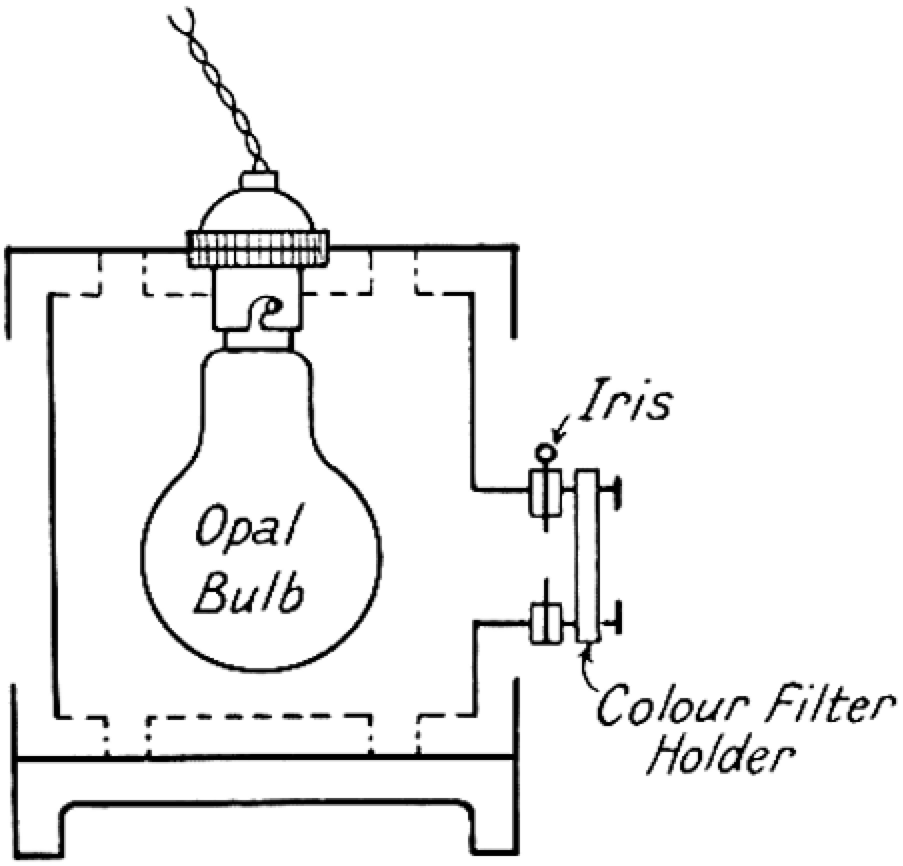


FIG. 79. SIMPLE MICROSCOPE LAMP

3. *The Source of Light may be Unsuitable.* If an electric lamp is used it should have an opal glass bulb inside a suitable housing with one or two windows, the size of the aperture being regulated by iris or other diaphragms (see Fig. 79).

If the electric lamp has visible filaments, the light should be diffused by a screen of "ground" or "opal" glass, or even by tissue paper in emergency. In any case, some scheme should be adopted for visual observation to obtain a small, bright, uniform patch of light about 1 in. diameter; it is helpful to have an iris or other means of reducing this diameter if required. The Pointolite tungsten arc may be adapted for visual observation by one or more of the means described above, but some "light filter" preferably with variable density will usually be required. Suitable "wedges" of neutral glass may be mounted to be moved oppositely so as to secure more or less dimming; an alternative is to vary the thickness of a layer of a suitable absorbing liquid.

The Theory of the Microscope. It was mentioned above that a very suitable illumination for a microscopic object consisting of a plane containing opaque and transparent areas would be the provision of a self-luminous surface immediately behind it. The transparent areas would then function as self-luminous sources, and the theory of the formation of the image would be similar to the simple case of the telescope. Subject to the absence of aberrations and stray light, the resolving power so obtained would depend only on the numerical aperture of the objective.

In practice, lenses suffer from certain imperfections, and also the object has to be illuminated with the aid of a condenser. It is an experimental fact that certain very delicate structures cannot be detected unless the illuminating cone is of somewhat smaller angular aperture than is required to fill the whole aperture of the objective. In such cases it is easily shown experimentally that the light entering the objective from the object is partly *diffracted* light, and any theory of image formation *must* take account of the effects of such diffracted light.

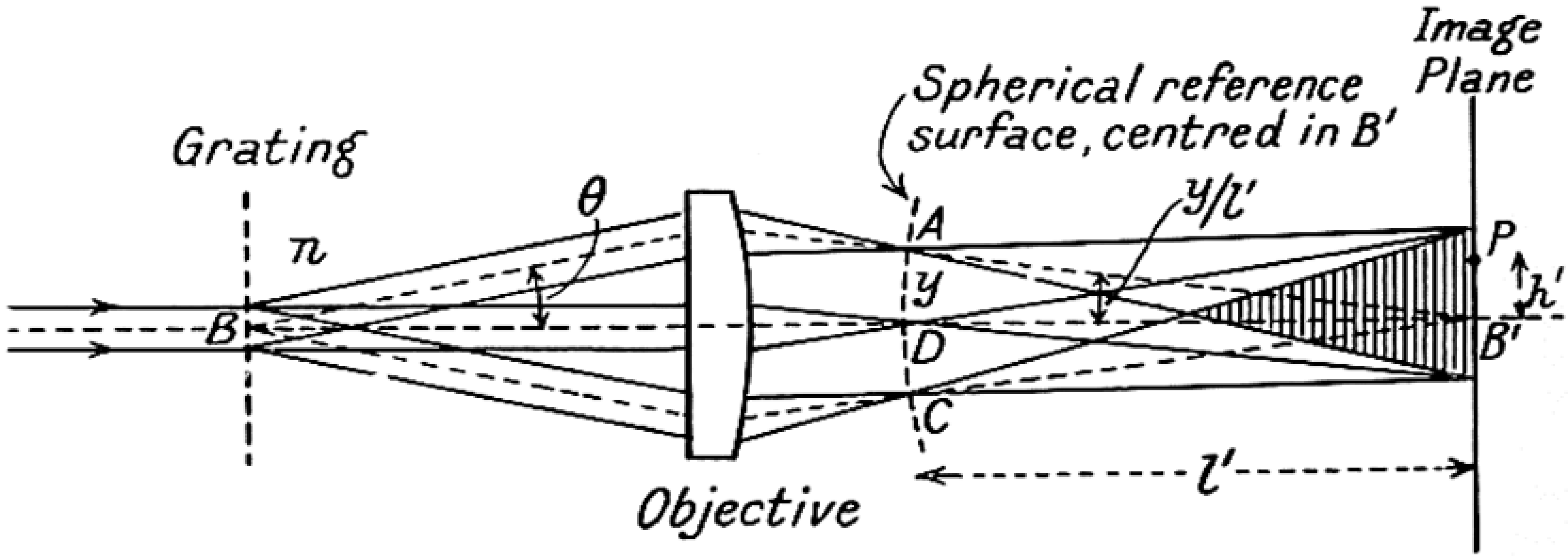


FIG. 80. DIFFRACTION IN THE MICROSCOPE (ABBE PRINCIPLE)

Now the theoretical investigation of the diffraction of light by material objects is generally a matter of the most difficult nature. It only becomes simple to calculate the main effects when the "object" is of a very simple character. A straight edge, a circular hole, a rectangular slit, or a row of similar apertures or structures in one plane are among the cases where simple methods are possible. This partly accounts for the fact that the question of the formation of the image of a "grating" (or row of slits or similar small apertures) is of such theoretical importance. It is true, of course, that very many natural objects show a regular structure, and this makes the corresponding theory of particular interest.

The thorough analysis of the mode of image formation in any condition of illumination is a very difficult and intractable problem, but light may be gained in considering specially simple cases, especially that of the grating.

The theory developed by Abbe and others[3] supposes that the light derived from the condenser and transmitted by the object may be resolved into systems of plane waves. This assumption was further examined by Stoney.[4] Thus in Fig. 80, if we concentrate attention on one such system represented in the diagram by one " parallel beam " travelling in a particular direction, we shall have diffraction occurring in the grating in the object plane; the illumination of the object is then said to be perfectly "coherent"; the objective will receive the direct and such diffracted beams of which the angular directions fall within its limits of aperture. Such beams will be concentrated by the objective to form sharp maxima near the back focal surface, and these maxima will have related phases, so that their effect in any other surface can be calculated.

The Abbe Principle. On the basis of the experiments which he made with specially ruled gratings, Abbe reached the conclusion that the necessary condition for the resolution of a regular structure was that the directly refracted light, and at least one of the diffracted beams should enter the objective.

In Fig. 81 we have pictured a parallel beam incident on a grating at an angle θ_o in a medium of refractive index n_o. The angle of transmission is θ_1 in a medium of refractive index n_1. The general formula giving the position of a diffraction maximum in the second medium is, writing x for the width of the grating element,

$$x(n_1 \sin \theta_1 - n_o \sin \theta_o) = p\lambda \text{ (see p. 369)}$$

where p is an integer and λ is the wave-length of light in air. If $p = 0$, this reduces to the law of refraction.

Now, if the incident parallel light is incident normally, as in Fig. 80, the angle of the first order diffracted beam is

$$xn_1 \sin \theta_1 = \lambda$$

or

$$x = \frac{\lambda}{n_1 \sin \theta_1} \quad . \quad . \quad . \quad (3.06)$$

If, therefore, the numerical aperture of the objective enables it to receive rays at a maximum obliquity of θ_1, this equation would give, for the minimum value of x enabling the first order diffraction maximum on each side of the axis to enter the objective,

$$x_{min} = \frac{\lambda}{NA}$$

On the other hand, the light may be incident obliquely as in Fig. 81, and it is clear that the most favourable conditions under which the refracted beam and first diffracted beam can enter the objective is that they should lie symmetrically on each side of the normal. In this case if the angle of the diffracted light is ϕ_1 as shown in the figure, the above equation becomes

$$x\,(n_1 \sin \phi_1 + n_o \sin \theta_o) = \lambda$$

Note the "plus" because ϕ_1 is measured on the other side of the normal. But if θ_1 is the angle of refraction,

$$n_1 \sin \theta_1 = n_o \sin \theta_o$$

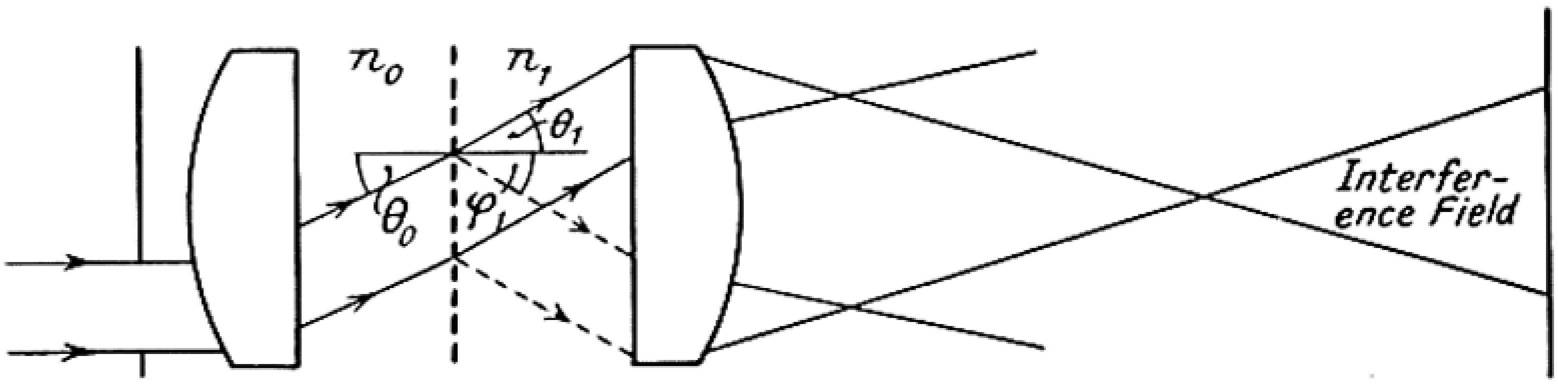


FIG. 81. DIFFRACTION OF AN OBLIQUE BEAM

so that the diffraction equation is

$$x\,(n_1 \sin \phi_1 + n_1 \sin \theta_1) = \lambda$$

and thus if $\theta_1 = \phi_1$

$$x = \frac{\lambda}{2n_1 \sin \theta_1}$$

And if the objective can receive rays as oblique as θ_1

$$x_{min} = \frac{\lambda}{2NA} \text{ (for oblique light)*} \qquad . \qquad . \qquad (3.07)$$

Now in the case of a full cone of illumination there will be some components of the light incident obliquely, and, if these secure resolution, the resolving power of the microscope will be given by the last formula, which closely agrees with that determined for a self-luminous object structure. On the other hand, if the illuminating beam is cut down to an extremely narrow angle and closely parallel to the axis, then, according to Abbe's principle, the resolving limit would be $x = \frac{\lambda}{NA}$, which agrees with experiment.

* If the numerical aperture of the condenser beam cannot exceed a value NA_c, which is less than NA_o the numerical aperture of the objective, the optimum condition then is $x_{min} = \frac{\lambda}{(NA_c + NA_o)}$.

Investigation of the Intensity Distribution in the Image. Let us consider the formation of the image of a grating of opaque bars separating equally spaced apertures of equal but negligible widths. Referring to Fig. 80, we have the case of a normally incident beam which is diffracted at an angle within the limits of numerical aperture of the objective which thus forms three maxima, one (D) due to the directly transmitted light, and one on each side of the centre (A and C) due to the first order diffraction. The grating is symmetrical with regard to the point B.

In Vol. I it was shown how to find the resultant of two vibrations of equal amplitude, but different phases. If the phase angles are $+\alpha$, and $-\alpha$ the phase of the resultant will be zero. Arguing on this simple basis we can see that if the grating is symmetrical with regard to the axial point B, and symmetrically illuminated, the phases of the disturbances sent in any chosen direction from grating elements on one side of the axis will have positive phase angles, while those from the other side will be numerically equal but negative. Hence, *the phase of the resultant disturbance in any direction will agree with that from* B.

Further, we know that if the objective is free from spherical aberrations, any disturbances originating from B will meet without phase differences in the conjugate image point B′, and they will, therefore, be in the same phase in a spherical surface ADC struck with B′ as centre. Therefore the diffraction maxima must have the same phase in such a spherical reference surface.

Their amplitudes may, however, have positive or negative signs. In the case where there is a grating aperture on the axis of symmetry, and the apertures are all very small, the amplitudes of the maxima will have the same sign; if the grating is moved so that a "bar" is on the axis the first lateral maximum will be opposite in sign to the central one.

Let AD = DC = y, and let the distance from the point D to the image point B′ be l'. Let P be a point (near the axis) in a plane through B′ perpendicular to the axis, and let PB′ = h'. Then if AC is small in comparison with l', the disturbance from A arriving at the point P will *lead* in phase by

$$\frac{2\pi}{\lambda}\left(\frac{yh'}{l'}\right)$$

while that from C will *lag* by an equal amount. If the grating apertures are all indefinitely small the direct and diffracted maxima all tend to have the same amplitude which we will write, a.

Hence the resultant of A and C in the image plane will be an amplitude of

$$2a \cos (2\pi yh'/\lambda l')$$

and adding the effect of D (same phase) we get the result

$$S = a\,\{1 + 2 \cos (2\pi yh'/\lambda l')\} \quad . \quad . \qquad (3.08)$$

This expression represents a curve (see Fig. 82) with positive maxima of value $3a$ for the values of $(\pi yh'/\lambda l')$ equal to 0, π, 2π, etc., and negative maxima = a for $\frac{\pi}{2}$, $3\frac{\pi}{2}$, etc. The intensities will, therefore, be proportional to $9a^2$ and a^2 respectively.

It can easily be shown that the separation of the main maxima given by

$$\pi h'y/\lambda l' = \pi$$

or

$$h' = \lambda l'/y$$

corresponds to the separation of the dioptric images of the grating apertures which would be formed if the latter were self-luminous. The angle of diffraction for the first order diffracted beam is given by

$$n_1 x \sin \theta_1 = \lambda$$

where $x =$ the grating element separation and θ_1 is the angle of diffraction. Now referring to Fig. 80 we see that the angle made with the axis

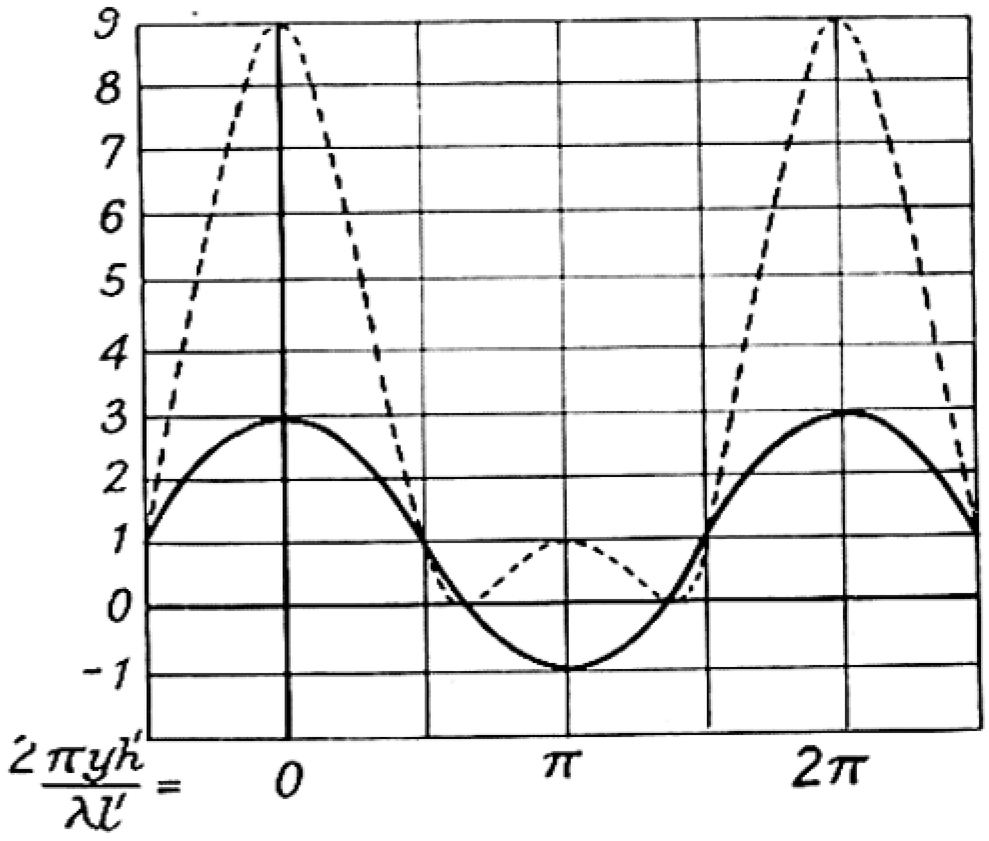


FIG. 82. AMPLITUDE AND INTENSITY

Dotted curve: relative intensity
Full curve: $1 + 2 \cos (2\pi y h/\lambda l')$ i.e. amplitude

by the extreme ray reaching the axial point in the image plane will be (y/l'). Hence, if h'_x is the image element corresponding to the dimension x in the object space, the optical sine relation gives

$$n_1 x \sin \theta_1 = h'_x y/l',$$

the image being in air. And hence $y = \lambda l/h'_x$, which gives $h' = h'_x$. It is important to notice, however, that the correspondence between object and image, true to scale in frequency, holds only for one image plane. Referring to Fig. 80, it will be understood that the relative phase of the disturbance from D at an axial point other than B′, will differ from the phase of the disturbances from A or C; hence the character of the resultant interference pattern will change according to the focus chosen; if the illumination is by a parallel beam, or completely coherent, there is thus no criterion by which the true focal plane can be chosen. However, the use of a beam of finite aperture will obviate the difficulty, as explained below. Going back to the case of full coherence it will be noticed that if further maxima are taken up by the objective the resultant becomes of the type

$$S = a\,(1 + 2\cos\phi + 2\cos 2\phi + 2\cos 3\phi + \text{etc.}) \quad . \ (3.09)$$

the number of terms depending on the number of maxima. This series approximates the more closely to a discontinuous succession of equal isolated very narrow maxima as the number of terms increases. The object was a grating of indefinitely narrow spaces. Hence we realize the importance of a wide angular aperture for the objective if the "image" is to bear the closest possible resemblance to the object. Fig. 83 shows the resultant of the first three terms of a cosine series which already has large values for $\phi = 0, 2\pi, 4\pi$, etc., and much smaller values elsewhere; the addition of further terms will add to the effect.

The representation of other types of object by trigonometrical series is further discussed in the appendix, p. 370.

The Diffraction Image. It has now been shown that if the object is a grating with narrow apertures coherently illuminated, the diffraction effects will result in an interference pattern in the image plane, with a succession of isolated intensity maxima representing the object. If the spaces are widened the phases and relative intensities of the lateral maxima will change, and thus produce a distribution of light more closely representing the relative widths of dark and light spaces as the numerical aperture of the objective is increased and more diffraction maxima are allowed to contribute their part to the image. This action was discussed by Conrady,[5] and an extension of his theory will be given below.

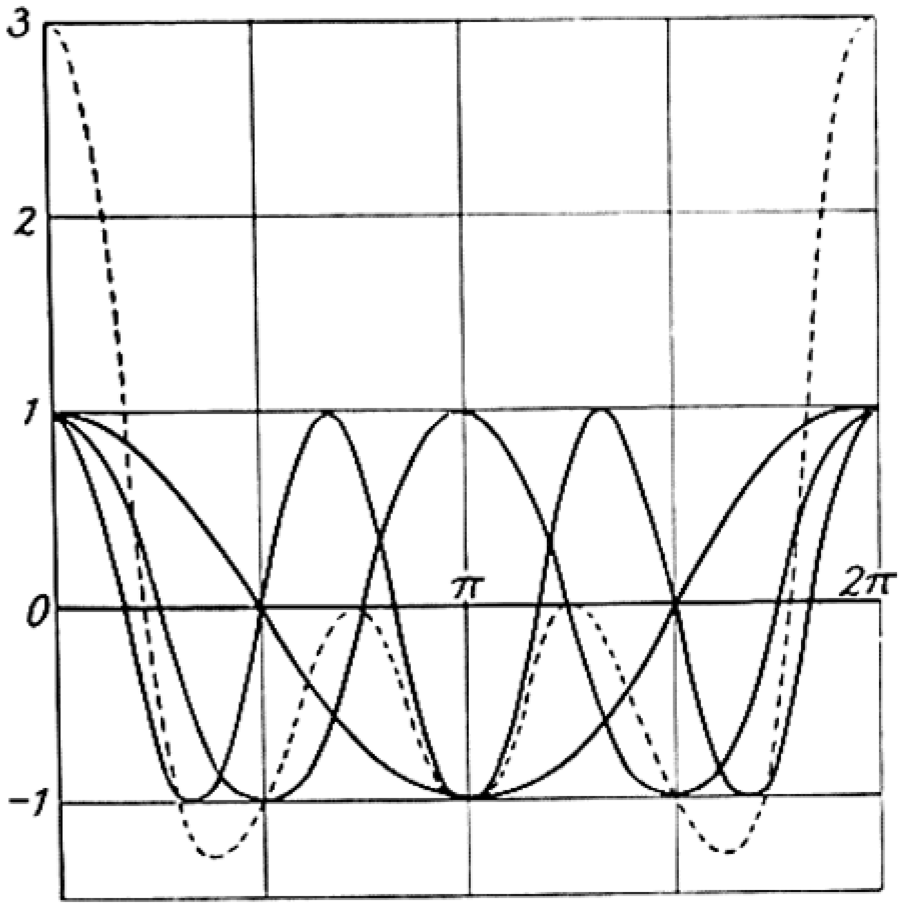


FIG. 83. FIRST THREE TERMS OF SERIES

Cos ϕ + cos 2ϕ + cos 3ϕ + etc.
Resultant: dotted curve

Notice, however, that the blocking out of the central maximum in the above case (first order only admitted) would have brought the expression into the form

$$S = 2a \cos (2\pi y h'/\lambda l')$$

in which the maxima are all of the same brightness, and have twice the frequency of the main maxima in the case of equation (α) above. Therefore if one observes the image plane while the central maximum is screened, the spacing of the bright maxima appears to be halved. A similar effect would be noticed if the first and second images were acting at first, and then the first order is screened.

Abbe himself performed many experiments of this type, and thereby somewhat alarmed microscopists, who felt that the conditions employed by him, viz. the use of a very narrow-angled illuminating cone derived from a small aperture in the condenser diaphragm might easily lead to artificial and erroneous results.

"Diffraction" came to be regarded as something likely to falsify the image, and therefore to be avoided if possible. However, it can be shown in further cases that the more diffraction maxima which contribute to the image formation, the more closely will the image resemble the object. Diffraction is, in effect, the very means of image formation and cannot be avoided; but its effect must be understood if the image is to be correctly interpreted.

Grating with Apertures of Finite Width. In Appendix II an expression is derived for the amplitude contribution for the light diffracted by a slit of width m_1 into a direction inclined at an angle θ to the normal (directions being defined by vectors in a plane perpendicular to the length of the slit); the formula is

$$A = K\frac{\sin U}{U}$$

Note that $U = (2\pi/\lambda)(\frac{1}{2}m_1 \sin\theta)$, which is the phase difference between disturbances derived from the edge and the centre of the slit respectively. It is further shown in the Appendix that if a grating has a great number of similar apertures the diffraction maxima will be found at the same angles, no matter what the width of the apertures may be, but the intensities of such maxima will be proportional to the amount of light which any single aperture is capable of sending into the relevant direction.

Now in the direction of the 1st order maximum, the phase difference between disturbances derived from the centres of adjacent apertures is 2π; hence the phase difference for the edge of a single aperture as

Table of Amplitudes (*see following page*)

Order	0	1st	2nd	3rd	4th
Amplitude	a_0	a_1	a_2	a_3	a_4
	1	$\frac{\sin 0{\cdot}8\pi}{0{\cdot}8\pi}$	$\frac{\sin 1{\cdot}6\pi}{1{\cdot}6\pi}$	$\frac{\sin 2{\cdot}4\pi}{2{\cdot}4\pi}$	$\frac{\sin 3{\cdot}2\pi}{3{\cdot}2\pi}$
	1	0·234	− 0·189	0·126	− 0·059

compared with its centre can at most approach π, since the apertures would then almost begin to overlap and the "bars" would disappear; but for the 2nd order maximum the phase difference between centre and edge will be doubled, and so on for the higher orders. Let us take for a numerical example the case of fairly wide apertures as compared with the bars in a grating so that U for the 1st order is $0{\cdot}8\pi$, or 144°.

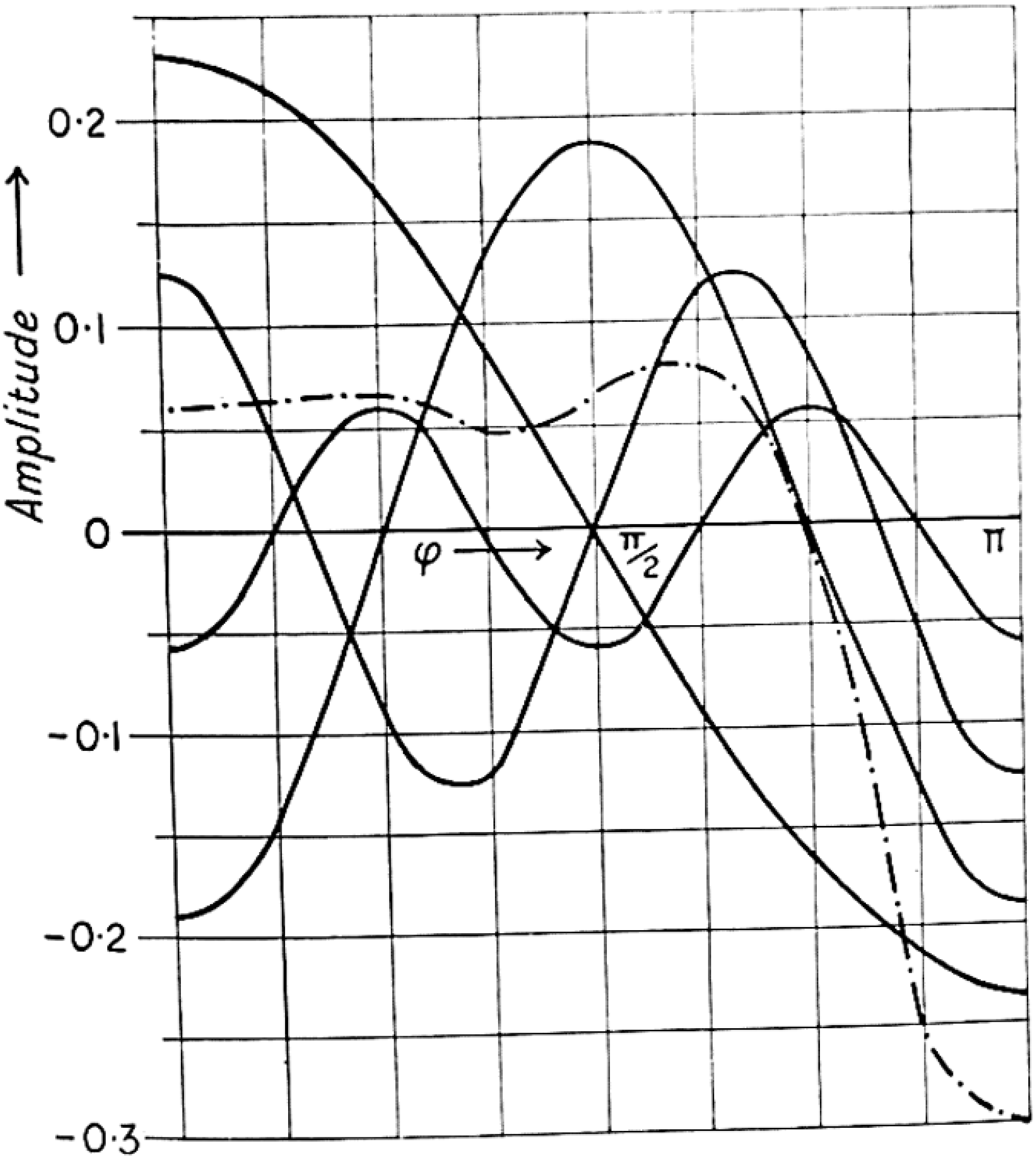


FIG. 84. RELATIVE CONTRIBUTIONS OF FIRST FOUR PAIRS OF DIFFRACTION MAXIMA

The sum is indicated by the dotted curve, for which the vertical ordinates have been halved

Then the amplitudes for the various maxima are given in the table on p. 120 (the constant K being treated as unity).

Approximate numerical values are given in the bottom row. It is worth noting that the amplitude of the first maximum will (in the case of a symmetrical grating with a space at the centre) *always* agree in phase with the disturbance from the centre point, no matter what the width of the apertures, since (sin U)/U is positive until U reaches π; however, the phases of the higher maxima can be either positive or negative.

The amplitude and intensity curves can now be plotted (Fig. 84) exactly as in the case of Fig. 82. The central maximum (zeroth order) gives a uniform amplitude of unity (not shown in the figure); the two 1st order maxima together produce a sine curve amplitude with maxima at $\phi = 0$ and π; the two 2nd order maxima produce a sine curve with *minima* at 0, $\pi/2$, π, etc.; and we see from the figure that the addition of all these begins to produce an amplitude curve with broad "high" parts and narrow low parts.

If the amplitudes of the zeroth order, 1st order, and so on, are a, a_1, a_2, etc., the full series* to be plotted is

$$S = a_o + 2a_1 \cos \phi + 2a_2 \cos 2\phi + \text{etc.},$$

where $\phi = 2\pi y h'/\lambda l'$.

By plotting the curves and adding the ordinates, or otherwise, it can be seen that the sum of a great number of terms except the first tends to be a rectangular curve having a height of $0{\cdot}25$ between $\phi = \pm\, 0{\cdot}8\pi$, a height of -1 from $\phi = 0{\cdot}8\pi$ to $1{\cdot}2\pi$, and so on; thus the amplitude in the image is found (Fig. 86), on adding the constant of unity for the a_o term, as $1{\cdot}25$ from $\phi = \pm\, 0{\cdot}8\pi$ and zero from $0{\cdot}8\pi$ to $1{\cdot}2\pi$, etc.; thus the amplitude distribution in the image resembles that of the object.

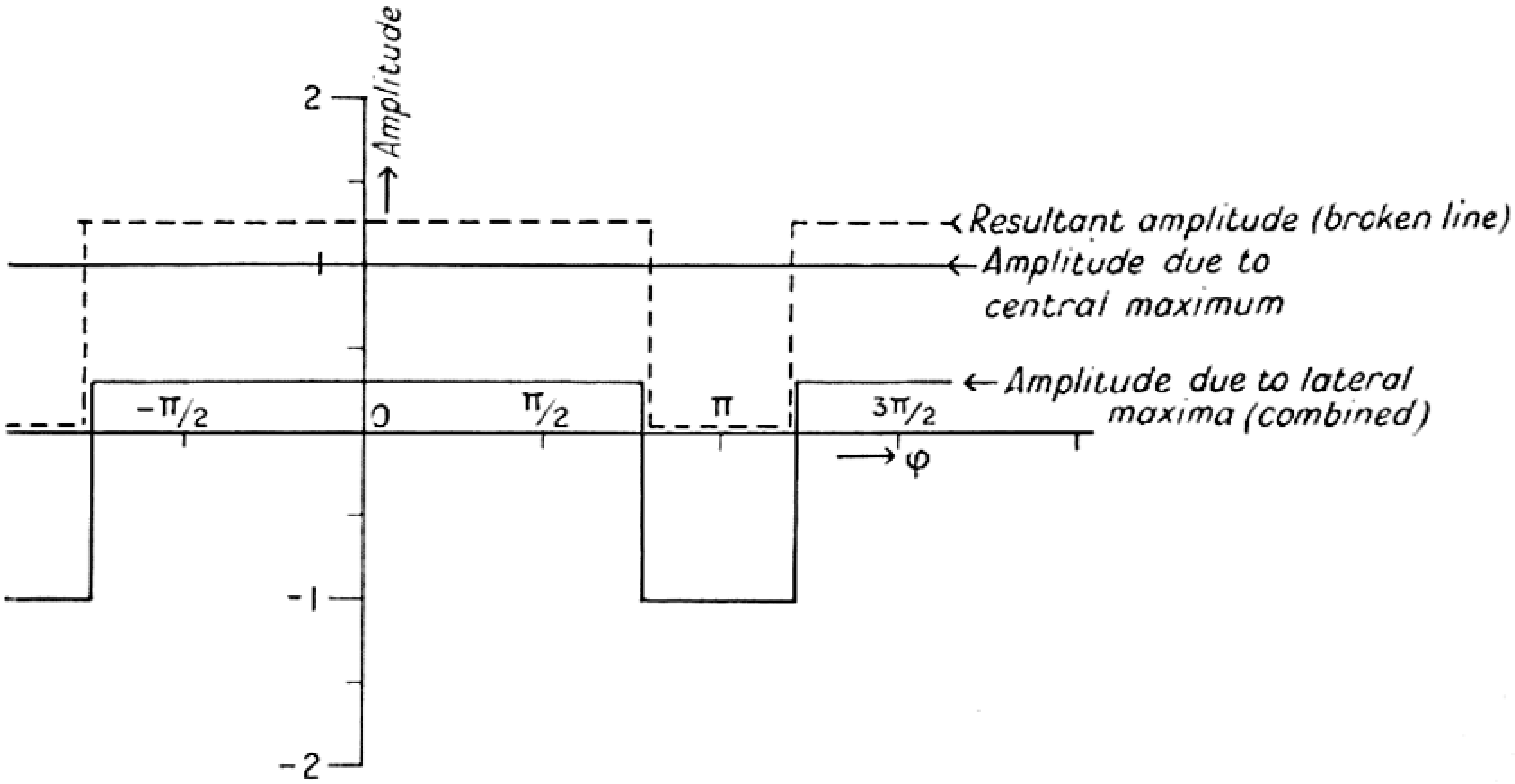


FIG. 85. FORMATION OF IMAGE WITH BROAD SPACES AND NARROW BARS

Let us now consider a *complementary grating* in which the bars and spaces are interchanged as compared with the case just considered. It will be obvious that if both these gratings were present *together*, there would be no grating, and hence no diffraction. Hence the diffraction maxima must have amplitudes equal and opposite in sign to those of the first grating, with the exception that the maximum of zero order which has the same sign, and owing to the smaller aperture it will have (in this case) one-quarter of the amplitude for the zeroth order in the previous case. Fig. 86 shows, in the same way as previously, how the interaction of the successive maxima produces an imitation of the amplitude distribution of the object with narrow spaces and broad dark regions. If the apertures of the second grating have a transmission factor smaller than unity, the amplitudes of the last table are all decreased in the same proportion, and in the case where both sets of apertures are present the summed amplitudes of the diffraction maxima, other than the central, will have the same proportions as before, but the central contribution will differ.

---

* The series is: $S = 0{\cdot}468 \cos \phi - 0{\cdot}378 \cos 2\phi + 0{\cdot}252 \cos 3\phi - 0{\cdot}1174 \cos 4\phi + 0{\cdot}0 \cos 5\phi + 0{\cdot}078 \cos 6\phi - 0{\cdot}108 \cos 7\phi + 0{\cdot}094 \cos 8\phi +$ etc.

Note that the terms in 6ϕ to 9ϕ inclusive will give positive contributions both at $\phi = 0$ and $\phi = \pi$.

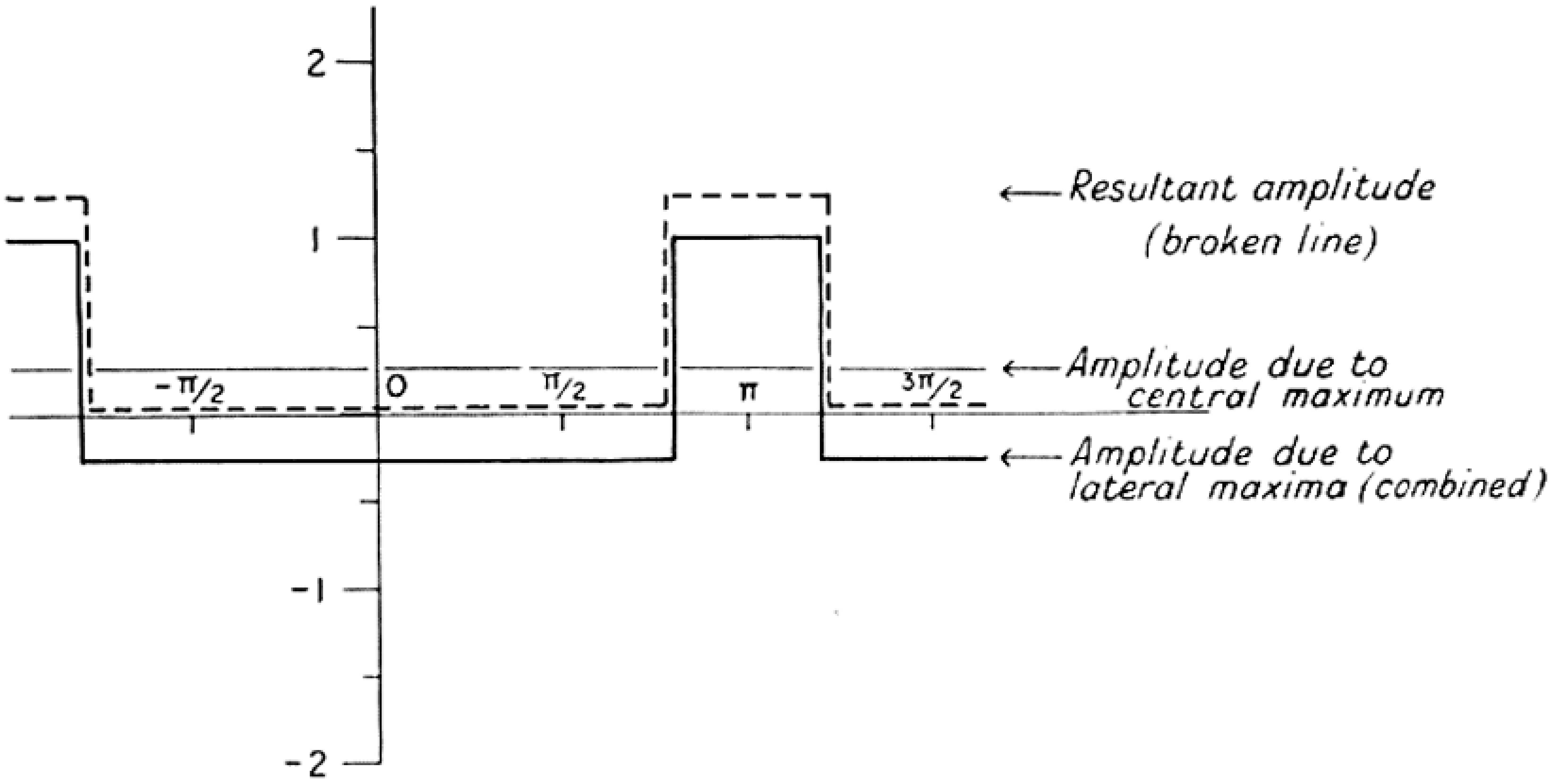


FIG. 86. FORMATION OF IMAGE WITH BROAD BARS AND NARROW SPACES

Phase Grating. Fig. 87 (*a*) is a phase diagram for the two complementary gratings discussed above; the maxima for the grating with the aperture at the centre being shown by full lines; those for the one with a bar at the centre by broken lines.

Suppose, now, that we replace the bars of the second grating by thin laminæ of refracting material, the reflection and absorption effects being negligible, and the retardation for every "bar" being π. It is clear that all the maxima for the second grating undergo a relative phase change of the same amount. The phase-amplitude diagram now resembles Fig. 87 (*b*). The diffraction maxima other than the central one all have doubled intensity; hence the combined amplitude gives a curve with an ordinate of 0·5 from 0 to $0·8\pi$, of -2 from $0·8\pi$ to $1·2\pi$, and 0·5 from $1·2\pi$ to $2·8\pi$, and so on; but the combined central maximum now has an amplitude of only 0·75. Adding this to the periodic terms (Fig. 88) we have a curve whose maximum and minimum ordinates are equally spaced above and below the horizontal axis, and with abrupt changes from the positive to the negative values. This obviously corresponds to uniform intensity in the image plane but accompanied by abrupt changes of relative phase in the regions of the image corresponding to the apertures and retarding bars respectively.

It is clear in the cases of grating laminæ with a different retardation that a similar effect will occur; though in the above case it is easy to follow the way in which the diffraction maxima produce the effect it is more complicated when the phase angles have values other than π or its multiples. However, it can be taken that when all the diffracted light is allowed to enter the objective, the image corresponds to the object both in relative intensity and relative phase, a proposition first clearly established by Lummer. It applies only for coherent illumination.

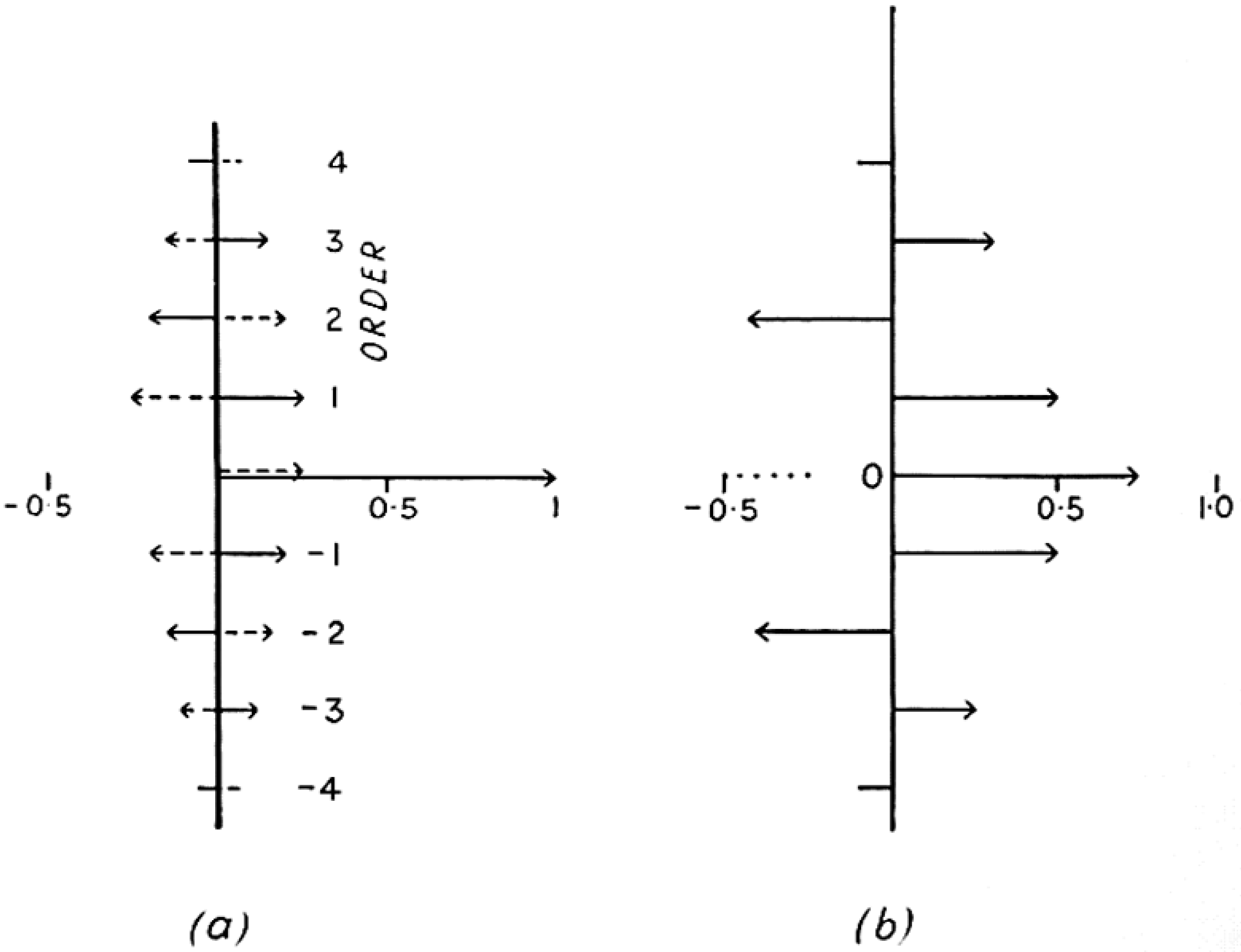


FIG. 87

(*a*) Amplitudes of diffraction maxima of successive orders
Full lines: amplitude grating with broad spaces and narrow bars
Broken lines: complementary amplitude grating

(*b*) Amplitudes of diffraction maxima for phase grating with same spacing as amplitude grating.
The retardation for narrow bars is π

Where a grating has bars which retard but do not absorb the light it is often called a "phase grating." We shall discuss the microscopical observation of such gratings below.

The Function of the Condenser. The foregoing discussions have assumed that the illumination of the object, whatever its nature, is by a parallel beam of coherent light. It has been shown, for certain periodic structures at least, that the wider the aperture of the objective, the more closely will the image resemble the object. If the angular aperture of the illumination is increased it can be regarded as the superposition of independent parallel beams at various angles; the diffraction effects for all such beams, produced by the object, will agree only in the one conjugate plane where the "image" is formed by the objective.

It was considered, with some reason, that the function of the wide aperture condenser would be to make the object behave to some degree as if it were self-luminous (Rayleigh) when the image formation could be discussed without bringing in diffraction theory. It was, however, erroneously concluded that if the image of the source of light were exactly focused by the condenser in the plane of the object, then the object illumination would be completely "incoherent," and all diffraction would be avoided. For this reason an immense amount of trouble was taken by some microscopists to secure the most exact focusing of the source in the object plane (so-called critical illumination) under the impression that only thus would a true image be obtained with the optimum definition. However, a paper[6] by the present author showed that even under such conditions, the image formation can still be described in terms of diffraction theory, and thus there is nothing critical about "critical illumination."

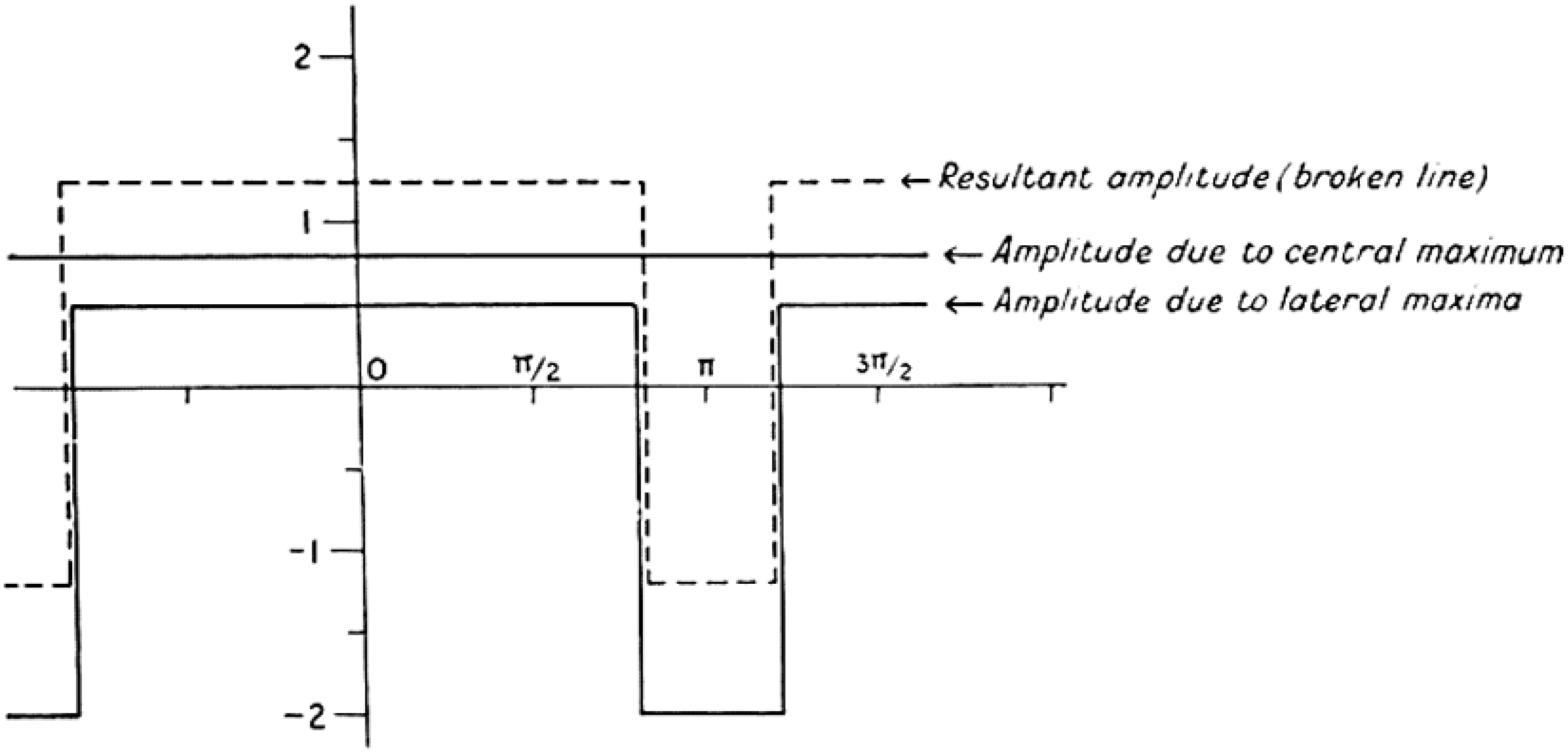


FIG. 88. FORMATION OF IMAGE OF PHASE GRATING WITH SPACING IDENTICAL WITH THE AMPLITUDE GRATINGS OF FIGS. 85 AND 86

It is quite true that this condition, and the use of a well-corrected condenser, do bring the illumination under good control, and the effects of stray light and glare are thereby more easily avoided.

We will consider a case in which a plane source of light is placed in the anterior principal focus of the condenser. The source can be imagined to be composed of infinitesimal elements, such as A, Fig. 89, which acts effectively as a point source, and gives a divergent beam of rays which enters the entrance pupil of the condenser, is collimated, and gives thus coherent illumination in the object plane. Let us suppose that the object plane contains a regular grating of bars and spaces; this diffracts the light and thus the objective forms the diffracted images of the element A in its posterior focal surface. Thus each "homologous" set such as A_1, A_0 and A_{-1} will act together in forming an image of the grating as explained above on pp. 117–25. The same applies to all the infinitesimal source elements, hence the focal surface contains a set of diffraction images of the complete source, and the number of orders present will depend on the numerical aperture of the objective.

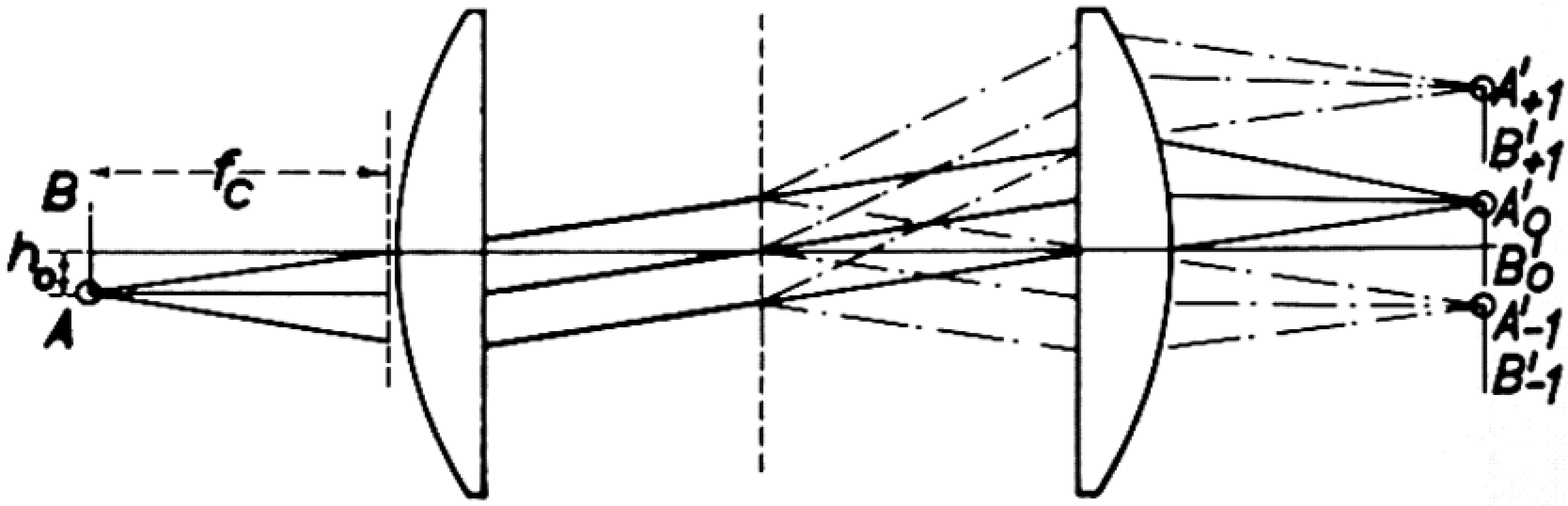


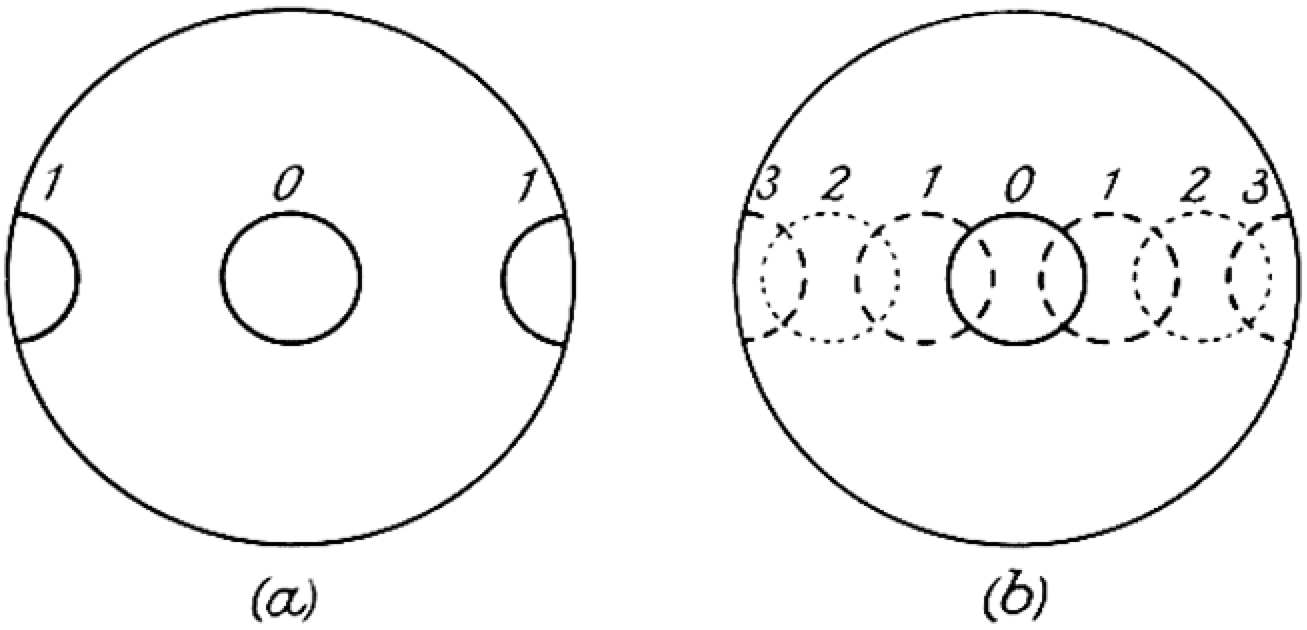
FIG. 89

Note that the numerical aperture of the angular cone of rays passing through a single point of the object depends in this case on the angle subtended by the source at the first principal point of the condenser; as shown in the diagram it is (very closely) h_0/f_c.

As a finite cone of rays passes through each point of the object-plane the system will, if properly corrected, produce a unique conjugate image-plane in which such rays re-unite; this introduces a factor not present in perfectly coherent illumination. Fig. 90 suggests the appearance on the upper focal plane of a microscope objective when a small stop is being used in the condenser, and when gratings of 15,000 lines and 5000 lines to the inch, respectively, are being examined by a lens of *NA* about 0·3. The "direct light" is indicated by 0, and the figures 1, 2, etc., suggest the successive orders of the lateral maxima.

Such maxima will give an interference system in the field of the instrument and it can be shown that the frequency of the maxima in the image plane conjugate to the object plane for the objective corresponds to the enlarged dioptric grating image. In order to deal with the practical problem, however, the superposed effects of all the various beams passing in different directions through the object plane have to be worked out by an integration which is further discussed in Appendix III.

Modern Diffraction Theory of Image Formation. Degree of Coherence. A distinction has been drawn above between the cases when the microscope object is self-luminous, and when the object is supposed illuminated by a parallel beam derived from a single source of light; in these cases the vibrations derived from the various points of the object plane are said to be completely *incoherent* and completely *coherent*, respectively. However, in the case when the object is illuminated by a condenser, so that (unlike the illumination by a parallel beam) each point of the object plane is traversed by a cone of rays with appreciable angular differences, we have an intermediate case, and the theory has to take account of the "degree of coherence." The principal authors concerned in this development have included Berek,[7] van Cittert,[8] Zernike,[9] and Hopkins[10]. We can only summarize, here, the definitions and some of the more important practical conclusions.



(a) (b)

FIG. 90

(*a*) Appearance of back of objective with "15,000 lines band" just resolved using small condenser aperture
(*b*) Appearance with "5000 lines band"; same condenser aperture and objective

It has been shown, for example, that if the illumination is mediated by a symmetrically disposed source of uniform luminance, the degree of coherence depends primarily on the angular aperture subtended by the source. The effective source may be a luminous surface at any distance, or may be the exit pupil of a condenser which is focusing the rays through the object plane; provided the angular cone of rays passing through any point in the object remains the same, so also does the degree of coherence. This again shows that the idea of "critical illumination" was erroneous.

The concept of the "degree of coherence" may be briefly explained as follows. Suppose that displacements from two elementary sources of light received at some point illuminated by both are respectively represented by—

$$A_1 = a_1 e^{i\theta_1} \text{ and } A_2 = a_2 e^{i(\theta_1 + \delta)}$$

thus the separate intensities are $I_1 = |A_1|^2 = a_1^2$ and $I_2 = |A_2|^2 = a_2^2$. The combined displacement is now

$$A_1 + A_2 = e^{i\theta}(a_1 + a_2 e^{i\delta}),$$

and the corresponding instantaneous intensity, found by multiplying the complex amplitude by its conjugate quantity is

$$\begin{aligned} I &= (a_1 + a_2 e^{i\delta})\,(a_1 + a_2 e^{-i\theta}) \\ &= a_1^2 + a_1 a_2\,(e^{i\delta} + e^{-i\delta}) + a_2^2 \\ &= a_1^2 + 2a_1 a_2 \cos\delta + a_2^2. \end{aligned}$$

Now in the case of actual independent sources of light there can be no constancy in the difference between their epoch angles, even if the separate intensities remain substantially constant within the limits of observation. The *observed* intensity is the time average, denoted by placing a bar over the quantities

$$\bar{I} = \overline{a_1^2} + \overline{2a_1 a_2 \cos\delta} + \overline{a_2^2}$$

The time average of the cosine term, which has rapid fluctuations in magnitude and sign, will be zero, and thus

$$\bar{I} = \overline{a_1^2} + \overline{a_2^2}$$

Hence if the light from two independent sources of constant intensity is added the observed intensity is the sum of the separate intensities.

However, if the two elementary sources mentioned above are not actual sources in their own right, but owe their light wholly to another actual source, then δ will be a constant depending on the optical path from the actual source to the secondary sources, and from these to the point illuminated. If, for example successive illuminated points in a plane show a continuous change of δ there will be an interference phenomenon. Let us take the case when $a_1 = a_2$, then I varies from zero to $4a_1^2$, i.e. four times the intensity due to a single source. The contrast (see p. 288) in the fringes will be

$$\frac{I_{max} - I_{min}}{I_{max} + I_{min}} = I$$

However, the light from the secondary sources may not be wholly derived from a single source. The intensity may then be represented by a modified formula, say

$$I = a_1^2 + 2\Gamma a_1 a_2 \cos\delta + a_2^2$$

where Γ can have some fractional value. In the case where a_1 and a_2 are equal

$$\frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{a_1^2\,\{(2 + 2\Gamma) - (2 - 2\Gamma)\}}{a_1^2\,\{(2 + 2\Gamma) + (2 + 2\Gamma)\}} = \Gamma \quad . \quad (3.10)$$

Hence the degree of coherence Γ determines the contrast in any interference phenomenon due to the two equal sources.

A theoretical investigation shows that if light from a condenser of full numerical aperture NA_c illuminates an object plane uniformly, the relative degree of coherence between the light displacements at points separated by an "optical* interval" z is given very closely by

$$\Gamma = \frac{2J_1(z)}{z} \quad . \quad . \quad . \quad . \quad (3.11)$$

The variation of $2J_1(z)/z$ is thus identical with that of the amplitude in a star image, see Vol. I, p. 97. (The same degree of coherence would be given if the illumination is given by a real source subtending the same numerical aperture.) This relation enables us readily to appreciate the difference between fully coherent illumination (in which $\Gamma = 1$ for any distance in the object plane) and the partially coherent light from a source of finite aperture. For example $\Gamma = 0$ for two points separated by an interval equal to the radius of the Airy disc when a condenser of the same aperture (imagined fully corrected) gives a star image, but for smaller intervals the degree of coherence may be much greater. Thus, for example, when $z = 2$, the value of Γ will be about 0·6. With a condenser of numerical aperture, say 0·8, the actual interval h corresponding to an optical interval z will be

$$h = \frac{\lambda z}{2\pi NA} \approx 0.2\mu$$

for light of wave-length $0{\cdot}5\mu$. The degree of coherence for points at $0{\cdot}2\mu$ apart is 0·6. At distances comparable with the "resolving limit" of microscope systems there is usually a marked degree of coherence, and this has a considerable influence on the images of details.

The results of the theory are applicable to cases where the illumination aperture is not a full cone, as, for example, in phase-contrast and dark-ground illuminations; the coherence factor can again be described in terms of the amplitude of the relevant diffraction image.

As will be explained below, the coherence factor does not appear explicitly in expressions for contrast, etc., with microscope systems using illumination of restricted aperture. However, it should be understood that no condenser can produce the effect of perfect incoherence; nor does the ordinary microscope ever work under conditions approaching perfect coherence. Some indication of modern developments of the theory of image formation is given in the Appendix, p. 370.

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* For definition of optical interval, see p. 374.

Observation of Phase Objects in the Microscope. If a phase grating coherently illuminated is observed with a well-corrected objective of very wide aperture, Lummer's principle would indicate that the object would tend to disappear, since in the ordinary way the eye is not sensitive to mere changes of phase in the image. It is a familiar observation, however, that when a diatom with its transparent siliceous structure is observed in practice, the reduction of the aperture of the illuminating beam is usually sufficient to show up the structure fairly clearly.

Going back to the theory of the image of the phase grating, it seems that any factor which will upset the balance between the direct light and the diffracted light, for example a relative change in the intensity of the direct beam, will exhibit an intensity modulation in the image. Moreover, the exclusion of any of the highest-order amplitude contributions from the image must result in periodic amplitude effects or "fringes" which will appear to be superposed on the true image. If the central maximum is entirely screened, we have one type of "dark-ground" illumination, and this at once exhibits intensity variations of a periodicity corresponding to the object structure.

The "phase-contrast" method of Zernike[11] introduces a phase-retarding plate into the path of the direct light only. This again upsets the balance on which the invisibility of the phase grating depends. The main advantage of the method, as will be seen, is the possibility of making the *brightness* modulation of the image correspond in some degree to the phase modulation of the object. The method is discussed in more detail below. (See p. 134.)

It can be confirmed by careful observation that when a phase-modulating object, such as is exemplified by a diatom, is observed with a well-corrected objective of wide aperture (the illumination having an aperture approaching that of the objective) and with carefully adjusted focus, the image will tend to show little contrast. Now if the aperture of the illuminating beam is reduced, it will make the image formation depend upon relatively small maxima between which there will be a certain set of phase relations. If the image is a little out of focus there will be a difference of optical path between conjugate points which will be roughly proportional to the square of the numerical aperture for an oblique ray; this will cause a phase difference in any diffraction maximum characteristic of such a route. Again, if the objective is imperfectly corrected for spherical aberration, or if a "wrong" tube-length is being employed, this will again introduce phase differences among the maxima which will upset the balance upon which the invisibility depends. Most actual diatoms will call for some difference of focus to study the different parts of the frustule; most objectives will have some small zonal spherical aberration. Hence if the reduction of the aperture of illumination tends towards a single-valued set of phase relations, other than those of the "perfect" case, the image of the phase object should easily show intensity modulation. On the other hand, when the aperture is large, there will be no single set of phase relations for the maxima, and the average of all sets is likely to produce a confused effect in which a single type of intensity modulation is less likely to appear. The better the objective, the more readily will a slight change of focus "bring the object into view."

Owing to the necessity of reducing the aperture to perceive a phase-modulating object, it is understandable that the resolving limit for such an object is likely to be considerably less than the theoretical limit for the objective when operating with an object exhibiting intensity modulation. Most of the better-known microscopic test objects, rulings, etc., are phase objects.

Effect of Condenser Aperture. The Abbe theory has now appeared as a part only of the inclusive diffraction theory. Modern analysis, while not based on the Johnstone Stoney concept of homologous points has shown that this method of describing the action of the microscope is a useful one. Let us consider the theoretical aspect of the observation of a typical microscopic object consisting of a regular structure so fine that, when the condenser aperture is very small, the first order diffraction maxima lie entirely outside the limits of the objective aperture. Fig. 91 suggests the appearance within the ring of the objective O. D indicates a direct beam, and D_1 a first order diffracted beam; as we know, the angular apertures of these correspond to the aperture of the condenser. As this aperture is increased, the first order maxima begin to appear, and the homologous points (H) of the Abbe principle may be imagined now to begin to set up their interferences. The conditions should, at first sight, improve till the condenser aperture is equal to that of the objective, but this is dependent on the avoidance of relative phase changes between such pairs of homologous points, a matter dependent on the correction of the objective. In the majority of cases the resolution begins again to be lost as the aperture of illumination is yet further increased; this is partly due to glare, and partly due to the loss of the secondary interferences owing to the above reason.

The above case is well illustrated by the familiar diatom, *Amphipleura pellucida*, which is "resolved" by a first-class microscope objective of *NA* at least 1·2 when working with a condenser giving a symmetrical cone of illumination of *NA* 0·9 or thereabouts, but not with a smaller (or much larger) aperture of illumination. This resolves the structure of 10,000 lines to the inch, but the (only slightly finer) lines perpendicular to the first can only be "resolved" by the use of oblique light, presumably because the variations of optical path involved in the second structure are much smaller, and the diffraction maxima are much weaker relatively to the direct light.

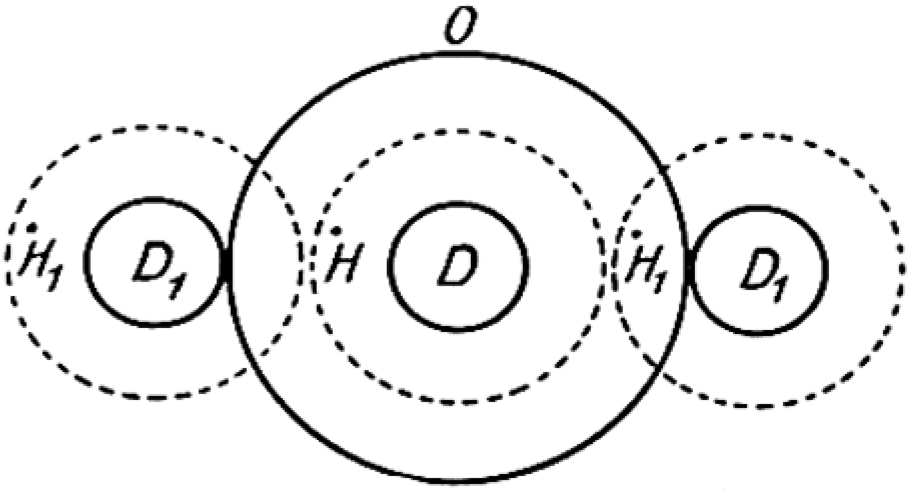

Fig. 91

It seems fair to conclude that since the majority of fine-resolution effects with the microscope are obtained with apertures of the illumination considerably smaller than the objective, the finer detail of the image of regular structures must be usually ascribed to interference effects due to the object as a whole. When, however, the conditions are such that the aperture of the condenser can be increased to equality with the objective, and when we are dealing with an object consisting merely of variations of opacity in one plane rather than of refractive index in a thin layer, then we may regard the image formation as closer to that of a self-luminous object, so that there will be an approach to point to point correspondence.

There are thus two ways of regarding and using the microscope. Firstly as a camera, secondly as a kind of interferometer. In the first case the aim is to obtain a "picture" of the object in the conventional sense, with a "one to one" correspondence between details. A precise meaning can only be attached to the picture when the object consists of a plane containing transparent and, more or less, opaque layers with definite geometrical distribution. When the object is three-dimensional or consists of variations of refractive index between two plane boundaries, the one to one correspondence loses much of its meaning. The effort to interpret the picture in a conventional sense becomes increasingly difficult, and the "image" is best regarded as giving "evidence" of structure (and that only when the illumination has been taken into account) rather than as a picture at all.

Practical Deductions. 1. *The objective must be well corrected.* The necessity of equal optical paths for various routes between corresponding objects and image points is clearly brought out by the analysis; especially for the case of an object with intensity modulation. Provided the objective is well corrected, the larger the numerical aperture the larger the number of diffraction maxima which will be admitted and the closer the geometrical correspondence between image and object.

2. *For an intensity-modulating object the numerical aperture of the illumination should ideally be raised to equality with the objective.* This applies to any structure resolvable with a narrow central beam. The most oblique beams are then accompanied by lateral maxima of still higher order, and the correspondence between object and image further improves. If, however, there is a deficiency in the spherical correction the image formation by the oblique beams will be dissimilar to the central ones and the integrated image will suffer in definition and contrast. The optimum aperture for the illumination will then be well below that of the objective.

3. *For a phase-modulating object contrast can only be obtained in the image in general by suitable restriction of the illuminating aperture accompanied by some change of focus (or other means of introducing optical path differences between the direct and diffracted light).*

Where the reduction of the illuminating aperture is adequate, there will be phase relations between the maxima which produce considerable contrast in the image of the periodic structure as the result of appreciable focus changes; thus the outline of an object seems to be well defined for one focus, and the "structure" is seen at another; this may occur also for intensity modulation when the objective suffers from spherical aberration.

4. *Where evidence of the finest detectable regular structure in the object is sought for, it is advisable to use an oblique beam of narrow aperture to illuminate the object.*

Under such conditions the one to one correspondence of object and image is spoilt in great measure, only the coarser features of the former being rendered by the limited angular aperture of the illuminating beam. If, however, there are regular structures present in the object, even periodic variations of refractive index, these latter have their best chance to produce an intensity variation of proportionate frequency in the image plane. The location of the interferences in the "picture" will roughly correspond to that of the corresponding structure in the object, but we must regard any element of the interference pattern as due to the whole of that structure, and abandon the idea of point to point correspondence.

Abbe pointed out that under the conditions of oblique lighting by narrow pencils, used with an object consisting of a layer containing media of varying refractive index, the optical path differences arising between disturbances traversing any neighbouring parts of the structure may vary with the inclination of the light. Hence the relative phases of the various diffraction maxima would be likely to show a corresponding variation, and this would be likely to result in a weakening of the interference phenomenon if wide-angled cones were used for illumination. It is a perfectly legitimate conclusion that we are more likely to find evidence of delicate and fine structures when using narrow cones of illumination, but the appearances in such a case are mainly secondary interferences and are not to be interpreted in any other way.

Phase Contrast Illumination. It was seen above that when the central maximum from a phase grating is given a relative change of phase by passage through a thin layer of retarding material, the balance between the direct and diffracted light is upset, and the phase modulation of the image becomes an amplitude modulation. Similar effects can occur even in the absence of any periodic structure.

Let the object plane be divided into a large number of elements of equal size. We will assume that the illumination consists of parallel light normal to the object plane. The phase of the light corresponding to each transparent element (immediately after transmission) depends on the relative retardation therein. Fig. 92 is a vector diagram in which the points marked in the circular locus suggest vectors OA, OB, OC, etc., showing the amplitudes (equal) and possible phases for such elements of the light. Increasing retardation corresponds to an anti-clockwise movement of a vector. According to the theorem of Lummer (see above, p. 124) the image plane will exhibit an exactly similar distribution of amplitudes and phases (supposing the optical system to be perfect, and that all the diffracted light contributes fully to the image). If now all these (*b*) *Spherical globules or other bodies of regular shape present in the object layer.* The laws of reflection and refraction allow the distribution of the emergent light to be calculated if the bodies are large enough.

(*c*) *Very small particles or filaments of material having a refractive index differing from the surrounding medium.*

In some cases the relative amount of light *scattered* into different directions by particles of various sizes can be calculated. Referring to Fig. 96, L represents a beam of light, and P a scattering particle.

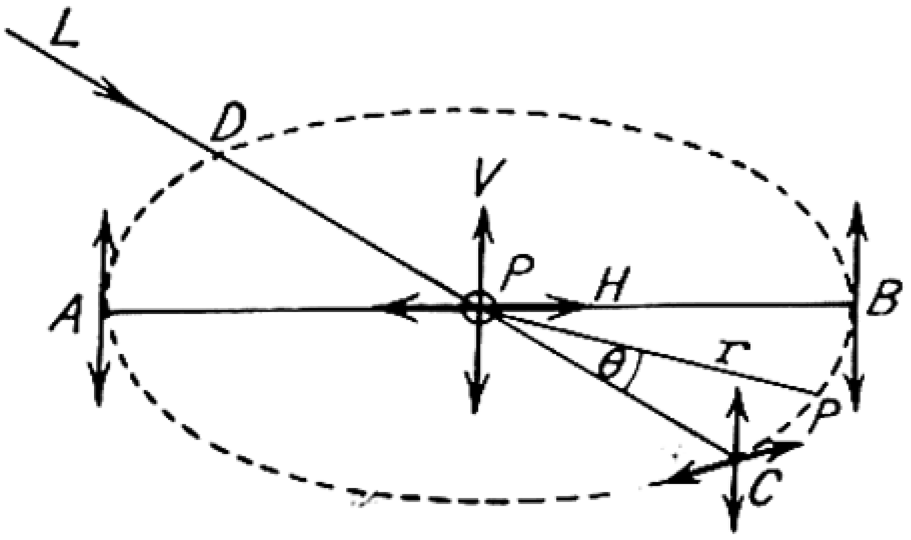

FIG. 96

In the theory due to Rayleigh,[12] the small particle "loads the ether" and becomes the seat of disturbances propagated in all surrounding directions. Fig. 96 is a perspective view. The electric forces in the wave-front are perpendicular to the direction of propagation, and may be resolved into horizontal and vertical components H and V. Both H and V produce equal effects at C in the line of propagation or at D (backwards), but only V can act at A or B in the direction perpendicular thereto.

Rayleigh's formula for the intensity of the scattered light in a direction making an angle θ with the incident ray is, if the incident light is unpolarized,

$$\frac{1}{2}\,A^2\,\frac{(D'-D)^2}{D^2}\,(1+\cos^2\theta)\,m\pi\,\frac{T^2}{\lambda^4 r^2}$$

where r is the distance of the particle from the point of observation, λ is the wave-length, T is the volume of one particle, m is the number of particles, D′ and D are the densities of the material of the particles and the medium respectively, and A^2 is the intensity of the incident light. Thus the intensity in the forward or backward direction will be double that at right angles to the incident light. The light scattered in a direction perpendicular to the incident beam is plane polarized.

With very small particles, the amount of light scattered is inversely proportional to the fourth power of the wave-length, accounting for the "blue" of fresh smoke, etc. With coarser particles the light becomes white to ordinary observation. The conditions of transition between the pure scattering effects and the refraction effects characteristic of larger bodies have not been fully explored, and this lack of knowledge causes much difficulty in the interpretation of phenomena observed when using dark-ground illumination.

Boundary Waves in Dark-ground Illumination. One important case of symmetrical dark-ground illumination is that of the exclusion of the central diffraction maximum by a suitable stop placed at the rear of the objective. The *kind* of image-formation for a grating object can be readily studied by reference to the foregoing methods, p. 119; at least for the case when the illuminating numerical aperture is relatively small, and symmetrical with the axis; and when the grating is so coarse that a great number of diffraction maxima can be taken in by the objective.

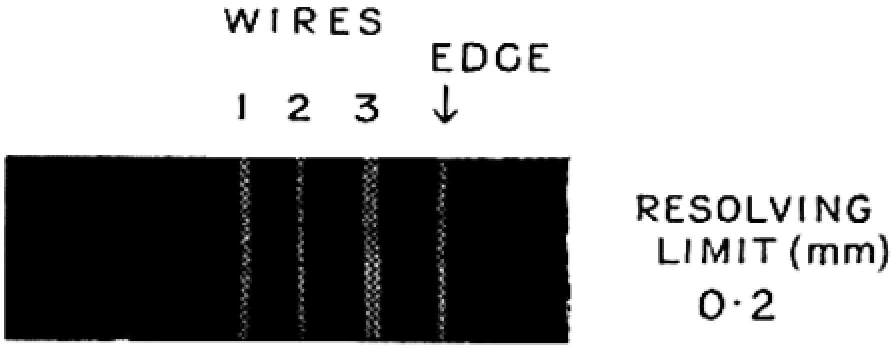

FIG 97. DIFFRACTED BOUNDARY LIGHT FROM WIRES

Wire 1 thickness 0·118 mm.
Wire 2 thickness 0·018 mm.
Wire 3 thickness 0·227 mm.

In the first place, considering the amplitude grating and its complement, since the diffraction maxima have the same amplitude (merely opposite in phase) in these two cases, the images should look exactly the same. Nothing can be inferred about the relative transmission of the different parts of the object. Again, taking the half-wave retarding phase-grating, we saw that the maxima were merely doubled; the dark-ground image again looks exactly the same, though brighter. In the use of an extremely oblique beam through the object, Fig. 95, the maximum of zero order then falls outside the aperture of the objective and, of the other orders, only those on one side of the direct light now enter the objective. This case can again be dealt with quite successfully by the Abbe principle as has been shown by Zernike; however, in such circumstances a description of the effects can be given in terms of the so-called boundary-wave effects. Apart from any periodic structure in the object plane it can be shown[13] that any edge (of change of transmission or optical path) in the object plane will in general seem to be a source of light. Hence this type of dark-field image usually shows a *bright border* to transition regions.

Fig. 97 shows a "dark-field" photograph of three very fine wires set up in a holder on an optical bench; they were illuminated by light from a slit (parallel to the wires) and viewed by a telescope of which the entrance pupil was within the shadow of an opaque screen. The resolving limit of the telescope is 0·2 mm. and the thickness of the wires 1, 2, and 3, were 0·118, 0·018, and 0·227 mm. respectively. The boundary haloes on both edges of the wire 3 are thus resolved. The edge halo of the screen is also seen.

Under such conditions the boundary waves from the opposite edges of a narrow obstacle show a phase difference which approaches π when the bar is made narrower and narrower; thus the "thread of light" given in the image by diffraction from a narrowing filament, unresolvable by the objective, gets dimmer and dimmer. Similar considerations apply to the light diffracted by any small opaque obstacle, colloid particles, etc.; but in such cases the shape of the particle and state of polarization of the light become of great significance.

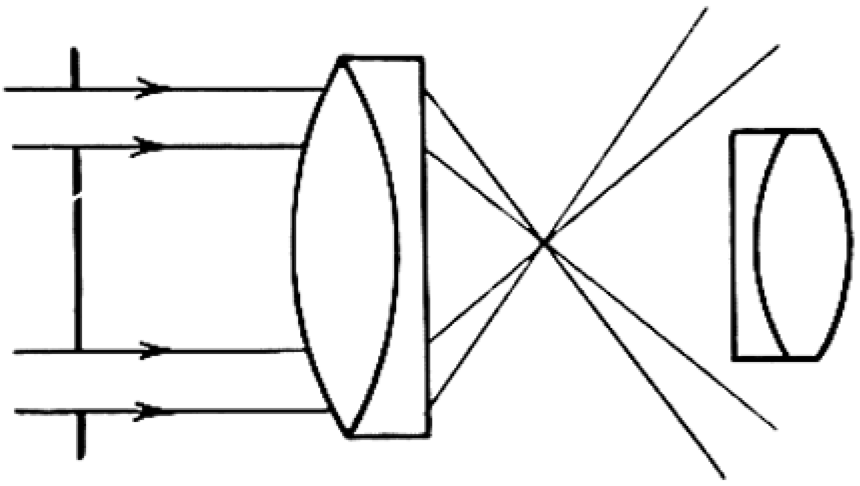

FIG. 98. USE OF ANNULAR STOP

When the "bar" consists of a narrow transparent strip with finite regions of optical path change at each edge, the light diffracted at such an edge depends on the sign of the change of path; if the prismatic action of the transition region tends to refract the illuminating ray towards the objective aperture the edge will appear bright, and *vice versa.* With a parallel illuminating beam any perpendicular filament of opaque or phase-retarding material can thus be regarded as a source of cylindrical wave fronts.

Unilateral illumination is used only for special purposes as explained below. Symmetrical dark-ground illuminators which produce annular illumination also give boundary-wave effects, the transition edges of objects being outlined by bright haloes. However, the complete diffraction theory of these cases becomes complicated. Optical systems for dark-ground illumination are discussed below. The conception of the boundary waves is still useful in discussing the image formation in phase-contrast procedure.

Optical Systems for Dark-ground Illumination. Referring to Fig. 98, it will be seen how the use of a central dark stop in the diaphragm of the condenser produces annular illumination, with a great gain in intensity as compared with the arrangement of Fig. 95. This method is often used with low power objectives.

Fig. 99 shows a section of a paraboloidal reflector; the top of the parabola is absent, so that the rays concentrated through the focus can proceed onwards. A solid truncated paraboloid is made in glass, and the top is oiled on to the slide, thus escaping the spherical aberration* arising by oblique transmission at a refracting surface, Fig. 100. Such paraboloids are still made and used.

The difficulty of manufacturing paraboloidal surfaces of sufficient accuracy led to the invention by Ignatowsky[14] and Siedentopf[15] of double mirror reflecting condensers.

Theory of Cardioid Condenser. To understand their action, we recall the formula expressing the relation between the length of the radius vector at any point on a polar curve, and the angle between the radius vector and the normal.

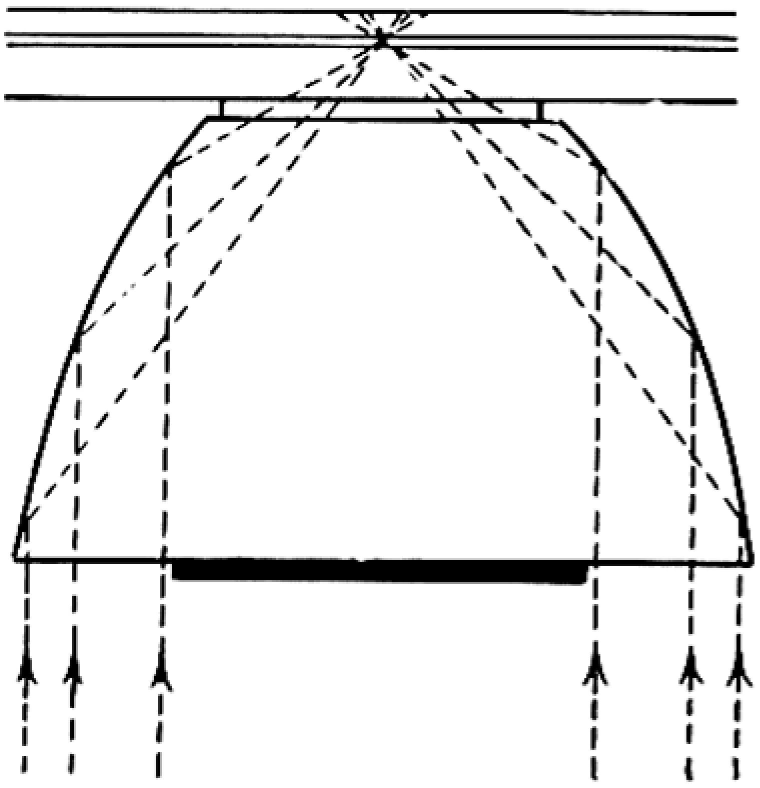

FIG. 99. PARABOLOID ILLUMINATOR

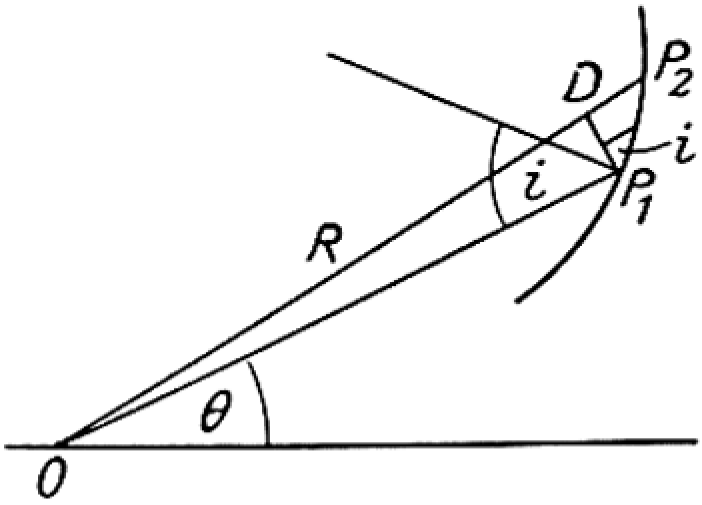

FIG. 100

The equation to the curve is $f(R,\theta) = 0$ and the angle is i.

Joining the origin O to two very close points P_1P_2 on the curve (Fig. 100) and dropping the perpendicular P_1D to OP_2, we find

$$\tan i = \frac{DP_2}{DP_1} = \frac{\delta R}{R\delta\theta}$$

Hence in the limit,

$$\tan i = \frac{1}{R}\left(\frac{dR}{d\theta}\right)$$

For the cardioid OPS (Fig. 101), whose equation is

$$R = r(1 + \cos\theta),$$

where $r = \frac{OS}{2}$, we obtain

$$\frac{dR}{d\theta} = -r\sin\theta$$

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* Vol. I of this book, p. 13.

and
$$\tan i = \frac{-r \sin \theta}{r(1 + \cos \theta)} = -\tan\left(\frac{\theta}{2}\right)$$

Hence if a ray started from O and were reflected at the cardioid at P, it would travel towards a point U on the line OS, for which OUP is an isosceles triangle. Take a point T at a distance $\frac{r}{2}$ from O and draw TV cutting PU in V, so that $\widehat{VTU} = \theta$. Then with T as centre and TV as radius, draw the circular arc VW; it is evident that if the ray PV were reflected at this arc, the final direction must be VY parallel to the line OS, since the incident and reflected rays then make equal angles with the normal to the surface. To show that the same spherical *locus* produces the same final direction, parallel to OS, for all rays, it must be shown that TV is independent of θ.

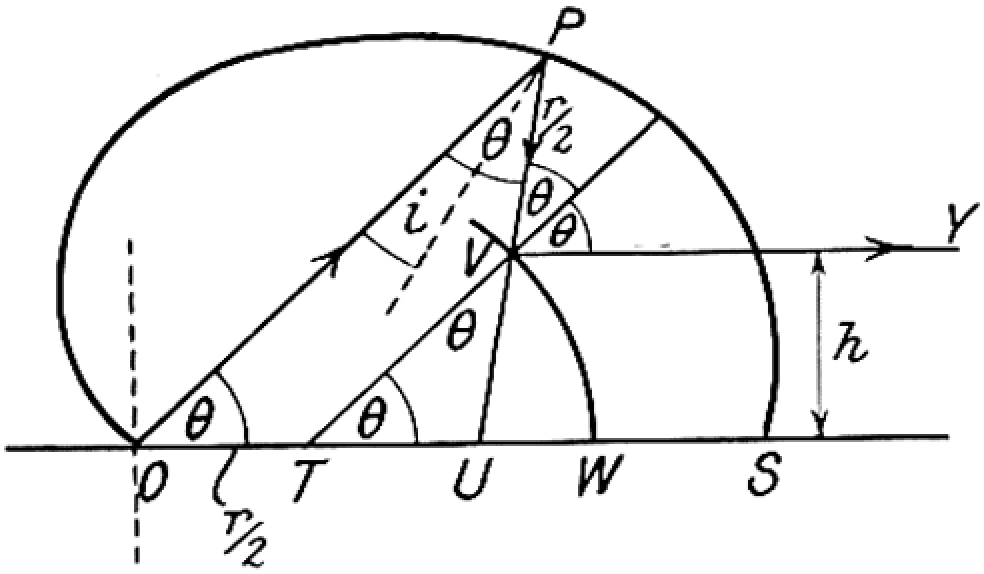

FIG. 101. THE CARDIOID

Now $TV = 2TU \cos \theta$, and $TU = \frac{OP}{2\cos\theta} - \frac{r}{2} = \frac{r(1 + \cos\theta)}{2\cos\theta} - \frac{r}{2}$

$$= \frac{r}{2\cos\theta}$$

Hence
$$TV = r.$$

If h = distance of ray VY from the axis OS, then evidently
$$\frac{h}{r} = \sin\theta$$

or
$$\frac{h}{\sin\theta} = \text{const.}$$

and the sine condition is fulfilled.

It, therefore, appears that if we send a parallel beam (from right to left in the diagram) so that reflection takes place at the sphere and then at the cardioid, we shall obtain an aplanatic refraction free from spherical aberration and coma, so that a distant source of light of reasonably small angular size will be sharply imaged in the plane, perpendicular to the axis, through O.

Fig. 102 shows the course of the rays through a cardioid condenser as made by Zeiss.

The rays meet the "spherical" surface first. In practice the second cardioid surface, being only a comparatively narrow one, can be represented sufficiently well in practice by a ring of the approximating spherical surface. The range of numerical aperture in the illuminating cone is about 1·2 to 1·33. If no direct light is to enter the objective the *NA* of the latter must be below about 1·05. Objectives of higher *NA* have to be stopped down.

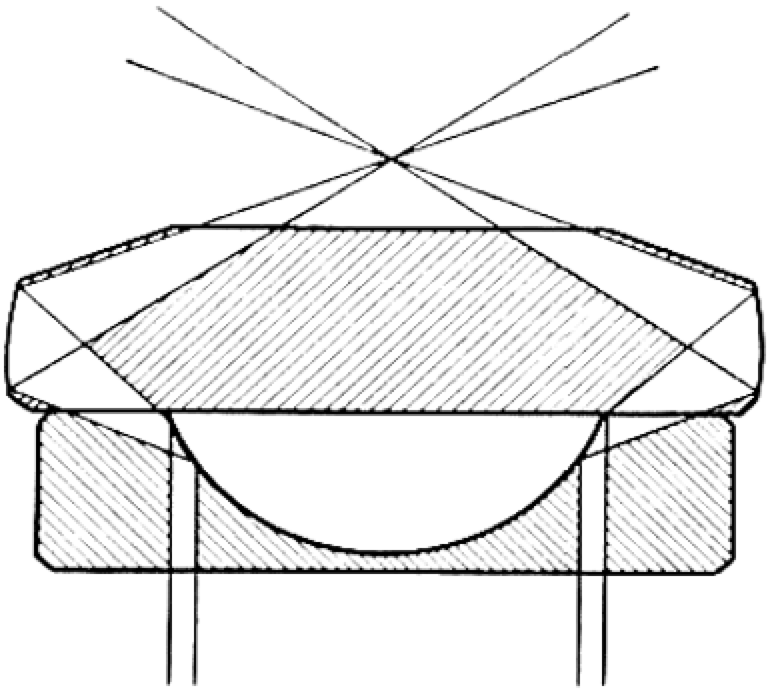

Fig. 102. Course of the Rays Through a Cardioid Condenser (Zeiss)

The *NA* of a ray perpendicular to the axis is equal to n sin 90°, i.e. to n, the refractive index of the medium. Hence if a ray in glass has an *NA* of, say, 1·4, it cannot enter a layer of water perpendicular to the axis, but will be totally reflected at the boundary since the refractive index of water is 1·333. Hence the *NA* of 1·333 is the highest which can be used to illuminate an object in a watery medium.

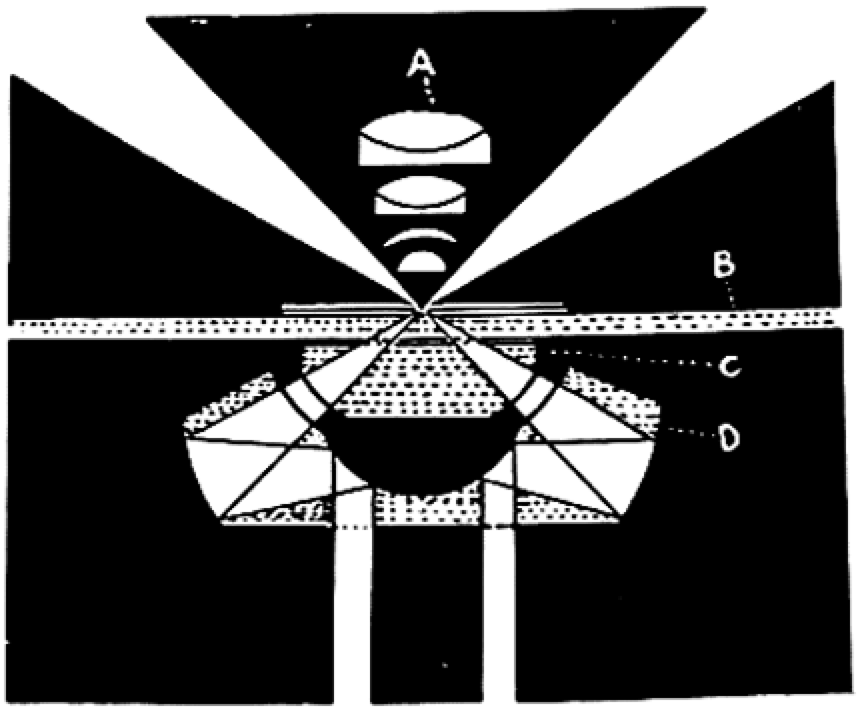

Fig. 103. Focusing Dark-ground Illuminator
(R. & J. Beck, Ltd.)

If the object is immersed in a medium of higher refractive index the *NA* of the illuminating beam can be increased, thus allowing objectives of higher *NA* to be used without stopping down.

When light is totally reflected at a glass-air surface, say, it is well known that a certain amount of light energy does pass into a very narrow layer (of thickness comparable with the wave-length of light) of the air at the surface. Advantage is taken of this in illuminating so-called "smear" preparations of bacilli with such dark-ground illuminators, the organisms adhering to the totally reflecting surface being thus brilliantly illuminated.

A number of other types of illuminator have been produced.

Fig. 103 shows the focusing dark-ground illuminator of Messrs. R. and J. Beck, Ltd., which allows control of the focusing point. With other types of illuminator it is necessary to choose an exact thickness for the microscope slide (usually about 1·2 mm.). The focusing type can be used with other thicknesses within reasonable limits.

The Slit Ultra-microscope.* This arrangement, introduced by Siedentopf and Szigsmondy,[16] has been employed chiefly in the study of colloid particles in various media. The illumination of the medium is concentrated into the region of a sharply limited image of a bright slit, and only such particles as lie within such a region are visible. Fig. 104 illustrates the optical system. The image of the crater of the arc is projected by the lens *f* on the micrometer slit *g*. The second lens *h* then projects an image of *g* into a point at the proper working distance from the objective *i*. Thus we obtain in the object medium a doubly reduced image of *g*. If the particles to be studied are disposed in a liquid, this is usually contained in a tube of square section *t*. The observing objective O is usually corrected for water immersion. The particles are seen when the region of illumination is brought by adjustment of *i* into the proper position with regard to O. A polarizing prism may be introduced into the illuminating beam for special purposes.

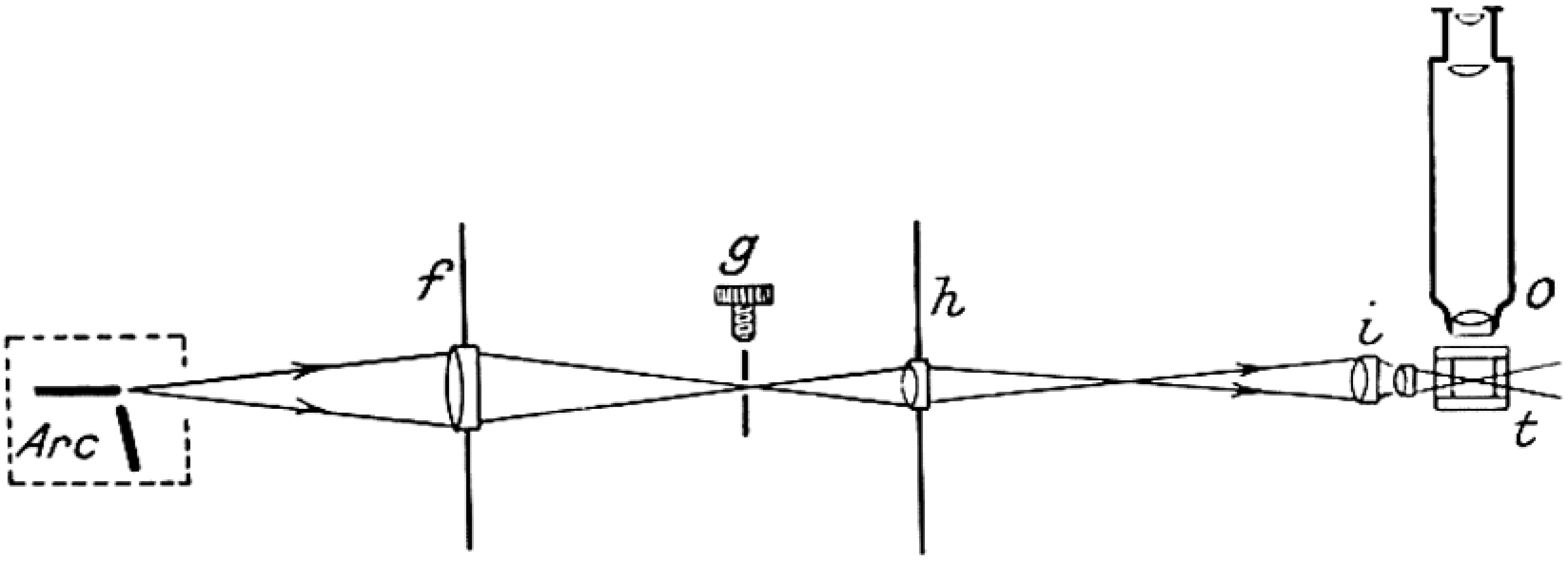

FIG. 104. THE ULTRA-MICROSCOPE
(Diagrammatic)

General Remarks on the " Cardioid " and " Slit " Ultra-microscopes. In these ways it is possible to make visible any particles, filaments, or differentiated structures capable of scattering sufficient light into the objective. Small objects thus seen appear bright on a dark background. If the dimensions of such objects are small in comparison with the resolving limit of the objective, the size of the patch of light representing the image will be determined by the considerations of diffraction.

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* Although the term *ultra-microscope* has become quite generally used it is not a very precise one, and is to be deprecated. It has sometimes been used in recent times to indicate the electron microscope in which the image is formed by passing a beam of electrons through an object, and focusing them by means of a magnetic or electric field. Much of the theory developed in regard to the optical microscope can be applied (with some reservations) to the electron optical case; but space does not permit of any discussion here.

The method is employed for the study of colloidal particles of diameters down to 4×10^{-6} mm. (Siedentopf), and light has been thrown on the phenomena of pedesis (Brownian movements) and the absorption of light by colloids in liquids and glasses, etc., more especially by making use of the one-sided illumination of Siedentopf and Szigsmondy.

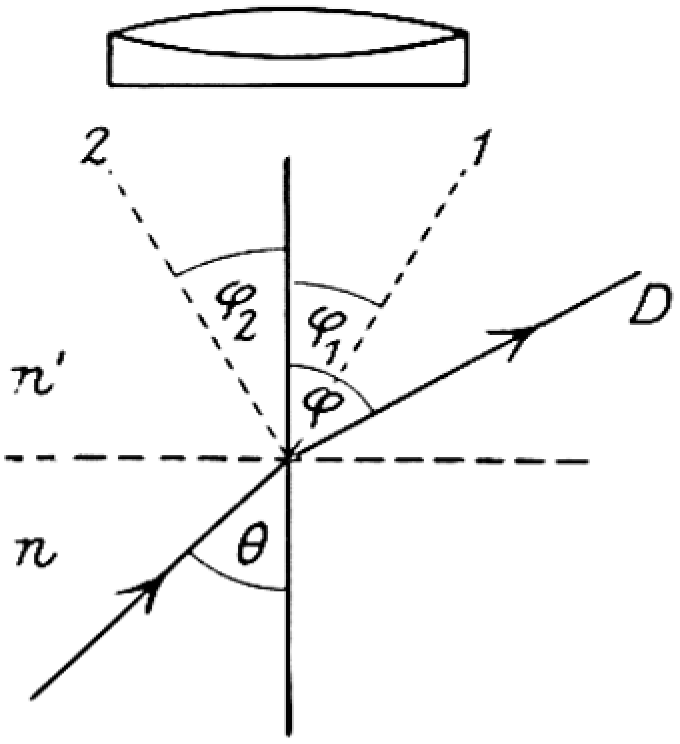

FIG. 105

Resolving Power for Oblique Dark-ground Illumination. The first investigation of this subject was due to M. J. Cross.[17] He applied the Abbe principle. Consider the illumination of a grating by an oblique beam of narrow angular aperture from the illuminator of a microscope. Let it be assumed then that in dark-ground illumination none of the "direct light" enters the objective, but that at least two diffraction maxima, say the first order and second order, must be formed at angles within the numerical aperture of the objective if the grating is to be "resolved." If the distribution of the diffracted beams is as represented in Fig. 105, the first order and second order being at angles ϕ_1 and ϕ_2, the equation for ϕ_2 is

$$n'x \sin \phi_2 + nx \sin \theta = 2\lambda$$

where x is the grating space, θ is the angle of the incident beam, and n and n' are the refractive indices on the two sides of the grating. In order to understand the optimum result we may imagine a grating of variable spacing. By broadening the spaces the second order beam will swing towards the direct beam D, and we could alter the angle between beams 1 and 2. By altering the angle of illumination we could then place 1 and 2 at equal angles with both sides of the normal, as shown in the figure, and thus it could be arranged that the first and second order maxima fall just within the objective aperture. This represents the optimum conditions for the objective with dark-ground illumination. Note that if we *increase* or *diminish* the illumination aperture, the maxima may not be focused by the objective.

If a_i is the numerical aperture of the illuminating beam, and a_o is the numerical aperture of the objective, the above condition is

$$x_d\,(a_o + a_i) = 2\lambda$$

If the first order maximum falls just within the objective aperture on the other side, the condition is

$$x_d\,(a_i - a_o) = \lambda$$

Subtracting these equations we obtain

$$2x_d a_o = \lambda$$

or

$$x_d = \frac{\lambda}{2a_o}$$

so that if the optimum conditions are obtainable, the resolving power may become equal to that for bright-ground illumination; but on adding the equations we find that

$$2x_d a_i = 3\lambda$$

so that in order to get these optimum conditions we must have

$$a_i = 3a_o \quad . \quad . \quad . \quad (3.12)$$

Since with the majority of cases the aperture of the illuminating beam will not greatly exceed that of the objective, and there will be few cases in which the absolute optimum is realized, the theory indicates that x_d must be about double the spacing resolvable with bright ground. This indicates that the resolving power *may* be halved.

Experiments by Siedentopf[18] have shown that this expected result is realized in the case of gratings, but Beck and various other observers working with objects such as diatoms have maintained that the resolution is not impaired. The reasons for the discrepancy in the observations is not yet adequately explained.

Owing to the drastic variation of diffraction angles with colour, it is usually the case that "dark-ground" images show brilliant chromatic effects when a lamp giving white light is employed as the source of illumination.

Interference Microscopes. Most of the light diffracted from coarse detail in a microscope object suffers little change of direction. Accordingly, when normal phase-contrast equipment is used such diffracted light passes through the phase plate and suffers no relative phase-change; thus the conditions for phase contrast vanish and no contrast is seen in the larger areas of uniform path except at the boundaries. The image resembles an outline drawing; only the finer details appear in true phase contrast.

It has frequently been suggested than an interferometer of the Mach[19]-Zehnder type might be converted into an interferometer microscope by the addition of the microscope lenses suggested in Fig. 106. In the plain interferometer, the mirrors A and D are "beam-splitters" using semi-reflecting surfaces, while B and C are fully reflecting; the interference effects can be regarded as equivalent to those seen in a thin nearly-parallel film represented by the mirror C and the image of B formed by reflection in D. Different sets of fringes correspond to the optical path differences due either to the various angular directions of the rays, or to the varying thickness of the equivalent film from whatever cause.

If the two branches of the microscope interferometer are equal, and the lenses also, other optical path differences can only arise in the object planes O_1 and O_2; the pattern seen in the eyepiece E will therefore represent the phase-contrast effects due to such path differences. It is not a difficult matter to secure such phase relations (see p. 130) that small relative retardations are well exhibited. The incident light in the interferometer need only be a ray-bundle of narrow angle, so that fringes of varying angular direction are not seen.

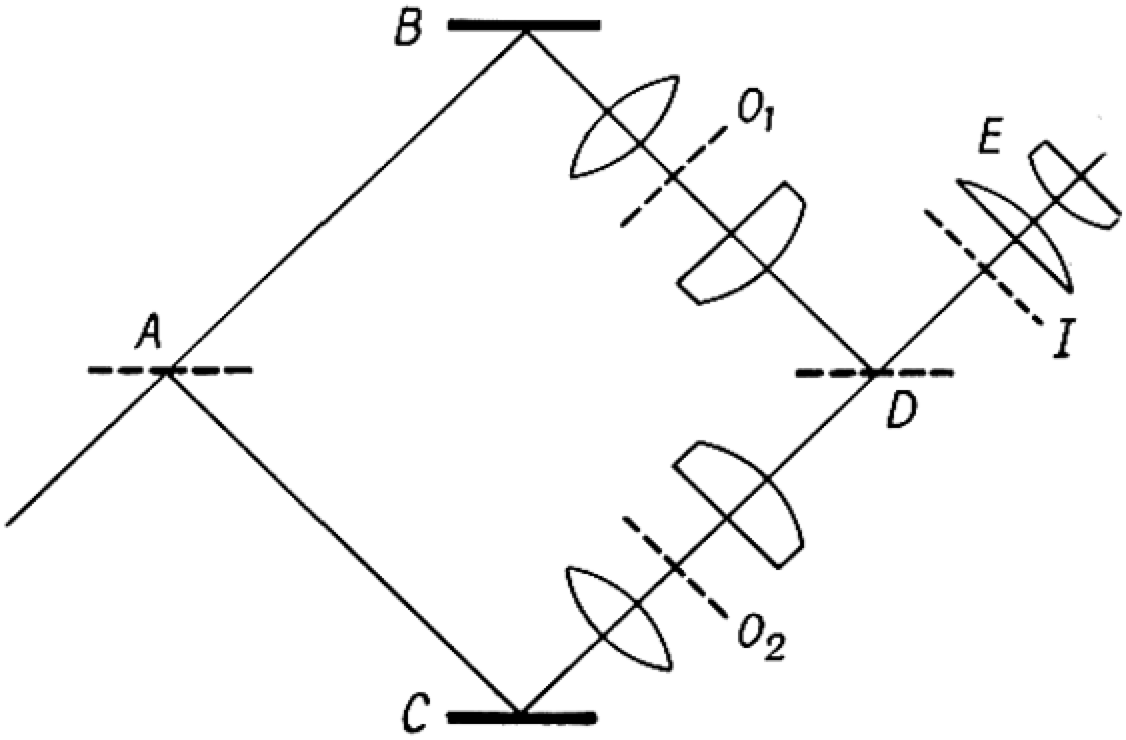

FIG. 106. INTERFEROMETER MICROSCOPE. THEORETICAL SYSTEM

The retardation of any element in the object-lamina depends on the angular direction of the transmitted ray; there is thus a mutual incompatability between the requirements of (*a*) a high numerical aperture of illumination for efficient microscopy, and (*b*) the maintenance of a precise value of relative retardation for such an element. A compromise is necessary.

The Linnik Interferometer. Unfortunately the difficulties of making and adjusting such apparatus are very severe, and the nearest approach to it is in the Linnik (Zeiss) (Fig. 107) and similar interferometers where the mirrors M_1 and M_2 (see Fig. 238) of the Michelson interferometer are each at the foci of matching microscope objectives. One "mirror" is the surface under test, the other is a comparison "flat." We may consider that the semi-reflector S projects an image of the system L_1, M_1 into the arm (2), where the virtual image M_1' forms an equivalent air plate with the mirror M_2. If the dry objectives work at numerical aperture A in air, $A = \sin \alpha'$, where α' is the angle between the axis (normal to M_2) and the direction of the most oblique ray; and if the thickness of the air plate is d, the difference of path in the two reflected beams for an angle of inclination α' is $2d \cos \alpha' = 2d\,(1 - A^2)^{\frac{1}{2}}$. However, the difference of path for the normal ray is $2d$, so that the discrepancy between the two values is

$$2d\,\{1 - (1 - A^2)^{\frac{1}{2}}\}$$

Now if the equivalent air plate is wedge-shaped, there will be one fringe at the apex, and the fringes will have fair contrast until the path discrepancy reaches an appreciable value, say, $\lambda/2$. Hence, if

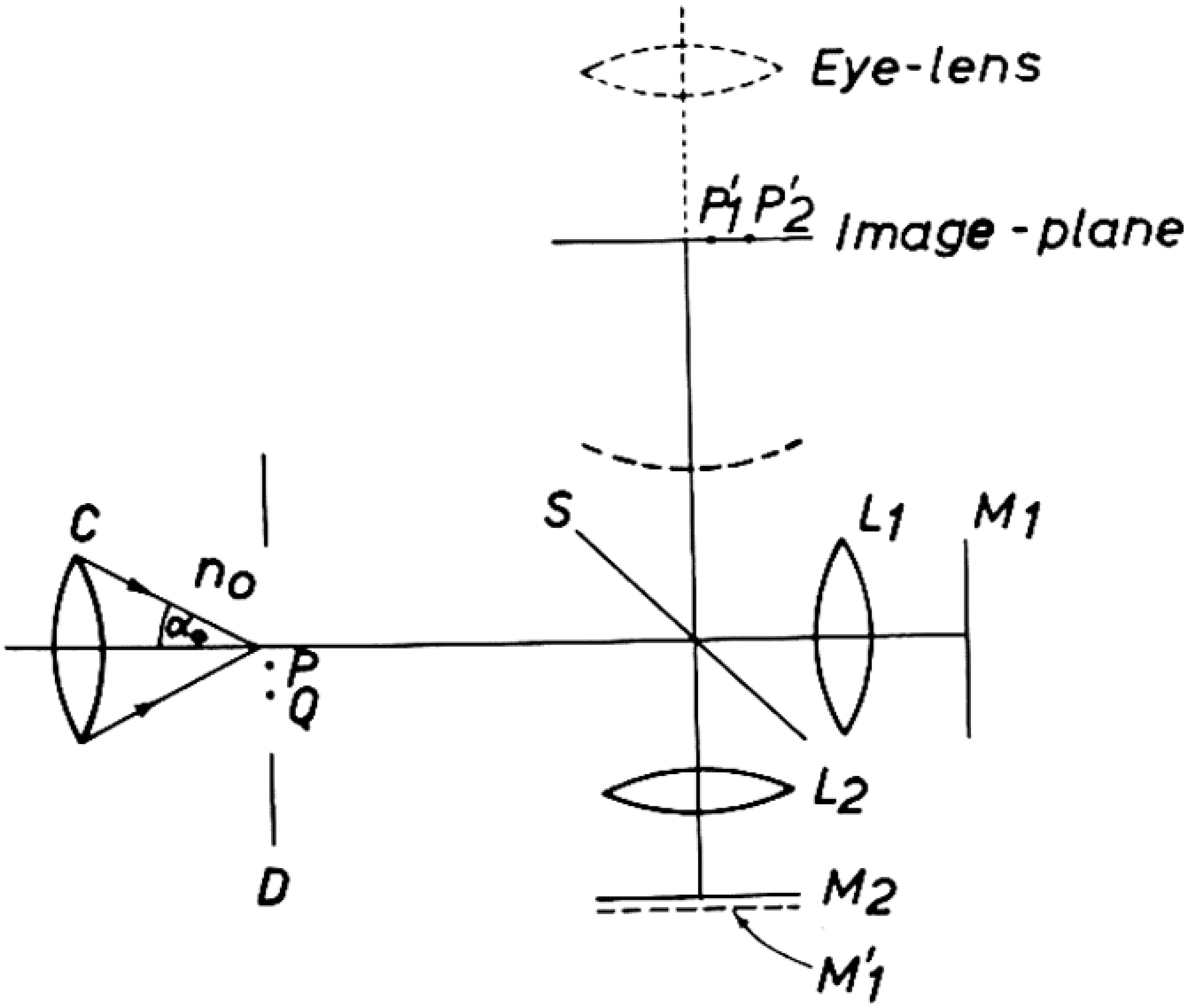

FIG. 107. LINNIK INTERFEROMETER (SCHEMATIC)

$$2d\,\{1 - (1 - A^2)^{\frac{1}{2}}\} \leqslant \lambda/2 \text{ we get } d \leqslant \frac{\lambda}{4\{1 - (1 - A^2)^{\frac{1}{2}}\}}$$

and the number of fringes beside the first is $2d/\lambda$. Hence the total number Z of fringes of good contrast is given by

$$Z \approx 1 + \frac{1}{2\{1 - (1 - A^2)^{\frac{1}{2}}\}} \quad . \quad . \quad (3.13)$$

The implications of this formula[20] as regards the limitations to be set upon the useful numerical aperture are worthy of study. The use of truly monochromatic light is assumed.

Another limitation on the contrast of the fringes is set by the necessity of adjusting the apparatus so that the maximum degree of coherence is obtained between the radiations interfering in the image plane. Suppose the plane of the diaphragm D (Fig. 107) in the instrument is illuminated by a source or condenser lens C the aperture of which may be supposed to give incoherent illumination. A point P in the diaphragm is imaged into the final image plane where its image should be observed by the eyepiece; owing to the possible lack of adjustment of the angles of M_1, M_2 (or etc.), the two images are separated by the interval $P_1'P_2'$. The actual disturbances interfering at any one point of the image plane are therefore derived from two separated points P, Q, say, in the plane of the diaphragm. If the separation PQ is ξ, the visibility of the interference fringes will be given by (see p. 128, and also a paper by Hopkins[21])

$$\frac{2J_1(p)}{p}$$

where $p = (2\pi/\lambda)\xi \sin \alpha_0$, assuming that the refractive index of the medium between condenser and diaphragm is air, and that the angular semi-aperture of the illuminating cone at P is α_0.

In the actual instrument, plane-parallel glass plates are mounted in both arms of the apparatus between the lenses and the semi-reflector; these can be tilted to ensure maximum coherence by making the pupil images coincident; the inclination and closeness of the fringes are adjusted by the same means.

Loss of fringe contrast can arise through lack of homogeneity of the light; this effect is minimized by securing exact equality of the optical paths.

Other Interferometer Microscopes. Such an arrangement as in the Linnik can only be applied to transparent objects in exceptional cases, and attention has therefore centred mainly on systems in which only a single microscope objective is necessary. A beam separation must take place, however, so that one part passes directly through the object while another (coherent) part takes an alternative route before the re-combination.

Lateral and/or angular separations can be obtained by the use of bi-refringent plates or prisms as applied to interferometry long ago by Jamin and adapted to the microscope by Lebedeff[22] in 1930; Rochon and other double-image prisms are used in systems proposed by F. H. Smith[23] in 1947, and Savart plates by Françon[24] in 1951.

Focal or axial separations can be obtained by bi-refringent lens systems also suggested by F. H. Smith. Beam splitting by the use of semi-reflecting surfaces has been applied in various ways by Merton,[25] Dyson,[26] Philpot,[27] and by Krug and Lau.[28] A great variety of possibilities is thus apparent, and the techniques are still in development.

Fig. 108 shows a system of the type used by F. H. Smith in the Baker interference microscope, in which L and M are doubly-refracting lenses whose separating action on a ray is similar to that of a Rochon prism. The lenses are shown diagrammatically as being cut from calc-spar and the axis direction is indicated by the shading. C and O are the condenser and objective respectively of the microscope (conventionally shown) and T is a Babinet compensator suitably orientated in practice; P and A are polarizing and analysing plates respectively.

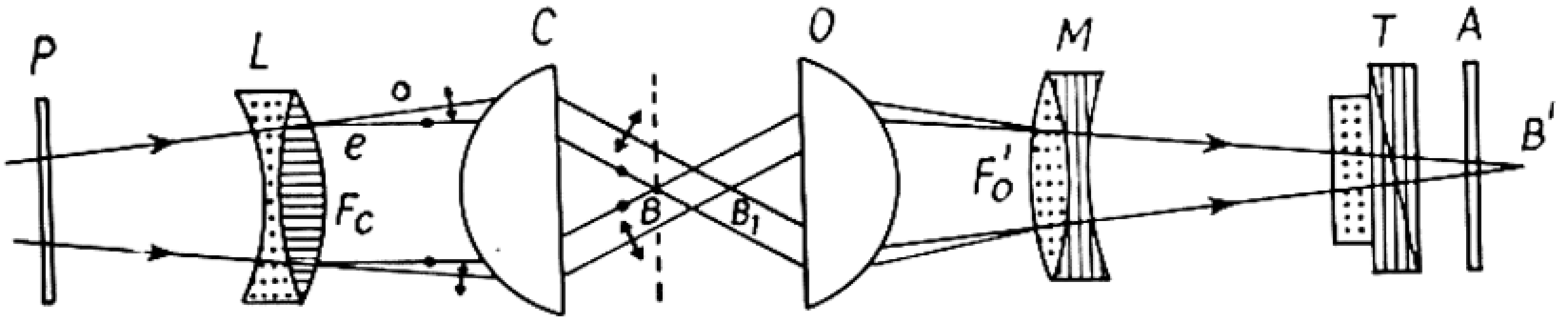

FIG. 108. BAKER INTERFERENCE MICROSCOPE
(F. H. Smith's axial separation type)

On entering L a ray is separated into ordinary and extraordinary rays, the latter ray only experiencing an increase of refractive index on entering the second (positive) member of L, so that the dioptric power of the lens is greater for the extraordinary ray. The condenser thus produces two images of the illuminating pupil, the one corresponding to the extraordinary ray in B and the other further along the axis at B_1. When these beams leave the objective O they would again focus at differing axial positions were it not for the compensatory effect of M, which can be given a small negative power greater for the more convergent beam; conjugate points of the illuminating pupil are hence brought to the same focus at B′, but only one beam contains the retardations due to a small object at B (say); the light in the other has passed *around* B. We thus compare the retardation at B with the mean retardation for the surrounding area. The analyser A separates out the parallel vibration components capable of interference. The compensator produces relative changes of optical path.

A lateral separation of the fields is produced by the alternative system (also due to F. H. Smith) shown in Fig. 109. A discussion of shearing interference effects will be found on p. 330 The beam-splitting device of Dyson is shown in Fig. 110, where A, B, and C are semi-reflecting surfaces, and A and C are the top surfaces of plates of equal thickness; the plate P separates B from the cover glass, the whole system being oiled together as in homogeneous immersion. A ray from the condenser is now separated into two components enclosing a parallelogram as shown. Only one path passes through the object, and interference can therefore take place on re-combination of the rays. The final image of the object formed by reflection in the plates appears to lie at M, where it would be inaccessible to a normal microscope objective.

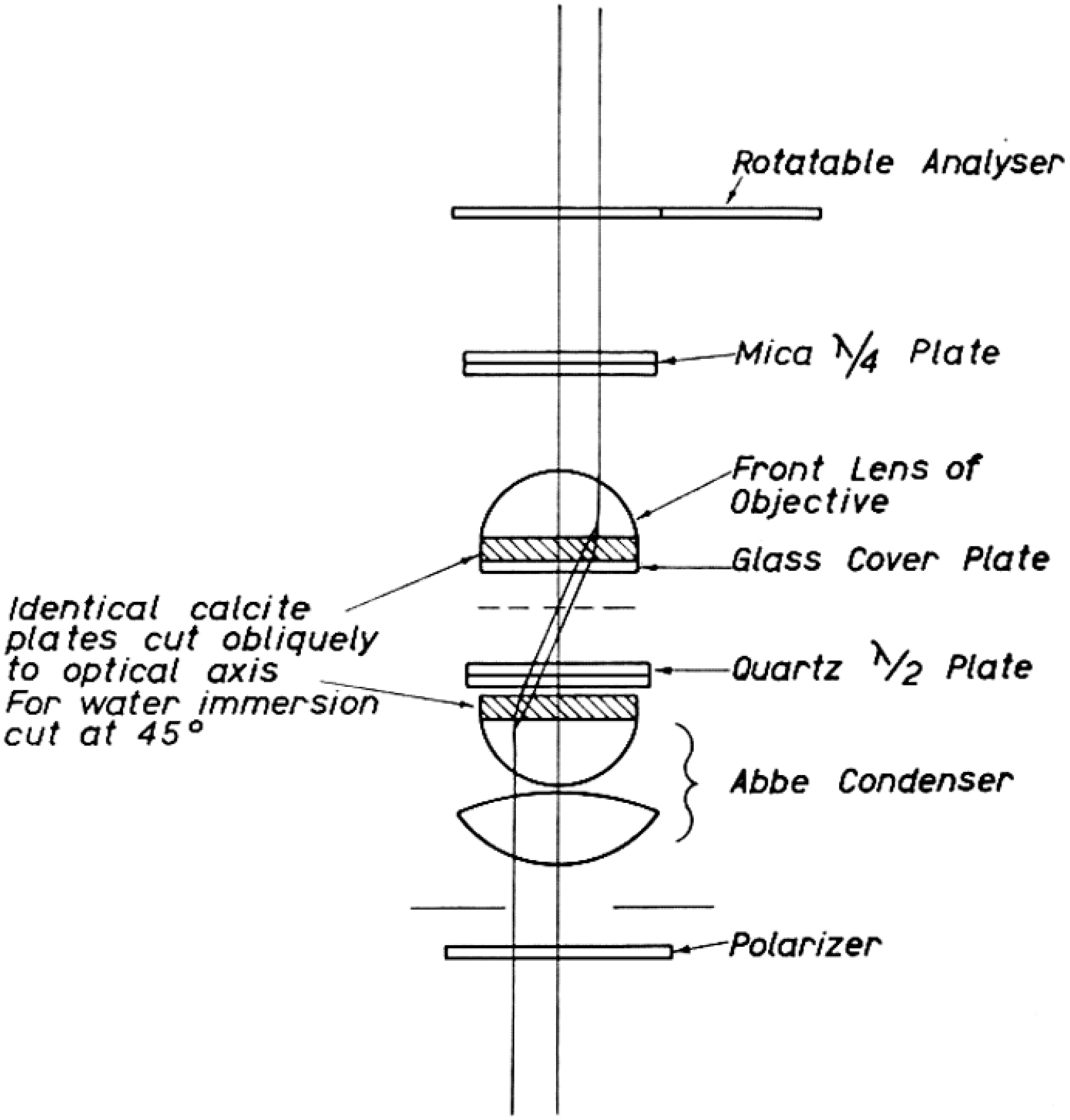

FIG. 109. SHEARING SYSTEM OF BAKER INTERFERENCE MICROSCOPE
(Lateral separation type, F. H. Smith)

However, the image at M is projected to M′ by the combined action of the upper concave surface S (fully silvered except for an aperture at the pole M′) and the semi-silvered surface C.

A projection system identical in principle, for obtaining a long working distance, was described by Dyson[29] before its application in the interferometer.

Multiple reflections can of course take place between the various reflecting surfaces, but such rays do not in general pass through M′

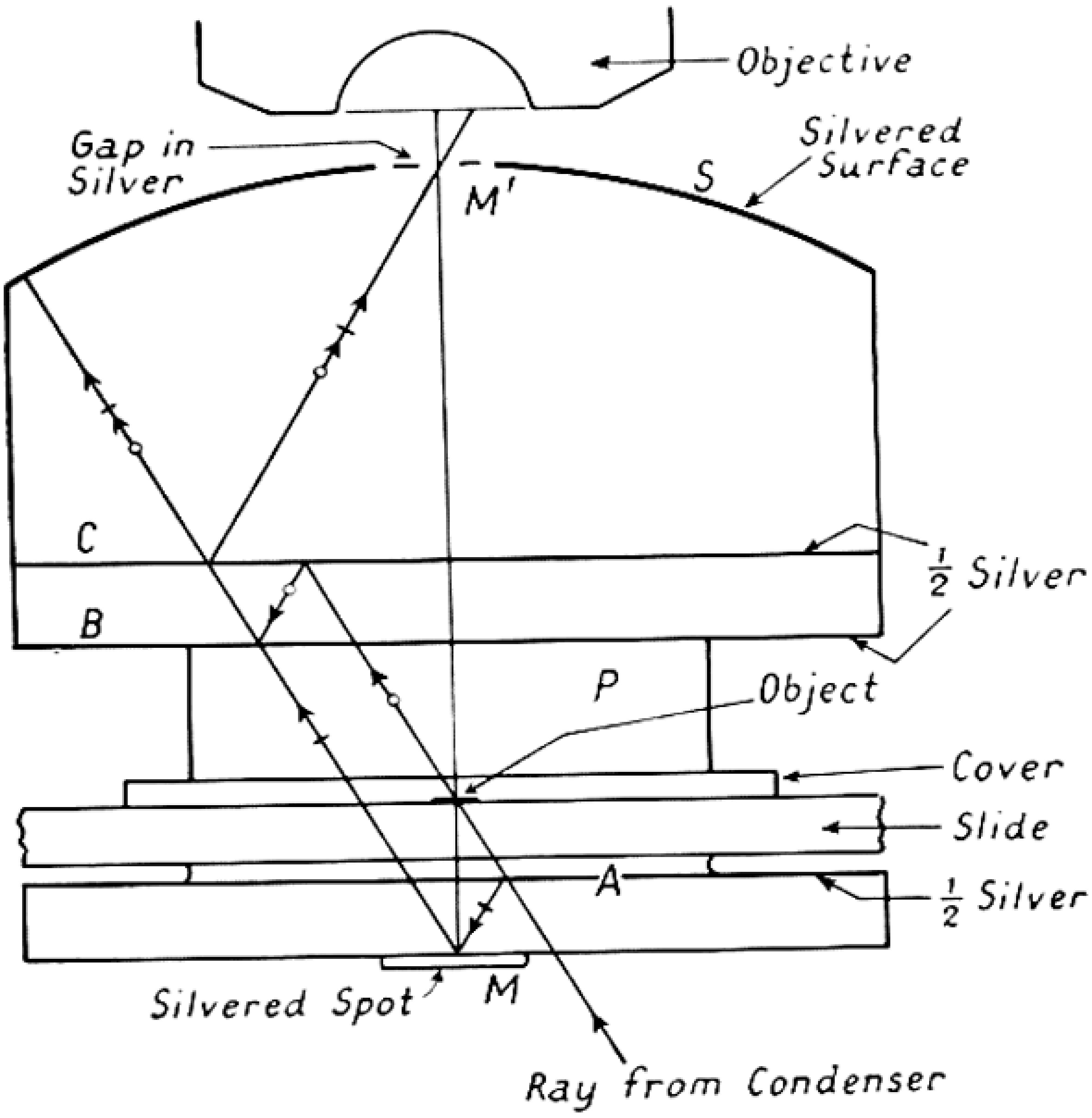

FIG. 110. COOKE INTERFERENCE MICROSCOPE
(Dyson's type)

since the corresponding images lie at other axial points; such light generally fails to enter the objective above M′.

If the equal plates are made very slightly wedge-shaped, variation of the relative paths of the beams is possible by lateral movement of one plate.

In conclusion it may be remarked that much work, both theoretical and practical, remains to be done before the vast possibilities of the various interference methods can be properly understood.

Microscopy with Ultra-violet Radiation. The expression for the esolving limit of a microscope objective

$$h = \frac{0{\cdot}5\lambda_o}{NA}$$

shows that when optical design has increased the angular aperture of the objective to the practicable limit, and has employed the practicable immersion media of the highest refractive indices, the only remaining way of increasing resolving power (diminishing h) is to employ radiation of shorter wave-length. The limit of transmission for stable glasses in the ultra-violet region is approximately $0{\cdot}3\mu$ (Vol. I, p. 245). Optical media available for regions of shorter wave-length are practically limited to quartz (crystalline and fused), lithium fluoride and certain other alkali halides. Crystalline quartz has only limited uses in lens systems owing to its double refraction; it can be employed in eyepiece lenses, or even in the back components of microscope objectives, but not in lenses where a great divergence of angular directions must be allowed for the rays. Present practice is mostly with so-called "monochromat" objectives constructed entirely of fused quartz, and using radiation corresponding to a single apparent "line" of the spark spectrum of cadmium, generally $\lambda = 0{\cdot}2749\mu$. The construction of such a lens is shown in Fig. 111 on a greatly enlarged scale. It can be freed from spherical aberration and coma in a very satisfactory way. Projection eyepieces of crystalline quartz are employed, and the image is registered by photography.

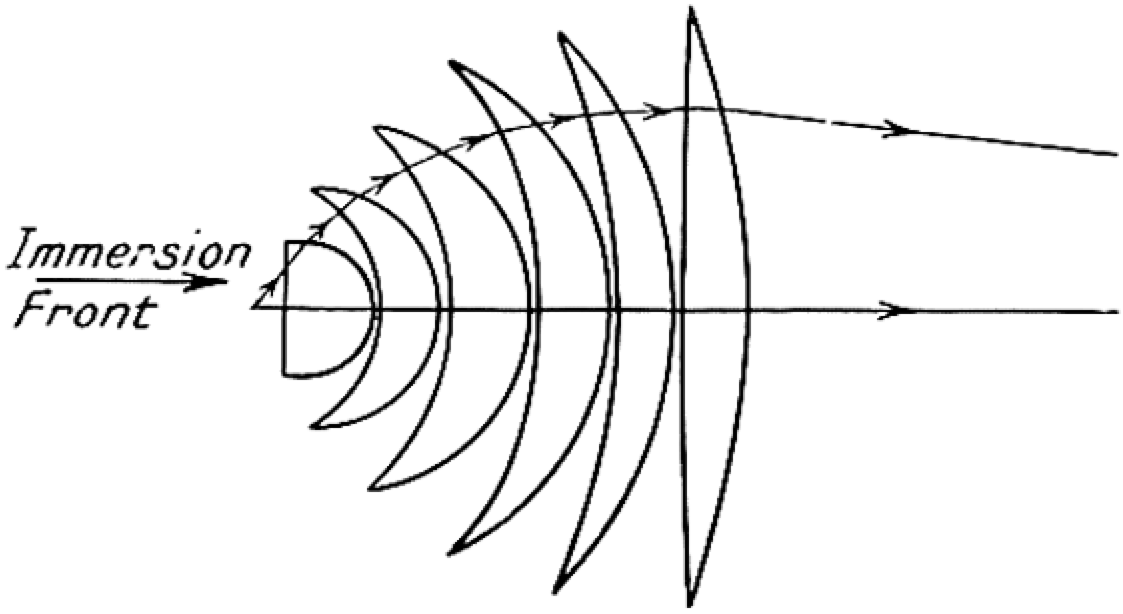

FIG. 111. MONOCHROMAT OBJECTIVE

Some success has been attained, by Trivelli and Foster, B. K. Johnson, and others, in the construction of achromatic systems giving substantially the same focus for wide ranges of wave-length. If suitable *regions* of wave-length can thus be selected at will the differential absorption of biological structures for the shorter wave-lengths of light offers a valuable method of exhibiting their structure (in microscopic sections, etc.), without staining.* The development of ultra-violet microscopy was begun by Köhler and von Rohr,[30] and carried on by J. E. Barnard[31] and others in this country.[32] Reflecting microscopes offer further attractive possibilities. See Chapter VIII, p. 340. The use of low *NA* ultra-violet objectives for photomicrography deserves more attention than it has hitherto received. The *same resolving power* can be obtained as with an ordinary objective of twice the *NA*. However, since the depth of field (see p. 98) is approximately inversely proportional to the square of the *NA*, a greater depth of field is obtained with an ultra-violet lens than with an ordinary lens of the same resolving power.

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* For a short treatment of the technique see *Practical Microscopy* (Martin and Johnson), Blackie.

Polarization Tests—the Optical Indicatrix. Microscopic tests, carried out with the aid of polarized light, are of great importance in the identification of minerals in small crystals, or in thin rock sections. Similar tests are now of increasing importance in biology and in colloid physics.

The section on crystal optics in Vol. I, Chapter VI, may be consulted for introductory study of the phenomena of polarized light. We may add here that in a doubly refracting medium there are in general three chief vibration axes; let them be $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

Light travels fastest when its vibrations are parallel to the direction $\boldsymbol{a}$, and most slowly when parallel to $\boldsymbol{c}$. The other direction $\boldsymbol{b}$ is a direction of intermediate speed.

The corresponding indices of refraction for rays having vibrations parallel to these directions $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, are α, β, and γ respectively, the velocities being proportional to $\frac{1}{\alpha}, \frac{1}{\beta}$, and $\frac{1}{\gamma}$. Thus it is possible to represent the refractive index for a vibration in any direction by a three-dimensional figure such as shown in Fig. 112. This is the so-called "optical indicatrix," an ellipsoid with semi-axes α, β, and γ. Any section of such an ellipsoid by a plane passing through the centre is, in general, an ellipse with different axes. From the ellipsoid we see, for example, that a disturbance propagated in the direction OB and vibrating parallel to OA has a velocity proportional to $\frac{1}{\alpha}$, while one vibrating parallel to OC has a velocity proportional to $\frac{1}{\gamma}$.

Consider a ray propagated parallel to the direction OC. The vibration, perpendicular to OC, can be resolved into components vibrating parallel to OA and OB respectively, for which the refractive indices are α and β.

Now since $\alpha < \beta < \gamma$, there will be two directions OD, OD′ between OA and OC and between OA′ and OC, for which the semi-diameter of the ellipsoid will be equal to β. The "ellipsoids" through

B and D and through B and D′ will, therefore, become circles, and the directions normal to these planes will be ray directions for which there is no difference of velocity for vibrations taking place in various azimuths. These directions of single ray velocity are the "optic axes" of the doubly refracting medium.

In any thin crystal section, or section of other doubly refracting medium, there will be fast and slow vibration directions corresponding to least and highest refractive indices. These may be represented by the "index ellipse" for the section.

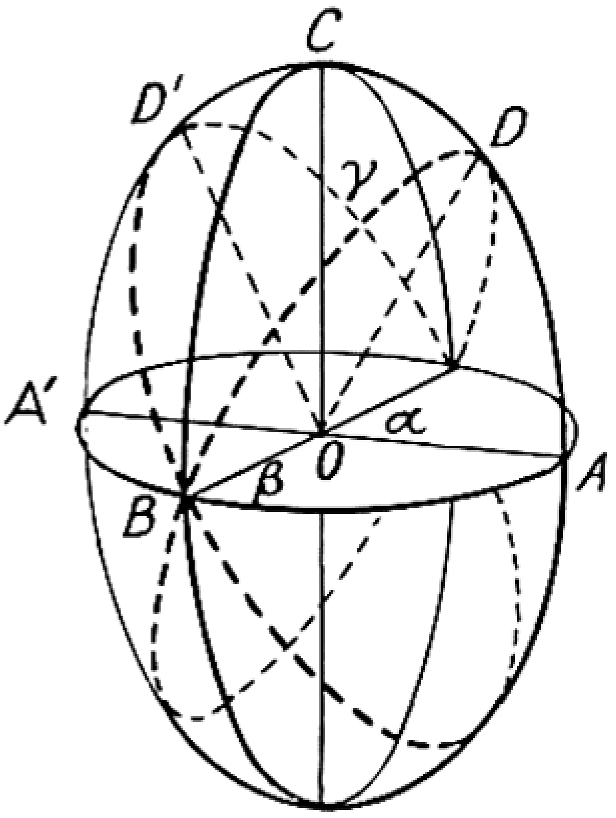

FIG. 112. THE OPTICAL INDICATRIX (FLETCHER)

The effects of double refraction associated with increasing paths through a crystal plate, the colour effects using white light, and the estimation of the double refraction with the aid of a wedge compensator have been described in Vol. I.

The Polarization Microscope. The polarization microscope is designed in the first place to permit of the application of such tests to very small and thin specimens. It is fitted (Fig. 113) with a sub-stage polarizer (Nicol, Glan-Thompson, or Polaroid, etc.), which can be easily swung in and out as desired. This is followed by a diaphragm and by a sub-stage condenser, usually of a two-component type, of which the upper highly converging system can be removed from the path of the light if it is required that the object shall be traversed by comparatively "parallel" pencils of rays. It is thus possible to illuminate the object with plane-polarized light. The objectives and eyepieces are of normal types, but it is considered a great advantage to have the widest possible field of view. An analyser may be inserted in the tube below the eyepiece. It is preferably of the square-ended variety, so that no displacement of the image is produced by its rotation. An auxiliary analyser with a divided circle to measure rotation ("cap analyser") may be fitted over the eyepiece with a swing-out movement.

Slots are usually provided above the objective and sometimes under the eyepiece, so that various compensators, such as a mica-quarter wave plate, quartz wedge, or Babinet compensator, can be inserted into the path of the light.

Petrological tests with the polarization microscope usually aim at the identification of various constituents of a specimen of rock, and for this purpose a thin section is made by grinding, and is cemented between a slip and cover-glass. It is desirable to know the thickness of the specimen, which is usually of the order of one-thousandth of an inch.

It is not within the scope of the present book to deal fully with all the numerous tests and criteria which can be applied for the identification of the various materials; we can only mention those which depend on the polarization of light. The doubly refracting materials are immediately recognized by the restoration of light between crossed Nicols. Suppose that an unknown doubly refracting substance is present, the crystallographical axes are likely to have all kinds of inclinations relatively to the section in different parts. Any element traversed by the light has its maximum and minimum refractive indices, represented by the major and minor axes of the ellipse representing the section of the indicatrix ellipsoid taken normally to the direction of the light. If the major or minor axis of the ellipse is parallel to the vibration direction of the polarizer, the light is not restored between the Nicols, but if the specimen is rotated through 45° the components of the vibration separated at the crystal surface are equal, and the colour is at maximum intensity. The highest order colour will be found in sections containing OA and OC where the *maximum* difference of refractive index is found; sections normal to OD or OD′ (the axes) would not show double refraction at all. As mentioned in Vol. I, p. 211, the Chart of Michel Lévy shows the colours characteristic of different thicknesses of crystal plates of varying bi-refringences, and in this way the maximum or sometimes the average bi-refringence of the mineral can be estimated which, together with other signs, is usually sufficient for identification. Alternatively, the bi-refringence may be compensated, and thus measured with the aid of the quartz wedge—or more readily with the Babinet compensator.

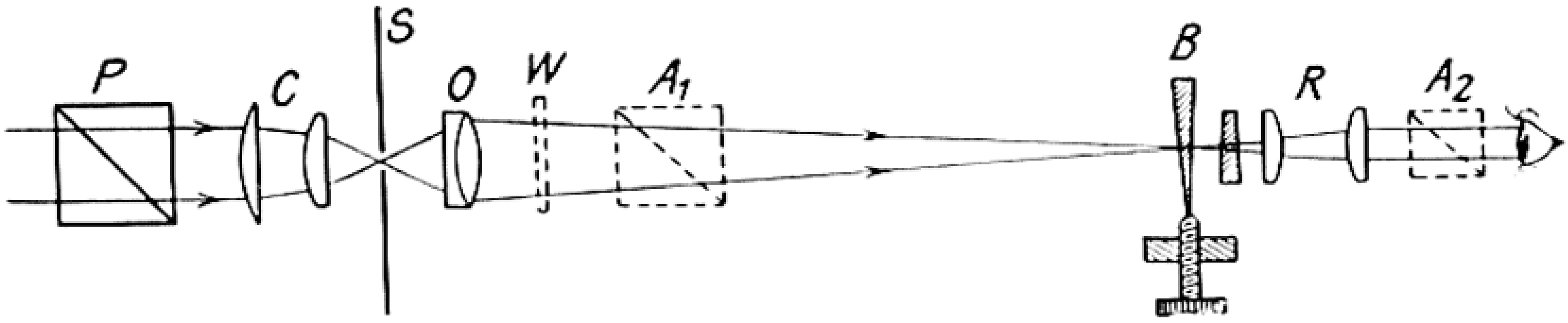

FIG. 113. OPTICAL SYSTEM OF POLARIZING MICROSCOPE FOR ORDINARY OBSERVATION

P = Polarizer
C = Condenser
O = Objective
W = Wedge or retardation plate
A_1 = Analyser used when compensator B is not in use
B_1 = Babinet or other compensator
R = Ramsden eyepiece
A_2 = Analyser used when compensator is in use

Extinction Directions. Another important test may concern the angle between a principal extinction direction for the section, and either the cleavage marks or crystal edges which are often recognizable. It is necessary in such a case to have a cross-wire in the field, marking the extinction direction of the analyser, and a means of measuring the rotation of the stage. In order to facilitate the extinction direction setting, various auxiliary devices can be employed.

Examination in Convergent Light. A system for examination of mineral sections in convergent light (the so-called konoscopic observation) is obtained in a simple way with the microscope, by using a condenser of power equivalent to that of the objective and removing the eyepiece, so as to study the appearances in the upper focal surface of the objective, where the so-called "stauroscopic" figures are found. A short discussion of the "ring and brush" appearances was given in Vol. I, pp. 217–219. The part of the slide illuminated by the condenser should be confined to the area of the specimen under test; this is easily arranged with the eyepiece in position.

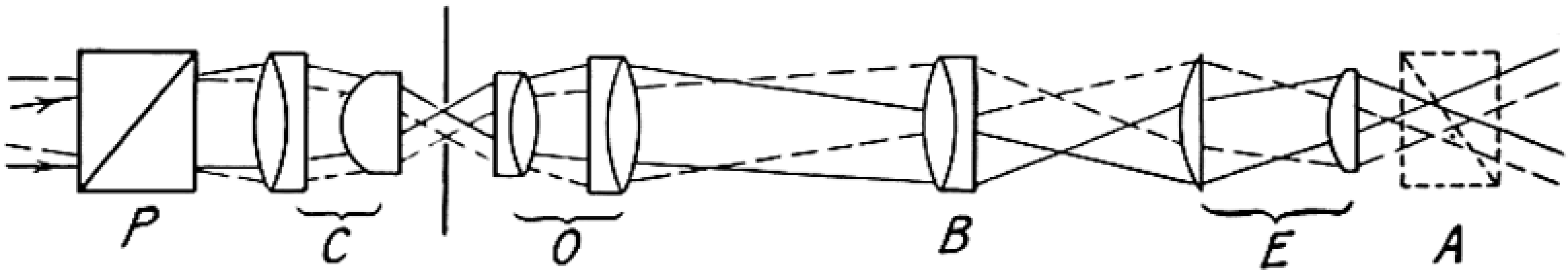

FIG. 114. SYSTEM FOR OBSERVATION OF CRYSTAL SECTIONS IN CONVERGENT LIGHT

P = Polarizer
C = Condenser
O = Objective
B = Bertrand lens
E = Eyepiece
A = Analyser

The use of a "Bertrand Lens" between the objective and eyepiece facilitates the observation of the upper focal surface of the objective, as shown in Fig. 114, from which it will be seen that the combination of Bertrand Lens and eyepiece forms a low power microscope projecting the image of this focal surface into the focal plane of the eyepiece. In the usual arrangement, this auxiliary lens is so fitted to the microscope that it can be pushed into position or withdrawn easily as required.

The brief mention given to petrological tests in this book by no means represents their great development. Owing to their commercial importance they have received a very great amount of study since Sorby, in 1858, took the initial steps in this branch of microscopy.

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CHAPTER IV

BINOCULAR VISION AND BINOCULAR INSTRUMENTS

STUDIES of binocular instruments[1] and binocular vision[2, 3] have formed the subjects of complete treatises, and it will not be possible in the limits of this chapter to do more than give the barest introduction, and to indicate the lines along which the study may be developed.

Physiology of Binocular Vision. The optic nerve in each eye leaves the eyeball in the region of the *optic disc*, and passes through the *optic foramen*, an opening in the end of the *orbit* (the conical cavity in the bone which contains the eyeball). The nerve then continues to the *chiasma*, or crossing point of the nerves from each eyeball. It is here that a separation or *decussation* takes place, the nerve fibres belonging to the nasal parts of each fundus passing to the opposite hemisphere of the brain, and those belonging to the temporal parts passing to the corresponding hemisphere of the brain; thus impressions on the right of each fundus are conveyed to the right hemisphere of the brain and *vice versa*. (See Fig. 115.)

In normal health of the visual system, both eyes are fixated on any object under examination, and a single visual *fused* impression results. The fixation is maintained by the motor muscles of the two eyes which are largely yoked in their action, so that the fusion of the images is maintained without conscious effort.

Corresponding areas of the two retinae are supposed to be connected by nerve channels to associated areas in one lobe of the cortex, and it was at one time thought that in this way precisely corresponding points in the retinae could be defined which when simultaneously stimulated would produce a single image point in the binocular field; however, this is found not strictly true, as will be explained below.

There is also an important inherent connection between the relative *convergence* of the visual axes of the two eyes and the *accommodation* of each eye, so that when a healthy pair of eyes views a near object, they converge to exactly the right amount to bring the fixated image to the fovea of each eye. This relation, however, is not of an inseparable character, and the convergence of the visual axes can be varied independently of the accommodation by special means, such as placing a weak prism in front of one eye, or altering the relative positions of pictures in a stereoscope. This, however, is apt to lead to strain and discomfort.

The normal condition of muscular balance in which the visual axes are parallel when the muscles of the eyes are at rest is known as *Orthophoria*; the condition of muscular imbalance, in which there is a lack of parallelism with the muscles at rest is known as *Heterophoria*. The lack of balance may be in the horizontal or vertical directions; there may be excessive or defective adduction or convergence, or it may be that there is an elevation of one visual axis relatively to that of the other. In such cases, fusion is only maintained by increased innervation of the weaker muscles, and when fusion is prevented by any means, as by the influence of drugs, *diplopia* or double vision will result.

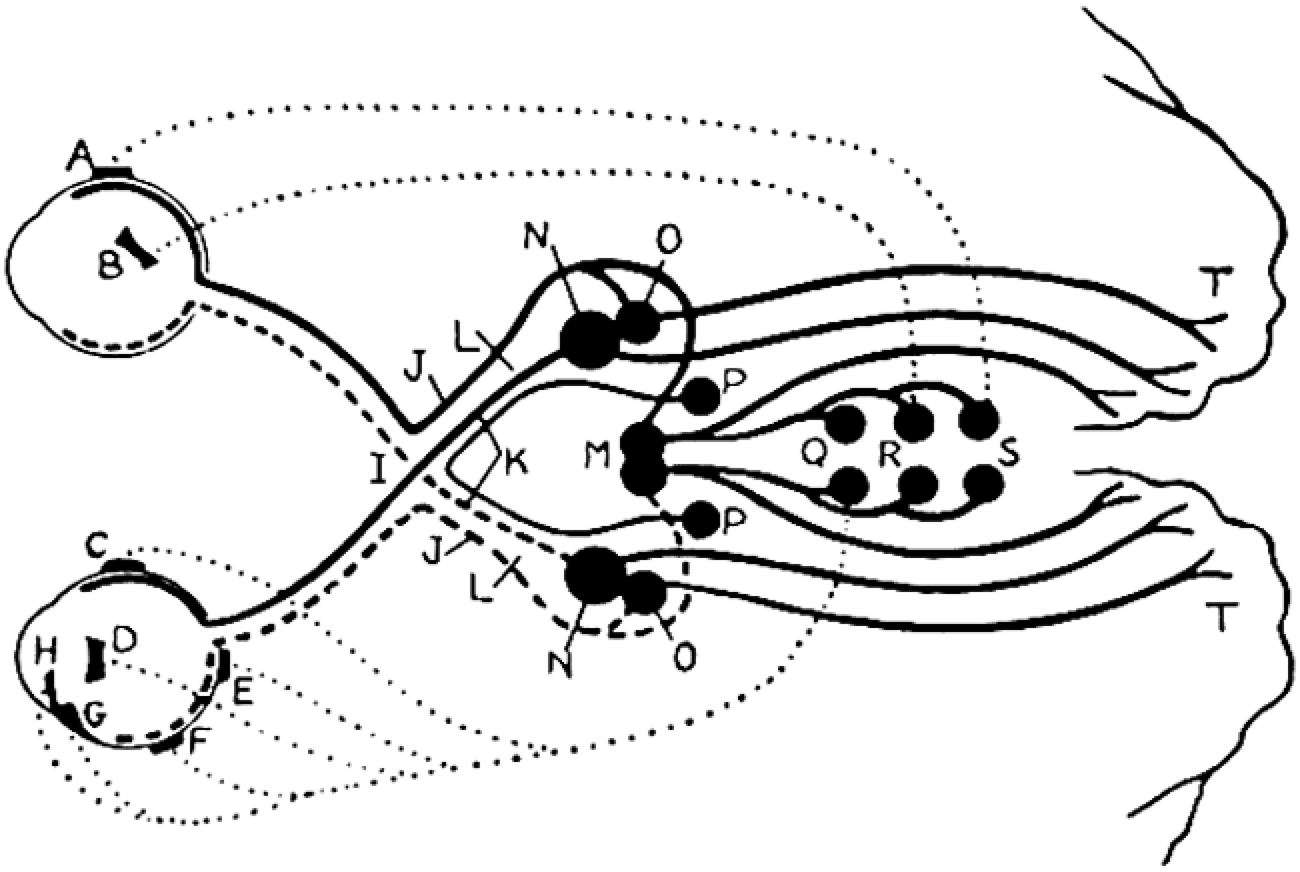

FIG. 115. DIAGRAM OF THE VISUAL TRACTS AND THE CONNECTIONS OF THE THIRD, FOURTH, AND SIXTH NERVES

A = External rectus (muscle)
B = Superior oblique (muscle)
C = Internal rectus (muscle)
D = Superior rectus (muscle)
E = Inferior rectus (muscle)
F = Inferior oblique (muscle)
G = Ciliary body
H = Iris
I = Chiasma
J = Uncrossed fibres of optic tracts
K = Crossed fibres
L = Optic tracts
M = Superior colliculus
N = Pulvinar
O = Lateral geniculate
P = Medial geniculate
Q = Nucleus of third nerve
R = Nucleus of fourth nerve
S = Nucleus of sixth nerve
T = Occipital lobes

In the case of *strabismus*, or squint, the muscular imbalance is so marked that fusion is impossible, and the visual axes of the two eyes cannot both be directed to one and the same object.

In testing for Heterophoria, use is made of the Maddox groove, consisting of a cylindrical groove ground in a piece of red glass and held before one eye; the glass is blackened with the exception of the groove. This eye then sees a lamp as a bright red streak perpendicular to the axis of the cylinder. The test chart consists essentially of a scale with a small aperture brightly illuminated in the middle, and the test is illustrated in Fig. 116; it is seen that the Maddox groove is held horizontally before the left eye, which then perceives nothing but a vertical streak which cannot possibly be fused with any part of the scale seen by the right eye. There is, therefore, nothing to bring the muscles out of the condition of rest, and the apparent position of the streak on the scale gives an indication of the *Esophoria* (eyes tending to turn inward) or *Exophoria* (eyes tending to turn outwards) which may be present. The vertical test may be made by turning both scale and groove into the vertical direction. In *Hyperphoria* one eye tends to turn upwards; in *Hypophoria* one eye turns downwards. Combinations of these conditions may be found. The amount is measured in prism diopters. (Vol. I, p. 306.)

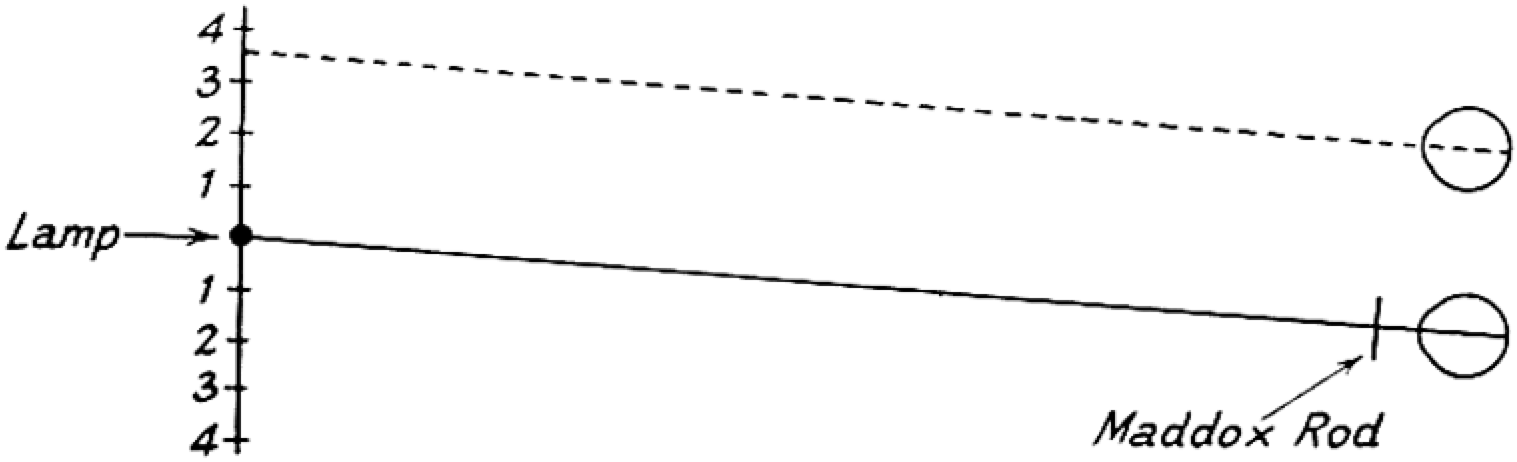

FIG. 116. TEST FOR HETEROPHORIA

Diplopia—Inhibition. It was mentioned above that when both eyes fixate the same object a single fused image is perceived. Conditions frequently occur in which different images are presented to the two eyes even in normal vision, while this is always the case in strabismus.

When both eyes are equally good and diplopia is produced, say by slight pressure with the finger on the eyeball at the rim of the orbit, both images are equally strong at first. By slightly increasing the pressure on the eyeball, however, the displaced image is weakened, and may then practically disappear from consciousness. Sometimes, by mental concentration on one of two almost equally strong images in diplopia, the other can be caused to be hardly noticeable. This effect can well be illustrated by putting two "infusible" patterns, say a circle and a cross respectively, into the two fields of a stereoscope.

Again, if different colours be presented to the two eyes, it is usually the case that there is a "struggle" between the two colours. Either one or the other is perceived almost at will. It is, however, sometimes found that if the colours are distributed in easily fusible patterns, then a binocular colour mixture in the additive sense may occur. Some observers, however, cannot see this effect.

Suppressions of this kind are related to well-known phenomena in the nervous system by which nerve pulses are inhibited, especially at the crossing points of nerve junctions; a strong pulse may, as it were, take up the whole road to the exclusion of any other.

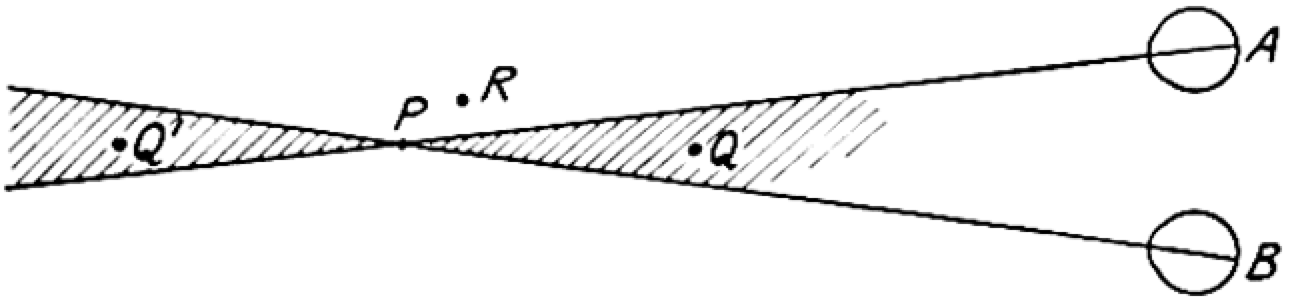

FIG. 117

It seems, in fact, as though one eye of the two is generally the one for which the pulse is the master. Let any one try this simple experiment. Make a ring with the thumb and forefinger; then with both eyes open raise the hand quickly so as to view, through the ring, some object on the other side of the room. It will usually occur that the ring comes into the line of sight of the "master eye."

Stereoscopy. Let the two eyes fixate a point represented in Fig. 117 by the point P. This can be represented in practice by a pencil. If another pencil is held at any point Q within the angle formed by the crossing of the visual axes, and either nearer or farther away than P, as at Q′, then Q or Q′ is usually seen doubled. But if the second pencil is transferred to R, outside this region of diplopia, it no longer appears doubled in consciousness, but shows the stereoscopic effect of a single object situated in space in a recognizable position in respect to P. This effect persists for a certain range of distances of R, both smaller and greater than that of P, but outside such a range the stereoscopic impression breaks down, and a doubled image with no stereoscopic effect is manifest. (See also p. 174.)

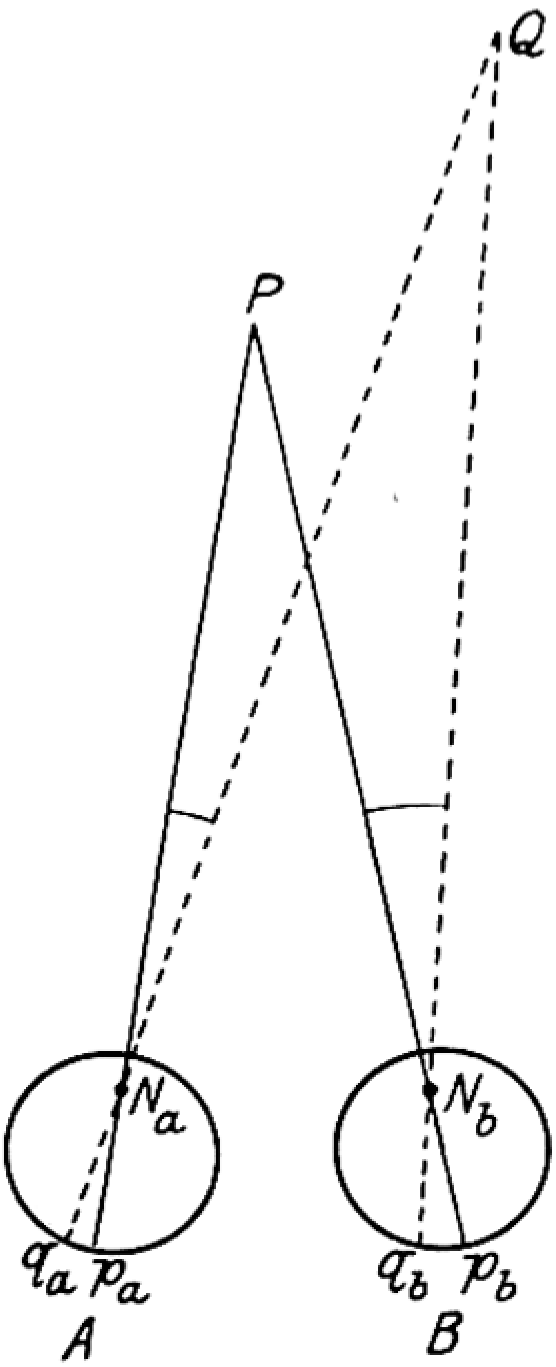

FIG. 118

The Presentation of Space to Consciousness. The visual image presented to consciousness when using either eye is that of a perspective presentation taken with the nodal point of the eye as centre. How this is effected by the retino-cerebral system is unknown. In common with a chick who can start to peck at food on emerging from the egg, we are endowed with this intuitive faculty of recognizing the geometrical relations of the positions of objects.

Let us now consider the binocular vision of two points P and Q at different distances from the observer, whose eyes are A and B (Fig. 118). Now the presentation of these points to eye A is effected through the image points $p_a q_a$ on the retina; similarly, the image points $p_b q_b$ determine the presentation to eye B. If both eyes fixate the point P, then p_a and p_b fall on the fovea in each eye respectively, while $q_a q_b$ fall at different distances in the two eyes. Let N_a, N_b be the nodal points of the two eyes. It is sufficient to consider one nodal point for each eye as in Vol. I, p. 153.

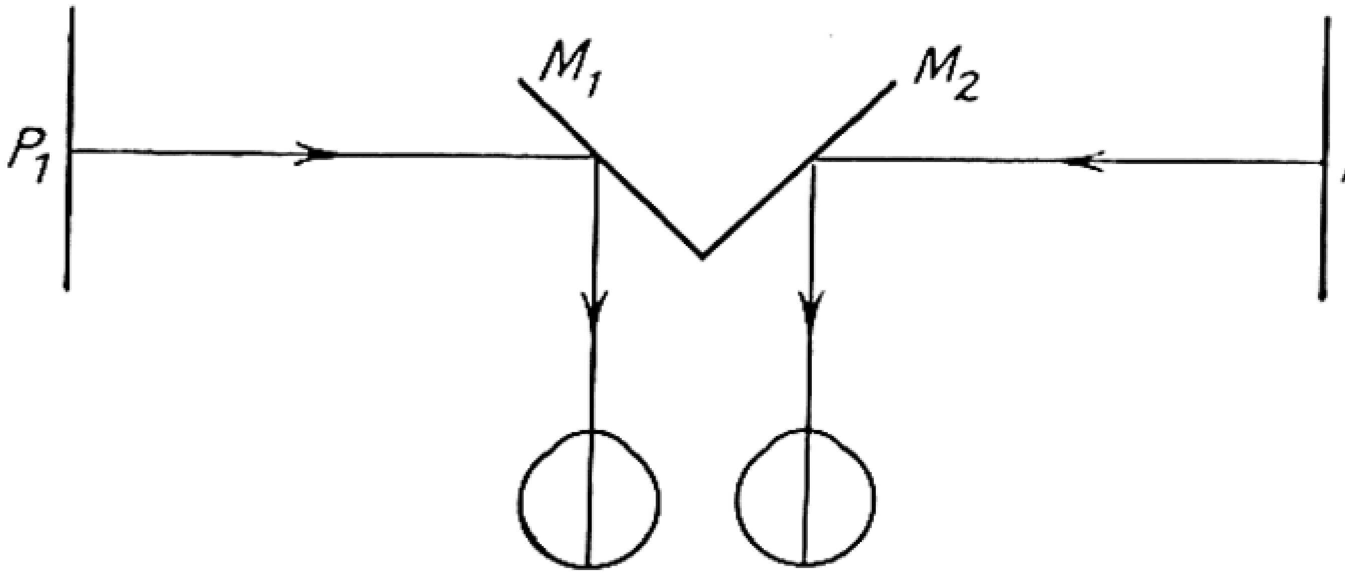

FIG. 119. WHEATSTONE STEREOSCOPE

Let f be the anterior focal length of each eye; then the difference of $p_b q_b$ and $p_a q_a$ is evidently (remember that the distance from the second nodal point to the retina is equal to the *anterior* focal length)

$$p_b q_b - p_a q_a = f\,(P\widehat{N_b}Q - P\widehat{N_a}Q)$$

provided the angles are not great. The difference of the two angles is the *binocular parallax*. If p_a and p_b are "corresponding points" of fixation, it used to be argued that the stimulation of q_a and q_b would produce a double visual image, or *diplopia*, unless they were also corresponding points. However, for any point q_a there is a range of possible positions for q_b within which a single image will result provided the "disparity" is not too great.

The Stereoscope. We ascribe a stereoscopic perception of relative distance to the difference in the perspective images presented to the two eyes. Acting on this reasoning, *Wheatstone*[4] argued that if a suitable perspective picture were presented to each eye simultaneously, then a sensation of a solid object should result; this is the principle of the *stereoscope*. Wheatstone's mirror stereoscope is shown in plan in Fig. 119; M_1 and M_2 are mirrors, reflecting the images of P_1 and P_2, the two perspective pictures.

The modern stereoscope (Fig. 120) is generally adapted for stereoscopic perspective pairs P_1 P_2 mounted at the interocular distance, and held in the focal plane of viewing lenses L_1 and L_2.

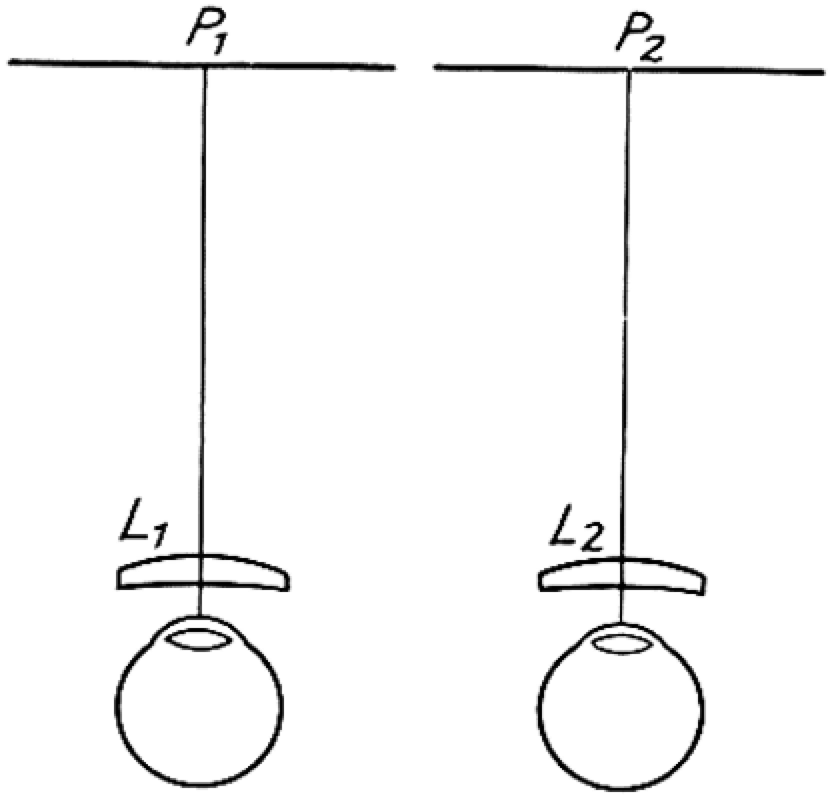

FIG. 120. THE STEREOSCOPE
(Diagrammatic)

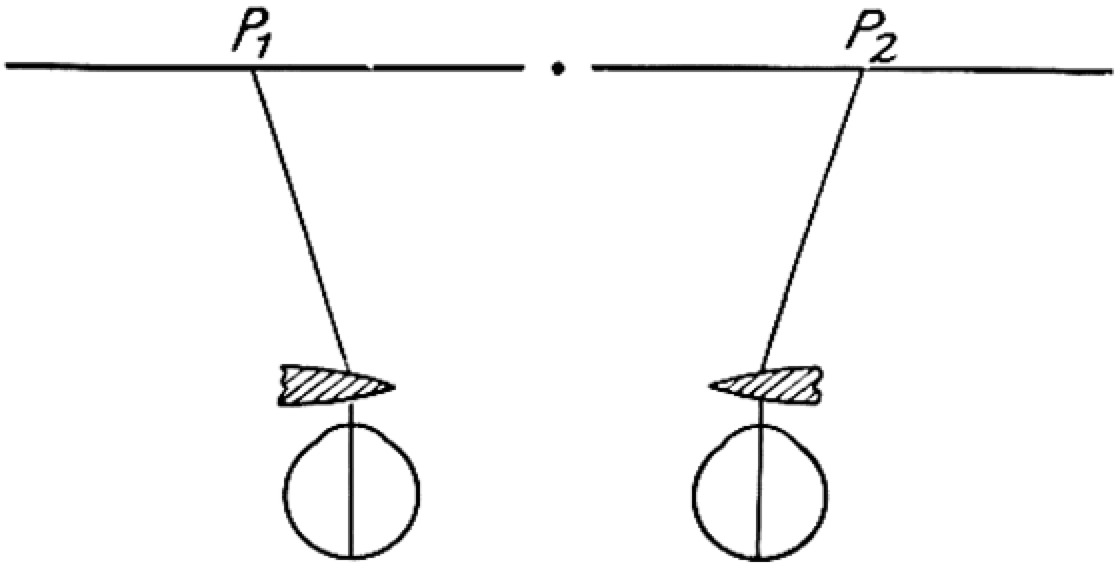

FIG. 121. BREWSTER'S STEREOSCOPE

If any pair of corresponding points in the pictures are "fused" when the visual axes are parallel, the accommodation which is at "infinity" corresponds perfectly to the convergence for these points, but there will be some deviation from the normal relation when points at different apparent distances are fixated. Wheatstone also used stereoscopes of this type. For the types of lenses used in modern stereoscopes see the Verant, also Albada's lenses, p. 11

Brewster's stereoscope (Fig. 121), examples of which are still in fairly wide use, was adapted for viewing larger pictures of which the centres were mounted at a greater distance than the ordinary interocular distance; this is effected by the lens-prisms. When such pictures are viewed by a modern stereoscope the inner parts of the viewing lenses are employed, so that the prismatic effect can again be obtained.

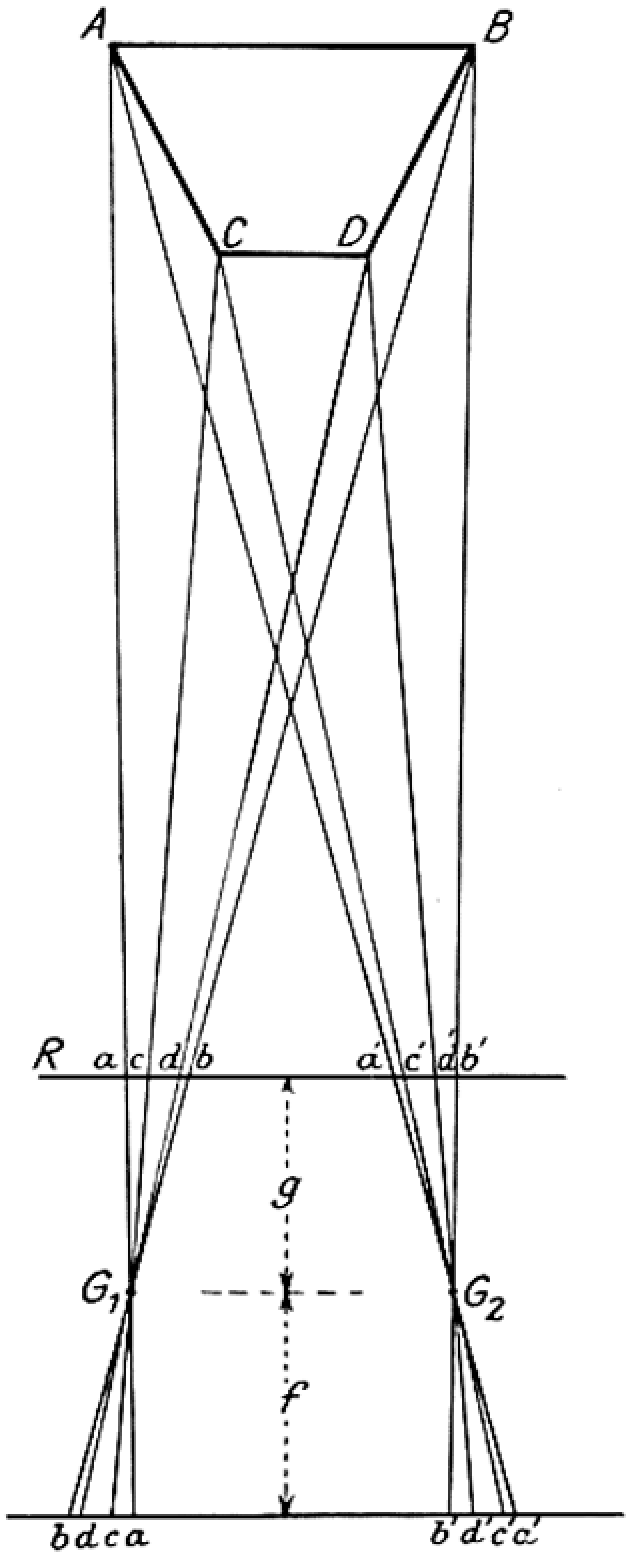

FIG. 122
PERSPECTIVE PROJECTION

Stereoscopic Perspective Pairs. Let G_1 and G_2 (Fig. 122) be two perspective centres situated at the ordinary interocular distance; think of them, for the moment, as the two eyes. We can imagine a projection plane R of glass on which, closing one eye at a time, we mark out the outline of a solid object ABCD; two perspectives are thus obtained of the kind suggested in Fig. 123. Another way of obtaining these perspectives would be to employ two cameras with lenses at G_1 and G_2 having the focusing distance of each lens from the back nodal point to the picture plane exactly equal to g, the distance from perspective centre to projection plane; the lenses are placed more particularly with their front nodal points at G_1 and G_2. The photographs thus obtained when mounted (duly reversed) in the plane of R will correspond geometrically to the drawings just imagined.

If two such projections are held in the original positions before the eyes, provided that g is great enough, most persons can then "fuse" the pictures without optical aid and obtain a visual sensation of a three-dimensional structure.

Difficulty is, however, experienced by many persons in securing this fusion of the two projections, because, when one of them is viewed, the relation between accommodation and convergence comes into play and causes the visual axes to converge in the projection plane rather than behind it. Success may sometimes be attained by looking downwards into a mirror at the image of a distant object; the visual axes then become parallel; if now the "stereoscopic picture" is quickly brought between the eyes and the mirror (not too close to the eyes, say, about 24 in.) it may be possible to secure the fusion. Another method is to look at the pictures through two pinholes, one before each eye; the accommodation is less definite and a fusion can be effected.

The function of the lenses in the stereoscope will now be more easily understood; the fusion of a pair of corresponding points can be effected without straining the relation between convergence and accommodation. We also secure the important advantage that by using short focus lenses we can deal with wide angle views; with the ordinary parallel-axis stereoscope the pictures have to be placed in the projection plane with their centres, say, 65 mm. apart, and therefore, their effective breadth cannot exceed this amount. It is clear that the correct angular projection of the perspective views cannot be obtained unless the focal lengths of the viewing lenses agree with those of the taking lenses. The importance of viewing a perspective presentation under the correct angle was explained in Vol. I, p. 4.

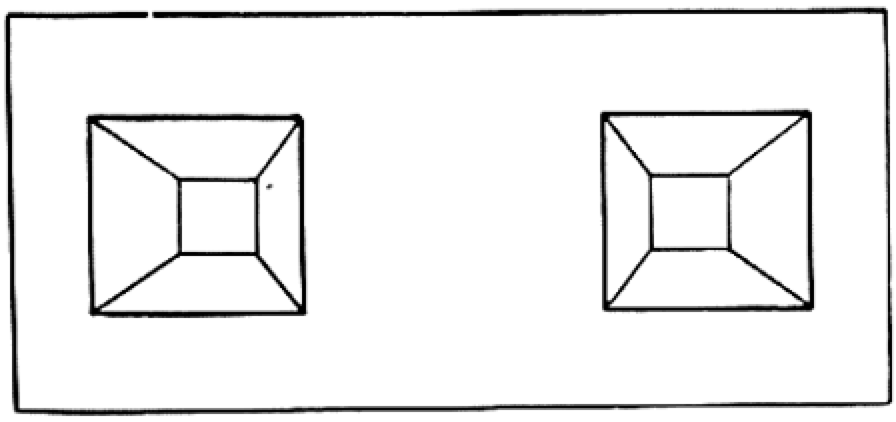

FIG. 123. PERSPECTIVE PAIRS

Stereoscopy and Convergence. The following experiment will illustrate some important conclusions. A stereoscopic reconstruction is obtained by fusing two pictures in a stereoscope; the pictures are two projections made in any suitable way, and are mounted separately so that they can be moved independently of one another. When fusion has been obtained they are moved slightly apart. It is found that in the duration of the movement the whole field appears to retreat; if they are made to approach each other the field appears to draw nearer; but such apparent movement from or towards the observer ceases immediately the pictures are still. and the whole field looks to be at the same absolute distance as before. The sensation of relative distance in stereoscopy is largely, but not completely, independent of the absolute state of convergence or divergence of the visual axes. See Ames, Ogle, and Gliddon.[5] Also there is considerable experimental evidence to suggest that the kinematic changes of convergence involved in ordinary binocular vision play a recognizable part in enhancing the acuity of space perception (W. D. Wright[6]).

Normal convergence should be obtained as nearly as possible in the ordinary use of a stereoscope. In ordinary vision, no appreciable change of accommodation is necessary in transferring attention from an object at infinity to one at a distance of not less than 3 metres, though there is a considerable change (p_o, see p. 164) of parallax. No changes of accommodation are involved when viewing a picture in a stereoscope. Hence if the relative parallax angles do not exceed p_o, there will be no abnormal strain on the accommodation-convergence relation.

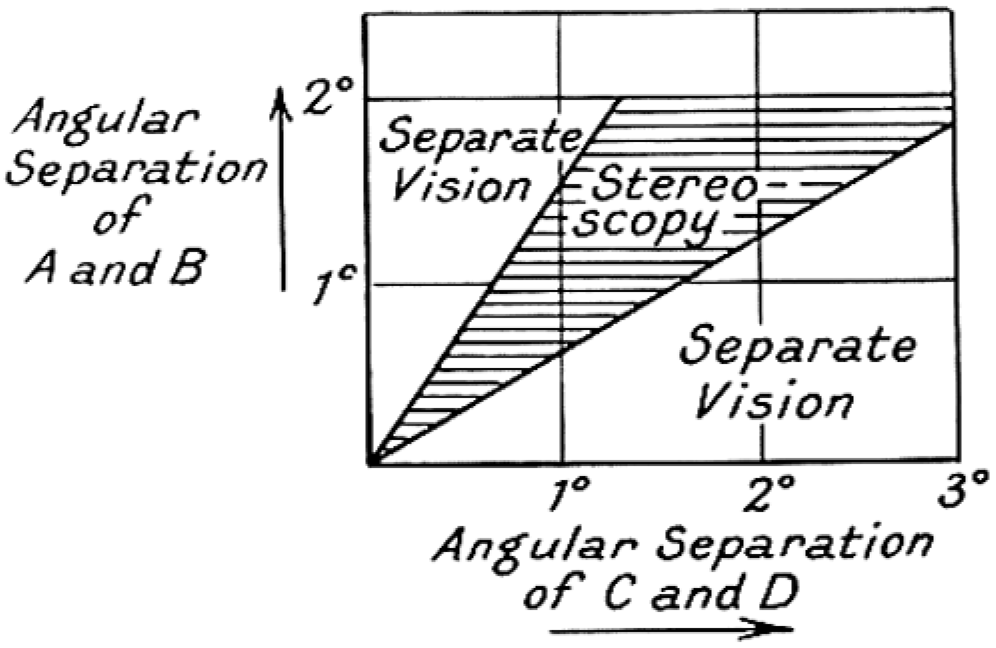

Fig. 124. Limits of Binocular Parallax for Stereoscopy

According to J. W. French,[7] if we are fusing the binocular images of two points subtending an angle of one degree to one eye, then the limiting binocular parallax is about half a degree, i.e. if two points A and B subtend an angle of 1° to the left eye, and two other points C and D in the same horizontal line as A and B are viewed by the right eye, and if then the images of A and C are fused, it will be impossible to fuse B and D into a stereoscopic image unless the angular separation of C and D to the right eye lies between about 0·5° and 1·5° (French gives 0·6° and 1·6° more exactly); for smaller or greater separations of C and D stereoscopic vision fails, and the images C and D are not fused. The limiting separations depend, according to the same writer, on the angular separation of A and B as shown in Fig. 124. In connection with these results the limited visual acuity even a short distance from the fovea has to be remembered.

Other experiments, and clinical experience of cases in which (by lesions in the muscular system) one eye has suffered a rotation around the visual axis, have indicated that (as mentioned above) there is a tolerance for retinal disparity in which a point in one retina can fuse with any point in a certain *area* of the other. Such areas are called *Panum's areas*. It might be thought that the characteristic sensation of "depth" would disappear if fusion is impossible. This is, however, not the case; both Helmholtz and Hering showed that stereopsis can occur in diplopia, and this has been further confirmed by Wright.

Production of Stereoscopic Effects by Projection and Otherwise. The essence of the stereoscopic sensation is the binocular parallax arising in the different perspectives presented to the two eyes; it cannot arise when both eyes view one and the same picture; but pictures in different colours may be projected on the same screen, as in red and blue-green. If then the eyes are furnished with colour filters transmitting only these regions of the spectrum, the two pictures (which may be two suitable perspectives) may be combined visually to produce stereoscopic results.

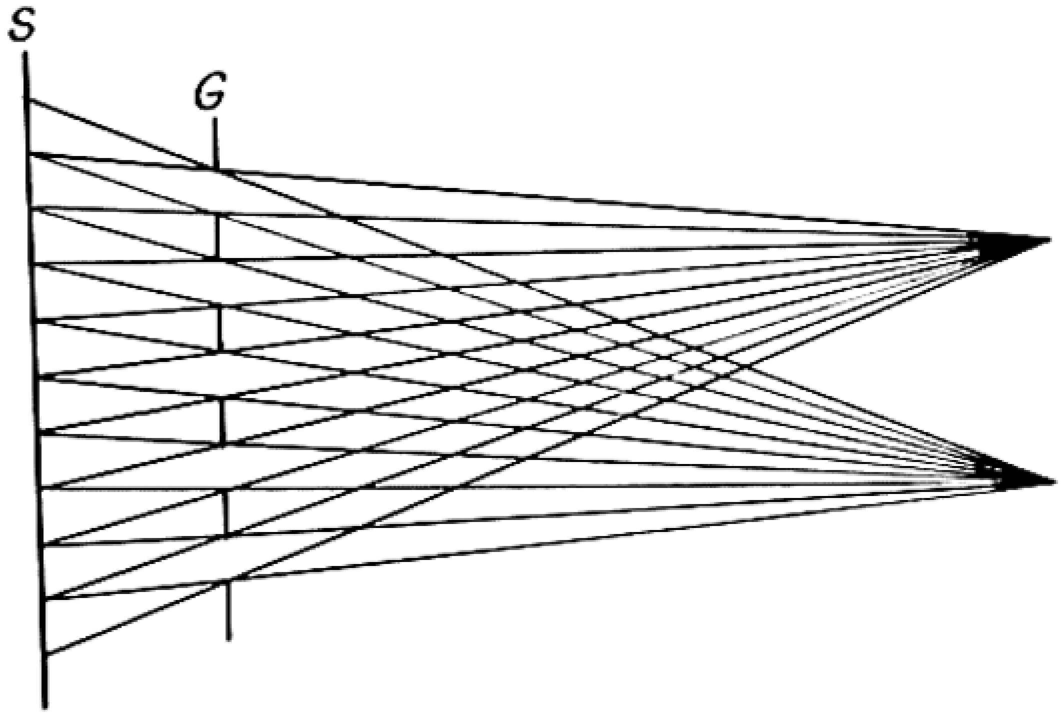

FIG. 125. GRID ARRANGEMENT FOR STEREOSCOPY

Alternatively, the two pictures may be printed in inks of red and blue-green to be viewed in a similar way (Anaglyphs).

If a suitable grid is held before a screen, the grid consisting of close equal bars and spaces of equal width, it is possible to arrange matters so that the two eyes view different strips of the same screen (Fig. 125). Thus two perspective projections can be made; one photographed in one set of strips; the other photographed in another set of strips; in this way a stereoscopic effect can be obtained. There seems, however, to be no escape from the necessity of a fairly close relation between the distances of the grid and the observer's head, so that the effect cannot be shown to many people at once.

Yet another possibility is to project the pictures alternately on the same screen, providing the observer with a rotating sector disc to obscure the eyes alternately, so that one eye views one set of pictures only. Perfectly satisfactory stereoscopic effects can be obtained in this way, but the necessity of the sector disc for each person has prevented the system from being used for entertainment purposes.

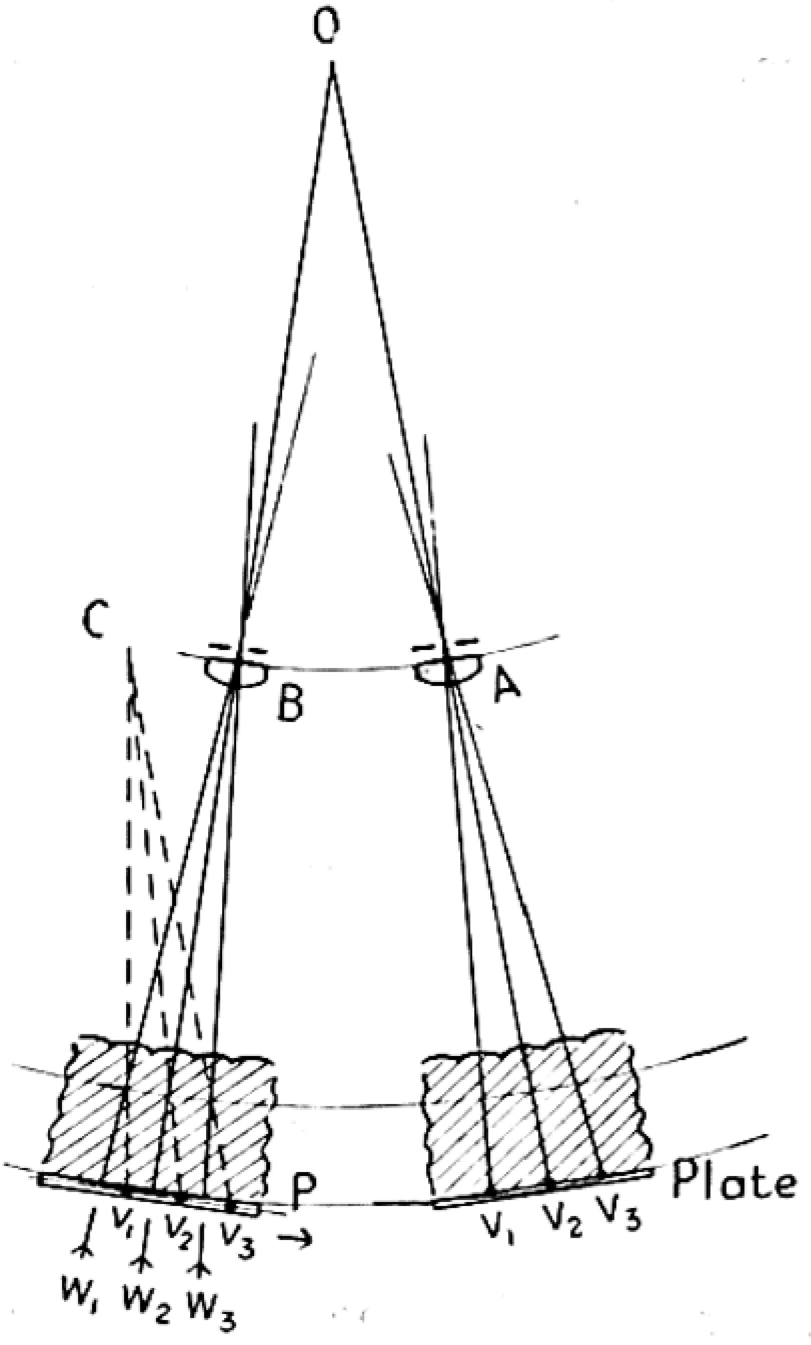

FIG. 126. PRINCIPLE OF LENTICULAR GRID METHOD

In another method the two perspectives are projected, on a metallized screen, in light polarized in perpendicular directions. The reflected light in the superimposed pictures is separated by polaroid screens held before the two eyes, so as to transmit only one picture to each eye.

Lenticular Grid Method for Stereoscopy. The use by M. Bonnet of a lens grid, similar to that employed in the Kodacolor system of colour photography, allows of the separation of perspectives for stereoscopy. The general principle of a similar method is illustrated in Fig. 126, where O is an object, and the camera lens swinging around O is shown in two successive positions, A and B; the axis of the camera points continually towards O. The scales of the diagram are, of course, distorted.

Now the plate P is covered by the cylindrical grid shown (greatly enlarged) in section. The curved surface of the grid element brings an astigmatic "image" of the pupil of the lens to focus on the plate; while a general image of the object is formed on the surface of the grid. The grid elements are so small in comparison with the focal length of the lens that the "line-process-screen action" now involved is not objectionable. At any instant, then, the parts of the plate being exposed will be in the linear images of the pupil, but the "exposure" will be proportional to the local image illumination. If, thus, a photograph is taken with the system in position A, the plate being developed and photographically reversed from the negative, and put back in its original position with regard to the grid; and if the camera lens is replaced by an eye looking toward the grid, this eye will see an image in which the angular perspective corresponds to a view of the object from lens position A, except that the image will be inverted. But an erecting prism can be used (for taking only) so that the view is normal.

Suppose now the lens and grid are moved before development together into position B, but the plate is moved at the same time a little to the right relatively to the grid. The pupil images W_1, W_2, and W_3 now fall on unexposed parts of the film. On development, reversal, and replacement as before, an eye replacing the lens at B can obtain the perspective view characteristic of position B. However, if the displacement of V_1, V_2, V_3 is suitable, an eye placed at C will be able to see the "A" perspective. If the positions B and C correspond to the left-hand and right-hand eyes in binocular vision a stereoscopic view can be obtained from the fusion of the two perspectives. The geometrical conditions to be satisfied will be easily derived. The relative movement of the plate can be avoided if the centre of rotation O is nearer than the object.

In practice the movement of the camera is given continuously, and thus no special positions for the viewpoints A and B are required provided that the distance from the grid is not excessively different from the normal; a single eye changing its position from A to B and beyond sees a continuously changing perspective. There are a good many possible variations of this type of procedure. No special stereoscope is required.

Vectographs. A similar effect but without perceptible grain is obtained from a technique (E. H. Land[8]) in which polaroid screens are incorporated *in* the picture. "A very thin polarizing surface, uniform in density and orientation, is prepared by orienting a concentrated suspension, in a plastic, of sub-microscopic dichroic crystals on the surface of a sheet of glass or plastic. To this polarizing surface is then transferred a resist of gelatine in which the desired photographic image exists as variations of thickness or as variation in hardness. The composite sheet is then subjected to a fluid reagent which acts through the resist to decrease the polarization of the polarizing layer, thereby forming in it an image in terms of degree of polarization." If this sheet is viewed through an analysing polaroid spectacle glass the contrast is a maximum when the polarized and analyser directions are perpendicular.

It is now possible to superpose on the first picture a second in which the perspective corresponds to the view point of the other eye; the second picture has its polarization direction in a direction perpendicular to the first. Hence, if polaroid spectacles are worn with mutually perpendicular polarizing directions for the two eyes the two pictures are separately seen; one by the left, the other by the right eye.

It does not matter that each of the polarizing sheets in the picture brings out the contrast of the other picture to some degree.

These pictures are called vectographs.

Interocular Distance. Binocular instruments must necessarily take account of the variations of interocular distance between different individuals, which ranges in adult men from about 56 mm. up to 72 mm. with a mean of 63 mm.; the mean for women is somewhat smaller, being about 61 mm. Particulars of measurements may be found in a paper by J. W. French.[9] Prismatic binoculars are usually made adjustable between the limits of 57–70 mm., but owing to the much larger eye lenses and exit pupils of Galilean binoculars there is not the same need for the provision of the adjustment, although the necessity of making the glasses usable by a person of small interocular distance limits the diameter of the objective and, therefore, the field of view.

Effect of Duration of Illumination. Dove[10] performed an experiment many years ago (1841) which demonstrated the occurrence of the perception of stereoscopic relief under the very brief illumination given by an electric spark, thus demonstrating that ocular movements or changes of convergence of the visual axes are not an essential element of stereoscopic sensation, for although the sensation lasts much longer than the spark itself, any movements of the eyes would not move the location of the retinal image in either eye. *Langlands*[11] has shown that an improvement in stereoscopic acuity occurs when the duration of the illumination rises above 0·1 sec.; this suggests that ocular movements, when possible, increase the stereoscopic sense. It has, however, to be remembered that ordinary visual acuity as well as intensity discrimination both improve with the duration of the stimulus up to some critical value when "flash" observations are made (Graham and Kemp[12]); but the total energy received by the functioning retinal elements is a determining factor, and it is difficult to separate this from the effect of duration *per se*. Further careful experimental work is required.

Photogrammetry. Perspective pairs of photographs of the land surface can readily be obtained by aerial photography, and these can be viewed in stereoscopic plotting machines in which the observer sees the image in spatial relief and can cause a "wandering mark" to move along a contour in apparent contact with the surface of the ground; in so doing a pen automatically traces the contour in a map of suitable scale. There is now a vast amount of specialist literature in this field on account of its commercial and military importance; however, a review of some of the visual problems involved will be found in an article by W. D. Wright,[13] and some apparent anomalies in the setting of the wandering mark have been investigated by D. A. Palmer ("Binocular Acuity in Optical Instruments," University of London Ph.D. Thesis, 1959).

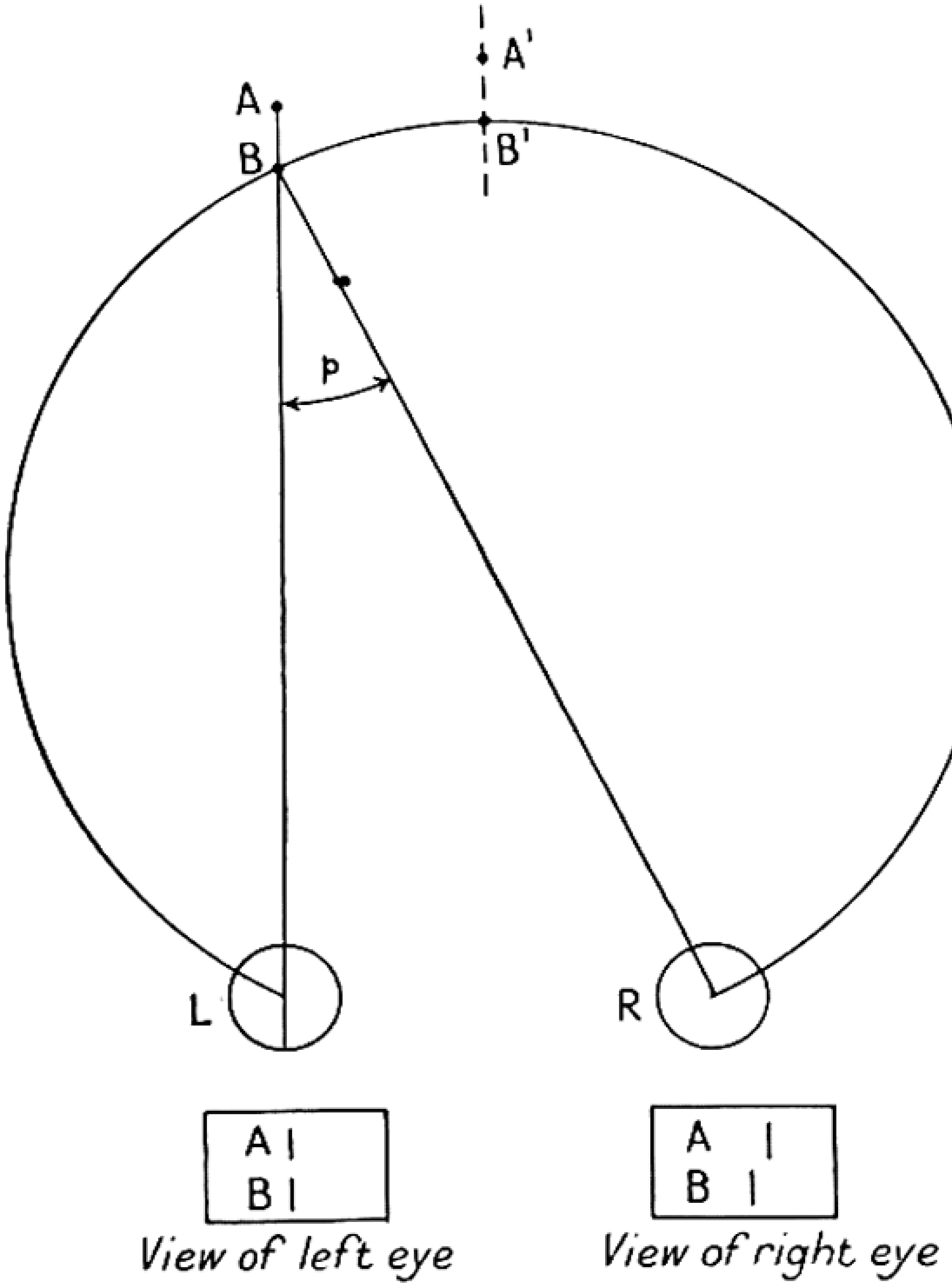

FIG. 127

Modern Views of Stereoscopic Vision. As mentioned above it seems inescapable that the origin of stereoscopic perception lies in the incongruity of the perspectives presented to the two eyes, and is closely related to the specially sensitive sense of contour acuity. Suppose, for example, that the two eyes view a pair of objects A and B, Fig. 127, such that A and B appear to the left eye to be in alignment, while they are separated to the other eye. Both experience and measurement suggest that the limit of the distance-difference perception will be governed by the limit of the acuity of the right eye in detecting the "contour step" between A and B.

Suppose, however, that the objects maintaining their separation unchanged are now moved into the median plane between the two eyes. (We can suppose that B is moved in the locus which keeps the total parallax angle constant, i.e. the "Horopter.") Then if the differential parallax of A and B was at the limit of perception in the first position, the parallax in the new position will be equal, though opposite in sense, for each eye, and will thus be half the original amount; it would therefore be below the limit for monocular perception. Experience shows, however, that there will be little difference if any between the limits of stereoscopic perception for the two positions of the test object; hence, in stereoscopy the visual system may integrate the differences (which would by themselves be sub-liminal) into a recognizable effect.

Owing to the existence of Panum's areas (see above) the horopter is not a mere surface in space, but has a certain volume. Experimental determinations of the locus of points which merely look to be at the same distance away" do not agree with the theoretical "circular" horopter and depend to some extent on the distance of the fixation point (Ames, Ogle, and Gliddon[5]).

Modern views of contour and stereoscopic acuity are summarized in a review by G.L. Walls.[14] One important factor which seems fairly well established in addition to the diffuse nature of any image contour with relation to the retinal mosaic (Vol. I, p. 150) is the occurrence of random movements in the axial directions of the eyes during fixation.[15] According to Krause such movements play an important part in the recovery cycle of the photo-chemical receptors. It would take too much space to go fully into the discussion, which is very speculative; but the view is that the estimate of the position of an image edge contour may be obtained from a statistical averaging of the impressions from the cones in the region of intensity fluctuation near the edge. It is well known that the precision of contour perception rises when the *length* of the contour line increases so as to bring more and more groups of cones into play. We may also recall that "the average of many length estimations made in whole inches may be accurate to a fraction of an inch"; thus the dimensions of one cone may not set a limit to contour acuity. It appears that in some animals at least the ratio number of cortical cells concerned in the apprehension of local sign for a particular part of the retinal image is perhaps 600 times the number of corresponding foveal cones (Talbot and Marshall[16]). However, many of these theories of vision are still not much more than a partial ordering of the experimental evidence, some of which still needs support. Recent work on visual acuity with brief stimuli suggests that image *movements* are *not* of major significance.

Stereoscopic Acuity. Many experiments have been made to determine the lower limit of binocular parallax which is capable of producing the sensation of stereoscopic relief. Under favourable circumstances a parallax of under 5 sec. of arc may be appreciated under conditions of steady observation; but 10 sec. of arc is usually required for instantaneous observations as in the experiments of Dove and Langlands. Such figures are, however, only valid for persons with excellent form vision and adequate training in stereoscopic observations. Some observers may find it difficult to detect parallaxes of whole minutes of arc without experience, and some appear to be lacking in the stereoscopic sense altogether. Pulfrich[17] has devised interesting test charts for use in a stereoscope; the elements of the chart embody various parallactic displacements, and they afford a ready means of testing the capabilities of various observers.

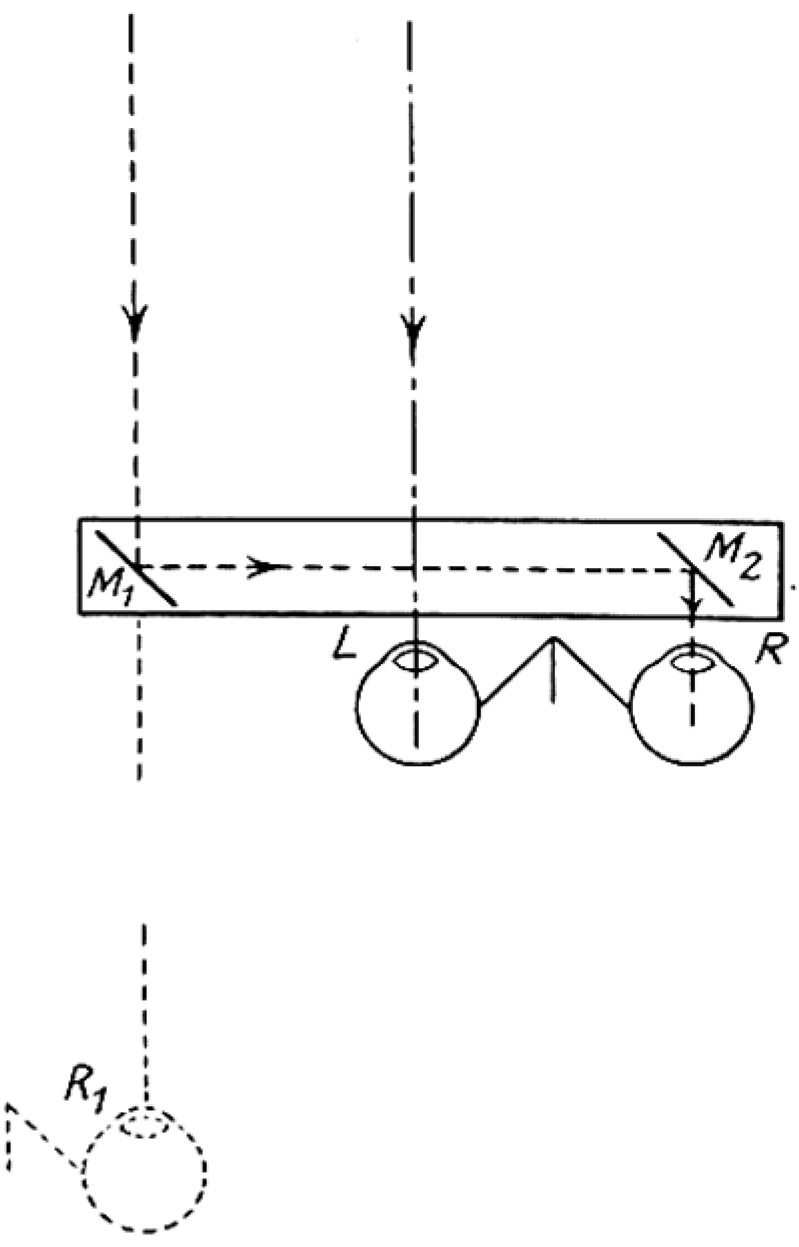

FIG. 128. STRATTON'S PSEUDOSCOPE

Pseudoscopic Effects. If the left-hand perspective projection is presented to the right eye, and *vice versa*, the "stereoscopic depth" sensation is reversed and opposes the ordinary interpretation of the perspective. The effect may be seen by cutting an ordinary stereoscopic pair and putting the left-hand picture on the right of the stereoscope, the right-hand picture on the left.

A similar effect can be produced by an arrangement of mirrors, as in the so-called "pseudoscope" of Stratton (Fig. 128). Two mirrors M_1 and M_2 are mounted together on a board at 45° to the direction of view of the two eyes L and R, so that rays from any object reach L directly, but must be reflected from M_1 and M_2 before entering R.

The perspective centre for the view presented to the right eye R is evidently at R_1, the image of the nodal point of R formed by the two mirrors. Let us draw the projected image of the right eye; then we may remind ourselves which is the nasal side by the convention in the diagram as suggested by von Rohr.

The experiment itself is a very easy and interesting one to make, and should be carried out by the student.

The Telestereoscope. Referring to Fig. 129, let A and B be two visual perspective centres, and let the separation be b. If the points P and Q are two points in the field situated near the line bisecting AB at right angles, and if their distances are r_p and r_q from the mid-point of AB, then we shall have, sufficiently nearly if b is relatively small

$$\widehat{APB} = \frac{b}{r_p} \text{ and } \widehat{AQB} = \frac{b}{r_q}$$

It was shown above that the parallax for the points P and Q depends on the difference of $\widehat{PAQ}$ and $\widehat{PBQ}$.

$$\text{Hence } \textit{parallax} = \widehat{PBQ} - \widehat{PAQ} = \widehat{APB} - \widehat{AQB}$$

$$= b\left(\frac{1}{r_p} - \frac{1}{r_q}\right)$$

$$= b\left(\frac{r_q - r_p}{r_p r_q}\right)$$

If the difference of distances is small so that we may write it δr, and the product of $r_p r_q$ as r^2, then

$$\text{parallax} = \frac{b\delta r}{r^2}$$

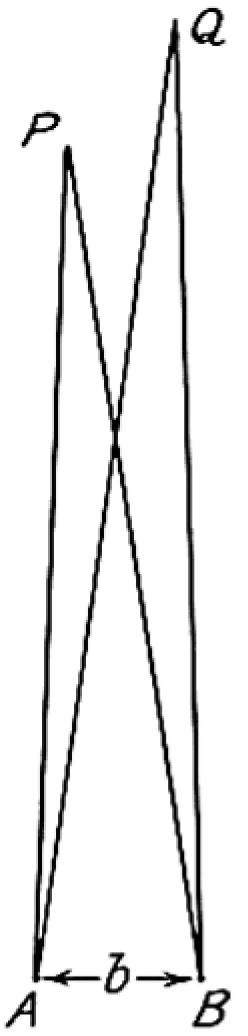

FIG. 129

We have made it clear that the stereoscopic phenomenon depends mainly on the binocular parallax, and it is in a sense immaterial how the perspectives are presented to the two eyes provided that corresponding points can be fused without upsetting too far the natural conditions of vision. By means of optical arrangements such as the pseudoscope above, we may present to the two eyes the perspective obtained from any desired centres, and the binocular parallax valid in vision is now to be calculated from the perspective centres actually in use. Thus in the telestereoscope (Fig. 130), the actual perspective centres are the mirror images of the nodal points of the eyes n'_a, n'_b formed by the double reflection. If the separation is now B the parallax for any pair of points is

$$\frac{B\delta r}{r^2}$$

The effects of such an arrangement were discussed by Helmholtz.

Stereo-telescopes. If Q is at an infinite distance, and P is a point at a distance r_o which can just be recognized stereoscopically with the ordinary interocular base b, as a nearer point, then the parallax must be the lower limiting value p_o, say, which is recognizable.

Hence since parallax $$= b\left(\frac{1}{r_p} - \frac{1}{r_q}\right)$$

we have $$p_o = \frac{b}{r_o}$$

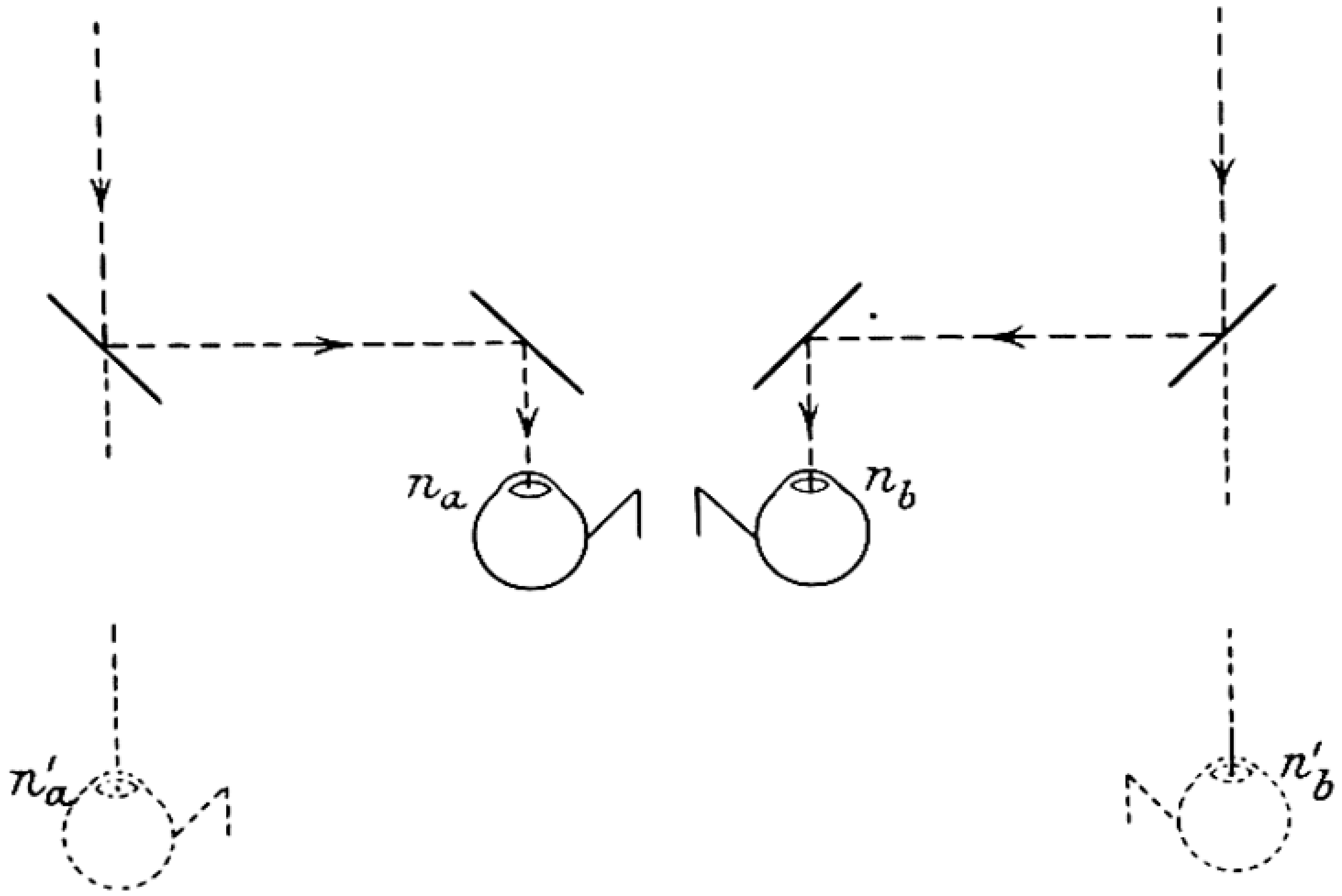

FIG. 130. SYSTEM FOR ENHANCED RELIEF (HELMHOLTZ)

The distance r_o is the "stereoscopic radius."

If the base length is increased to B, the stereoscopic radius R will now be given by

$$p_o = \frac{B}{R}$$

since the stereoscopic radius is increased proportionally to the effective base length.

In the stereo-telescopes we not only have an arrangement of reflectors to increase the base length, but an erecting telescope system is also held in front of each eye. The effect is, therefore, to magnify all visual angles by a factor m, say, which is the magnifying power of the telescope.

If the binocular parallax $\frac{p_o}{m}$ valid at the actual perspective centres is now presented to the eyes magnified m times, it will, therefore, be p_o and thus recognizable. The equation for the stereoscopic radius R_m is now

$$\frac{p_o}{m} = \frac{B}{R_m}$$

The parallax for any given pair of points at an approximate distance r, and difference of distance δr, is now

$$\frac{mB\delta r}{r^2}$$

and if this is equal to or greater than the lower limit of discernible

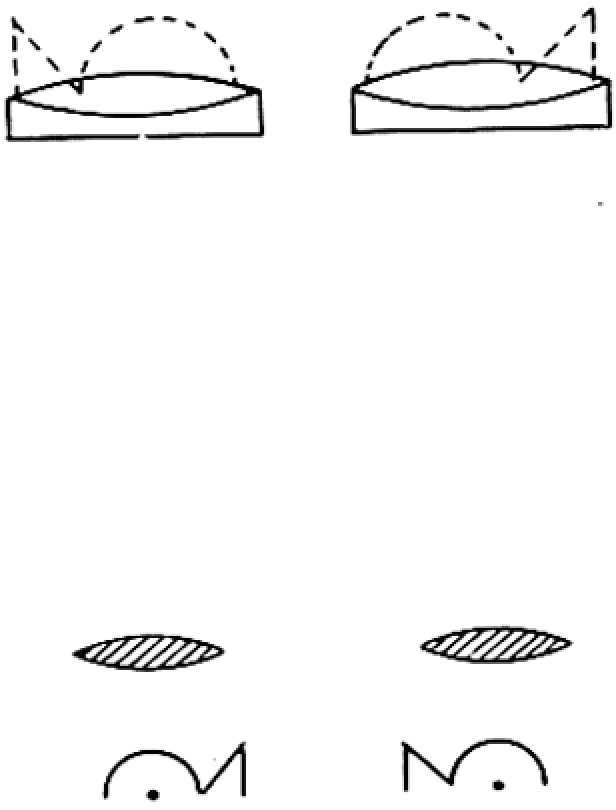

FIG. 131 (*a*)

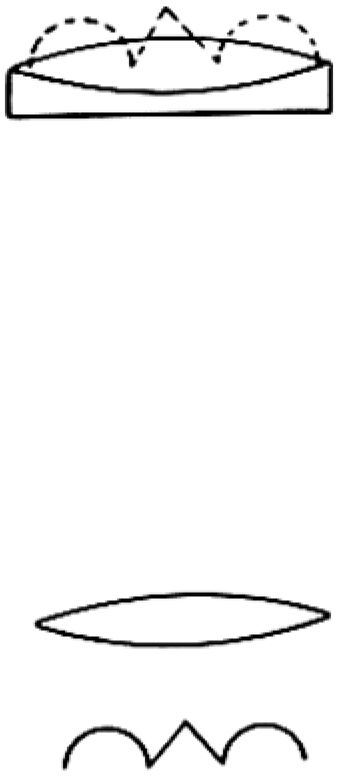

FIG. 131 (*b*)

parallax p_o, the stereoscopic separation of the two points will be recognizable; then the limiting condition is

$$p_o = \frac{mB\delta r}{r^2}$$

so that

$$\delta r = \frac{p_o r^2}{mB} \quad . \quad . \quad . \quad . \quad . \quad (4.01)$$

Binocular Telescopes. In considering the perspectives presented to the eyes, it is frequently helpful to imagine the image of the eye centre projected by the optical system into the object space. Fig. 131 (*a*) is intended to represent a non-erecting binocular telescope. Both the eyes are diagrammatically represented in the way suggested by von Rohr, the nasal side being indicated. Since the eye pupil is placed in the exit pupil of the instrument on each side, we consider an inverted image of each eye to be projected by the eyepiece system into the corresponding entrance pupil. It is quite clear that the effective projections will be reversed as well as inverted, and that the space image will be pseudoscopic. The erecting system

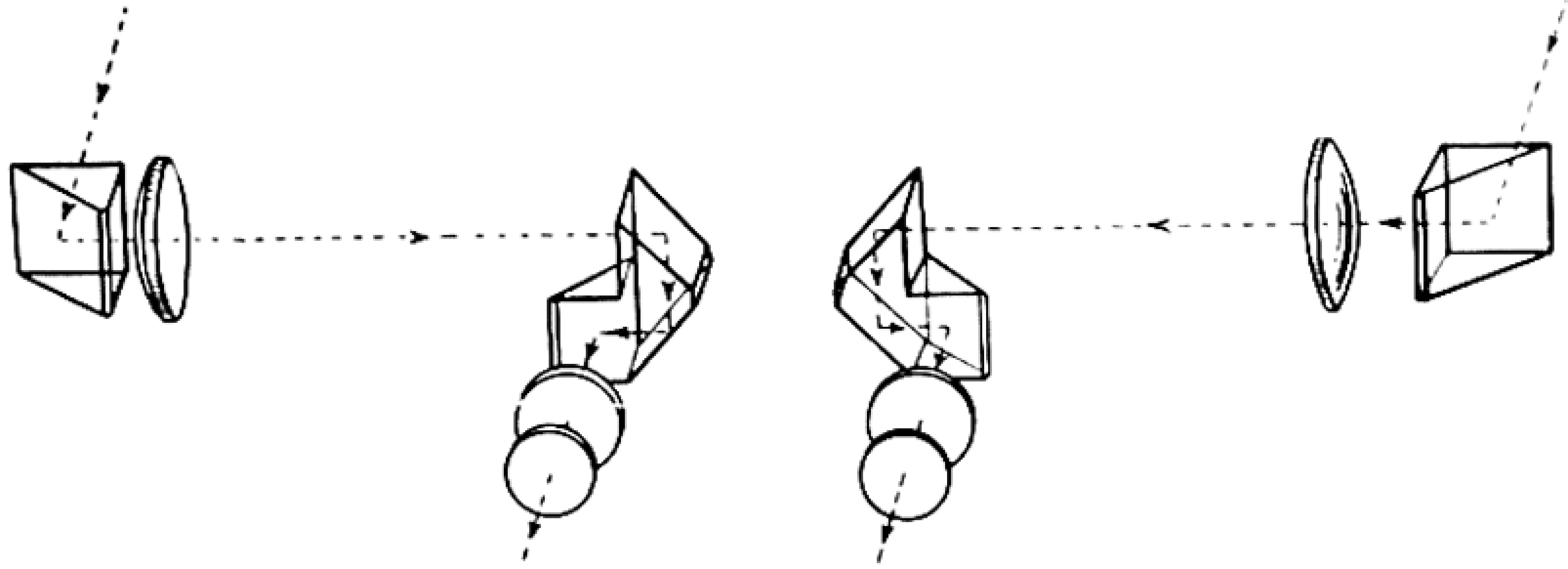

FIG. 132. STEREO-TELESCOPE SYSTEM

in ordinary binoculars is therefore necessary for this reason also. We may note that if both eyes use the same optical system (Fig. 131 (*b*)), and if we consider the effective centres, we find that the relation of the two eyes is the same in the projected image in the entrance pupil, and the effect cannot therefore be pseudoscopic, though there will still be inversion and reversion.

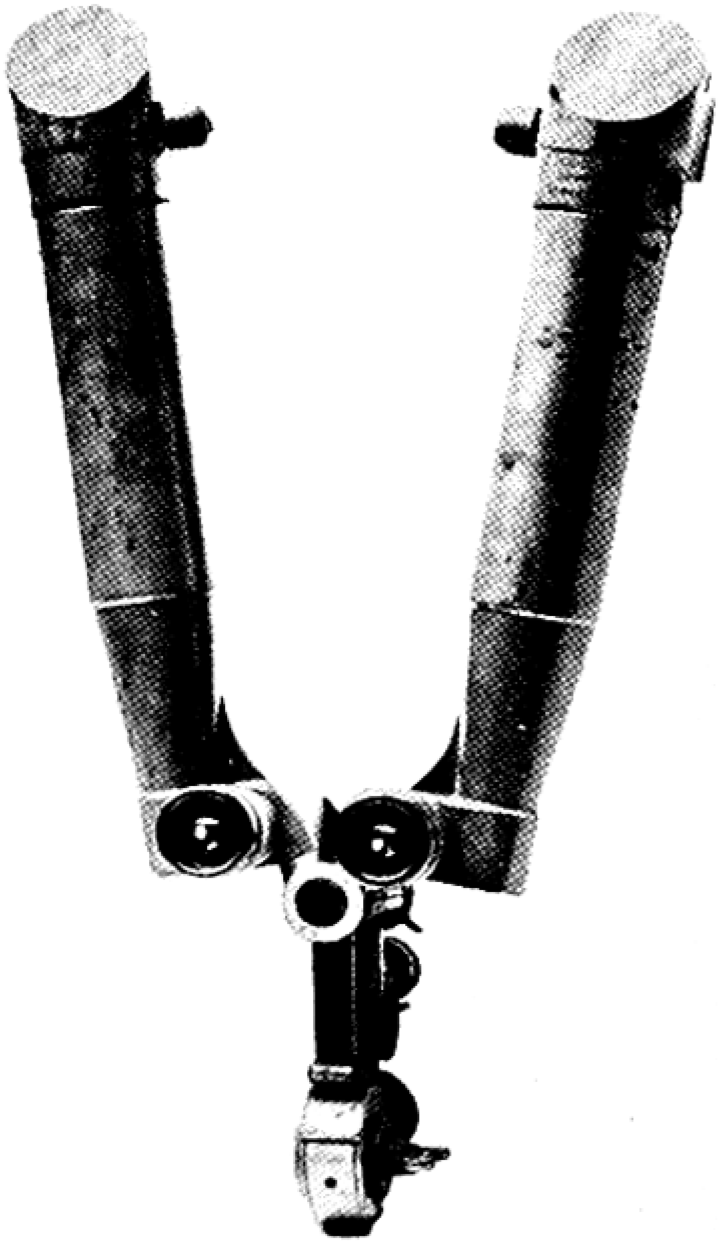

FIG. 133. MILITARY STEREO-TELESCOPE

The chapter on the telescope includes a number of diagrams of prism erecting systems for telescopes in which two parts can be separated; if the ray direction between these parts is at right angles to the initial and final directions, such a system will be useful in obtaining a binocular system with a long base and enhanced stereoscopic effect. Fig. 132 is a diagram of the arrangement of the optical parts in a stereo-telescope, and Fig. 133 shows a military instrument. The two arms of the apparatus can rotate about an axis parallel to the final directions of the optical axes of the eye pieces, and, as will be understood from Fig. 133, this enables a given interocular distance for the axes to be obtained either for the closed or open positions of the arms.

Adjustment of the Axes. As mentioned above, the eyes are subject to the habit of a fairly close relation between accommodation and convergence. If in using a binocular telescope the accommodation must be for "infinity," the corresponding normal direction for the visual axes would be parallel. It is not easy to produce or tolerate any considerable divergence of the visual axes; seven to eight minutes of arc may be managed without discomfort. Much greater amounts of convergence can be tolerated with the accommodation still at infinity; 20 to 25 min. causes little or no discomfort.

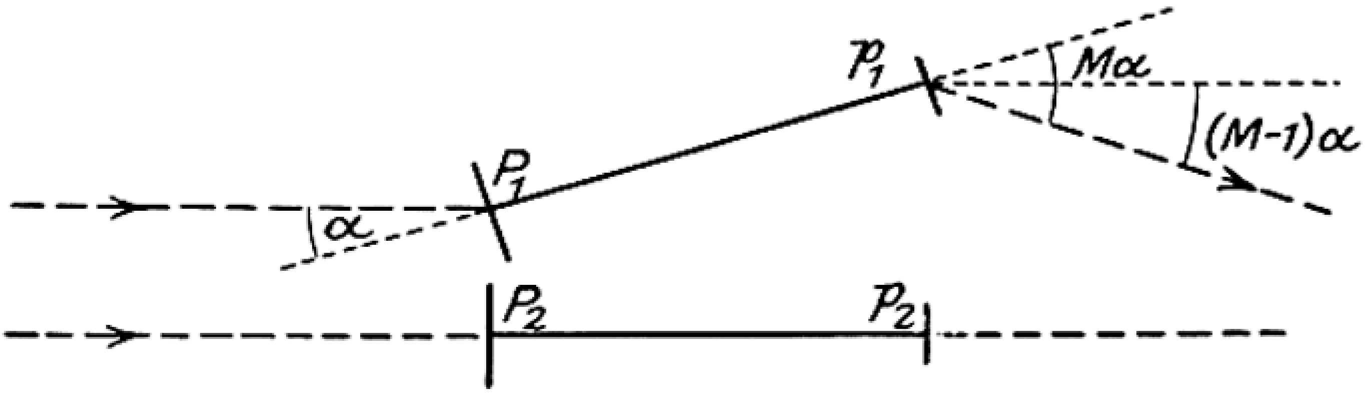

FIG. 134

Very little allowance is possible in the variation of the vertical directions of the eyes; the enforced Hyperphoria of one eye should not exceed seven to eight minutes of arc. These limits set a severe demand on the mechanical construction of binocular instruments, which therefore have to be mounted with their axes very closely parallel.

Let P_1 and p_1, P_2 and p_2 (Fig. 134) be the entrance and exit pupils of the two members of a binocular system mounted with the optical axes at an angle α. The systems are presumed to be erecting systems, so that it is clear that if parallel rays enter the centres of the two entrance pupils P_1 and P_2, these rays will emerge from the centres of the exit pupils at a mutual angle of $(M-1)\alpha$, which must not exceed the limits laid down above if comfort is to be preserved in use.

Thus with a pair of binoculars of magnifying power ×6, the limit of divergence of the optical axes towards the observer (requiring divergence of the visual axes) is, say,

$$\frac{7\frac{1}{2}\text{ min.}}{6-1} = 1\tfrac{1}{2}\text{ min.}$$

The same applies to the vertical variation. The limit of convergence of the optical axes is about

$$\frac{22\frac{1}{2}\text{ min.}}{6-1} = 4\tfrac{1}{2}\text{ min.}$$

These are the tolerances generally treated as standard ones.

Correspondence of Magnification. Another important matter in connection with binocular instruments is the necessity of the exact correspondence of the magnification in each part. Any inequality in this respect will produce very unpleasant effects to vision, and will upset the stereoscopic presentation of the binocular space image. Another defect which cannot be tolerated is that of imperfect erection of the images, which will also upset the stereoscopic effect, if present, and prevent the effective fusion of the two parts of the field. Particulars of methods of testing binocular telescopes will be found in a paper by Williams.[18]

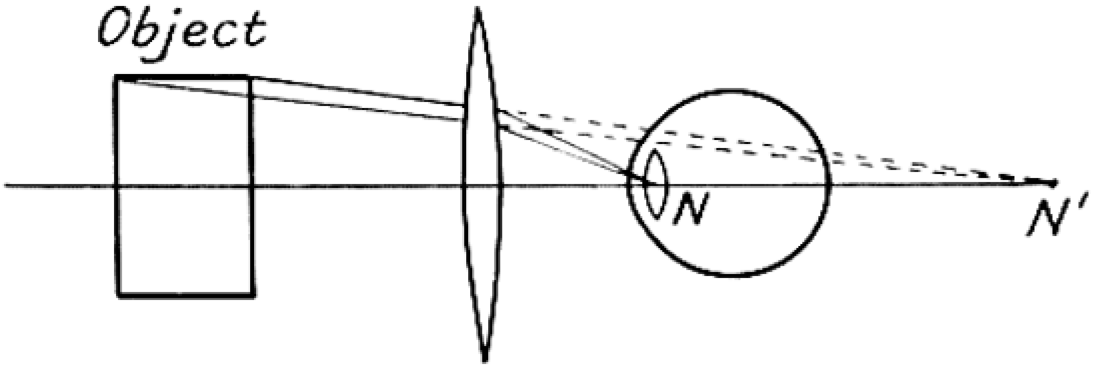

Fig. 135. Perspective Effects with Magnifier System
Effective viewpoint on same side. Natural perspective

Space Presentation with Binocular Instruments. As mentioned above, a ready way of obtaining the position of the effective centre when using any instrument before one eye is to find the image of the nodal point of the eye as projected by the instrument. Take, for example, the simple cases represented in Figs. 135 and 136. A rectangular framework is being viewed with the aid of a lens; N is the nodal point of the eye, and N′ its image projected by the lens. In the first case the eye is close to the lens (the case of the spectacle lens), and the positions of N and N′ are not far removed from each other; the framework is evidently seen under natural perspective, since the side nearer to the eye is seen under the greater angle. In the second case, the perspective centre is projected beyond the lens and framework; the latter is seen under unnatural perspective, since the side farther from the eye is seen under the

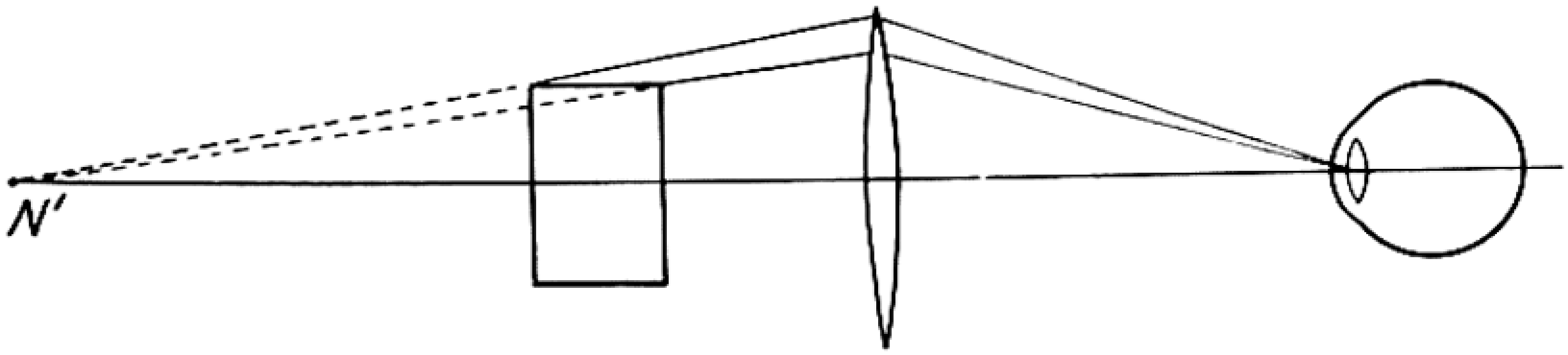

Fig. 136. Perspective Effects with Magnifier System
Effective viewpoint on far side. Unnatural perspective

greater angle. The student should draw for himself the diagram of the case when the perspective centre is projected to an infinite distance; objects of equal height then subtend equal angles at the eye. Von Rohr calls these three types of perspective by the names "Entocentric," "Hypercentric," and "Telecentric"; they are all possible with monocular optical systems.

When we consider binocular systems, we may have two general cases of space presentation through the stereoscopic sense, i.e. *orthoscopic*, in which the two perspective centres have their natural relation; *pseudoscopic*, in which the centres are reversed as explained above; and sometimes the system may project the two perspective centres into one point, when the effect is "*synopic*," and the stereoscopic sense cannot exist since binocular parallax must be absent. Theoretically, as von Rohr points out, the combination of the three kinds of perspective with the three binocular possibilities makes nine modes of space presentation possible with binocular instruments.

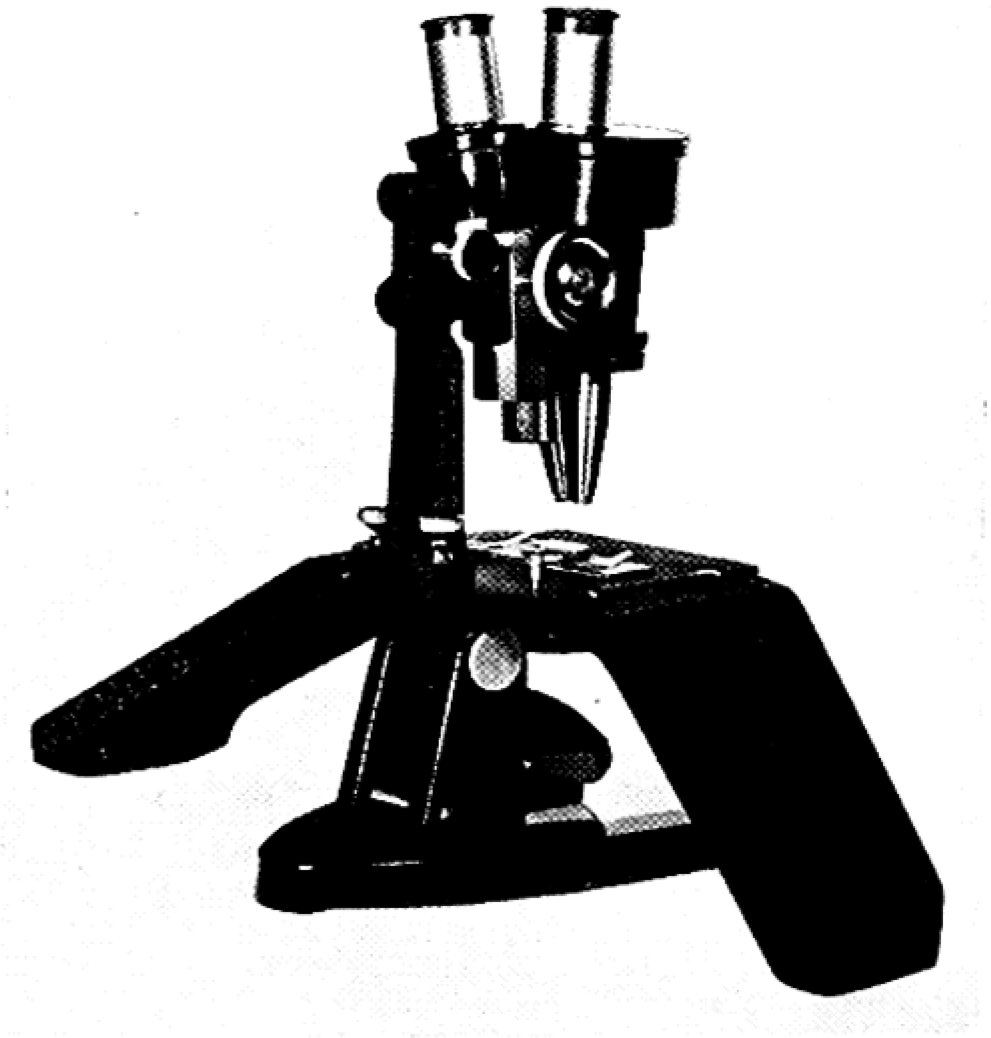

FIG. 137. GREENHOUGH MICROSCOPE

Most of the traps in this respect are avoided in the familiar instruments in which Entocentric perspective unites with orthoscopic presentation, but designers of new systems have to bear in mind the possibilities of error.

Binocular Systems for Microscopes. Owing to the necessary proximity of the objective and the object, two independent optical systems are only possible with the lowest powers, as in the Greenhough microscope, Fig. 137. The illustration shows a type in which the two objectives are mounted in separate tapering tubes. In more recent constructions paired objectives are mounted in one block of material bored with holes having suitably inclined axes. The same block can house two or three objective-pairs which can be brought into play by rotation of the block around a fixed axis into suitably clamped positions.

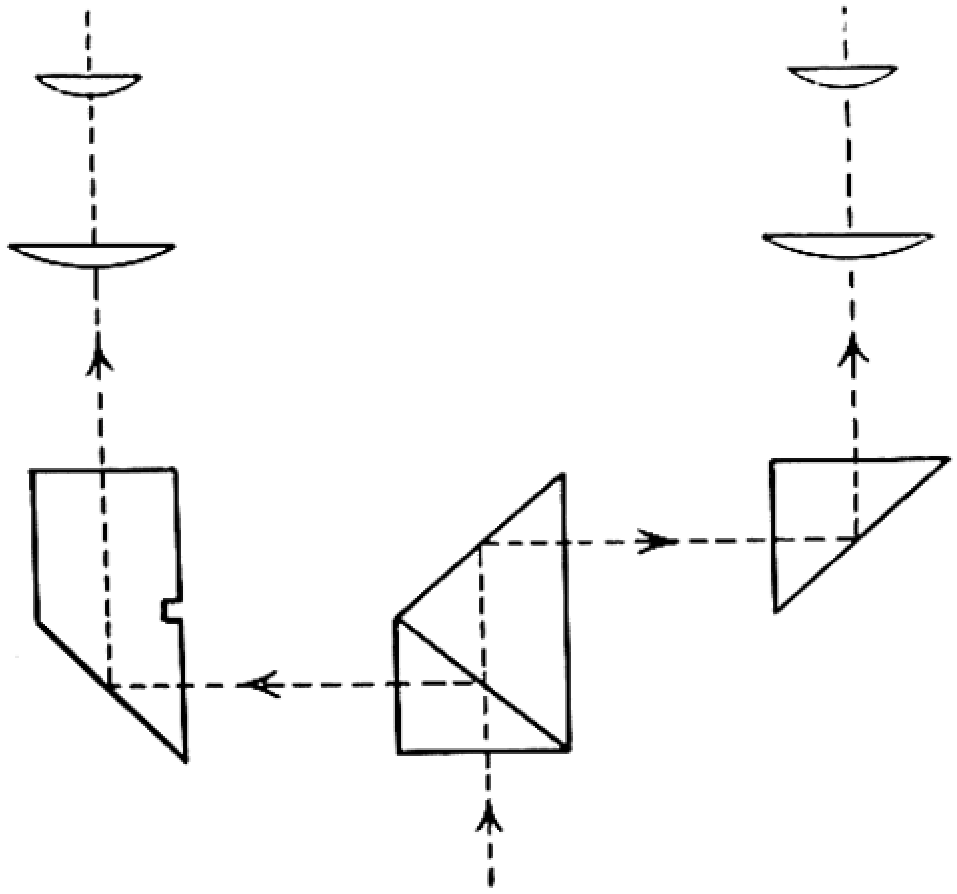

FIG. 138. BINOCULAR EYEPIECE SYSTEM

If the argument used above in the case of the binocular telescope is followed, it will be seen that each eye pupil will be projected by the system roughly into the position of the objective, and that the natural disposition of the nasal and temporal sides will be upset unless each system is erecting. It follows that an erecting device must be included in such a double-barrelled instrument. Use is made of the eccentricity of the eyepiece (when a prism erecting system is used) to obtain interocular distance adjustment, each member being rotatable about the axis of its objective.

When higher powers are to be employed, only a single objective is possible, and the beam from the objective must therefore be divided if a binocular system is used. If the whole aperture of the objective is to remain in action for each image so as to retain the utmost resolving power, then the division must be effected by means of a partly transmitting and partly reflecting surface.

The division of the field need not be effected *immediately* above the objective; various systems are now in use which can be adapted to the draw-tube of the instrument as binocular eyepieces. The early "stereoscopic" eyepiece designed by Abbe was of this kind; the separating surface was a thin air film. In order to bring the two exit pupils to the same level, two special eyepieces of differing construction were used. One worked as a "Ramsden," the other as a Huygenian eyepiece. In this arrangement the brightness of one image was twice that of the other. Modern arrangements have a more symmetrical design.

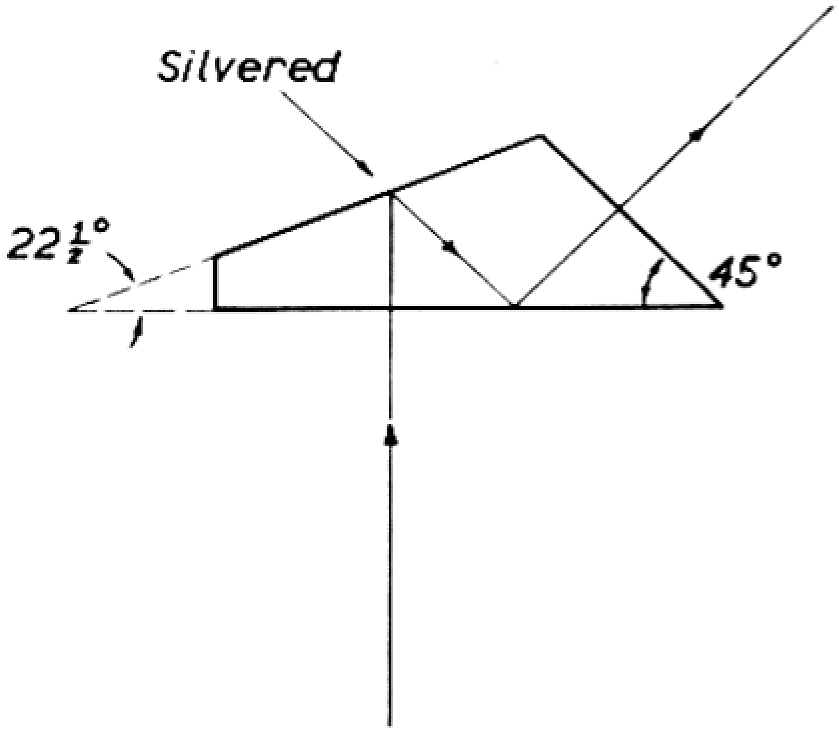

Fig. 139. Prism for Producing Inclination of Microscope Axis
(Another variant is seen in Fig. 32)

Fig. 138 shows the principle of an arrangement used by Cooke, Troughton, and Simms, and other makers in which the glass path in each arm is equal. A specially built body is usually required for systems of this kind. Other makers have brought out similar systems in which the eyepieces are given a backward inclination, so that a comfortable position for the head may be maintained even when the axis of the objective is vertical. See Fig. 139.

Messrs. R. and J. Beck, Ltd., have produced a binocular eyepiece of this kind with converging tubes. Many observers consider that convergent vision is the more natural with the "downward look" position of the head usual when using the microscope. A binocular microscope by Reichert is shown in Fig. 140.

Production of Stereoscopic Effects. No stereoscopic effect can be obtained while the whole aperture of the objective is in use for each eye. Let us consider the effective perspective centres obtained, say, with the Abbe eyepiece. The essential optical conditions are represented in Fig. 141 (*a*), which shows (conventionally) the two

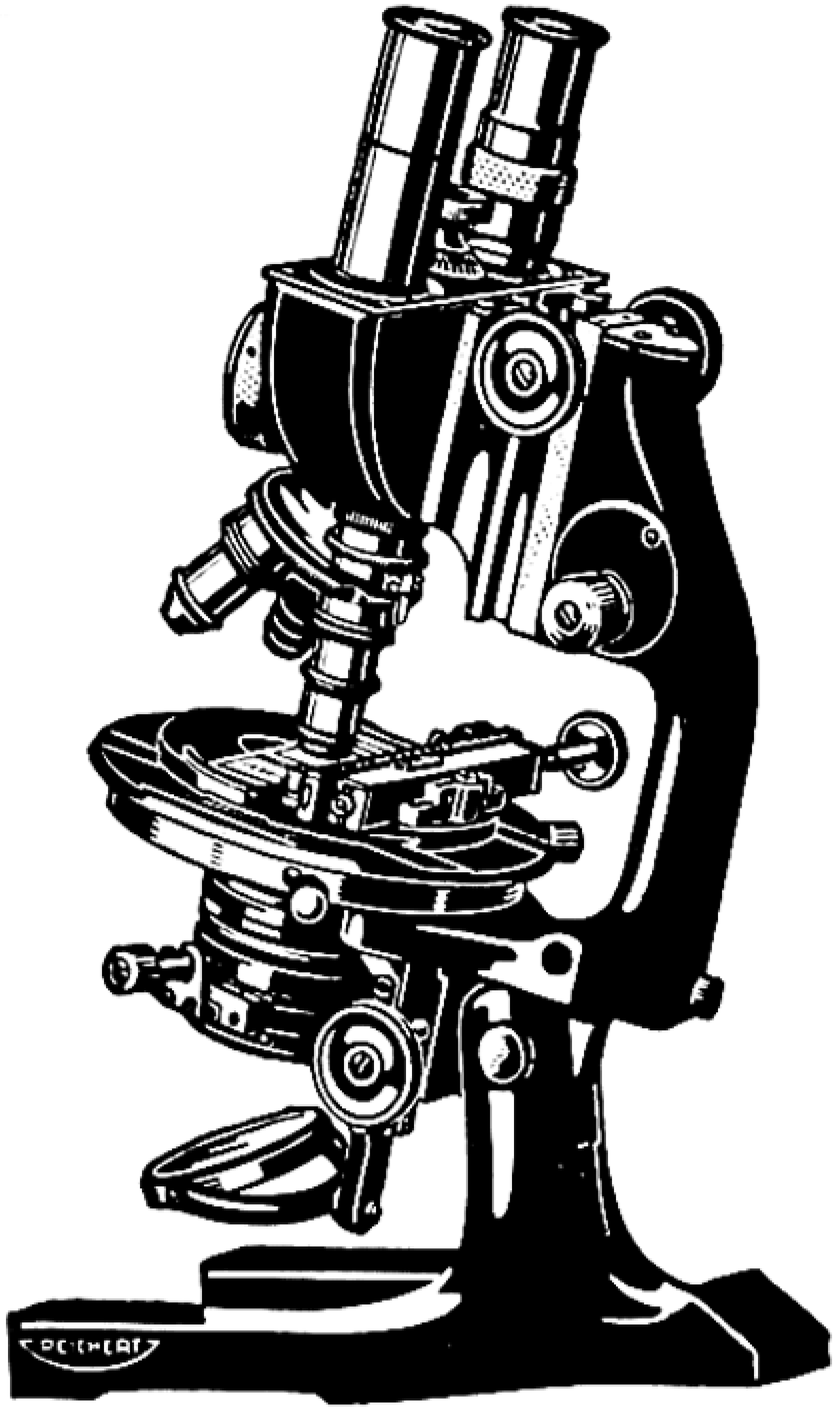

FIG. 140. BINOCULAR MICROSCOPE

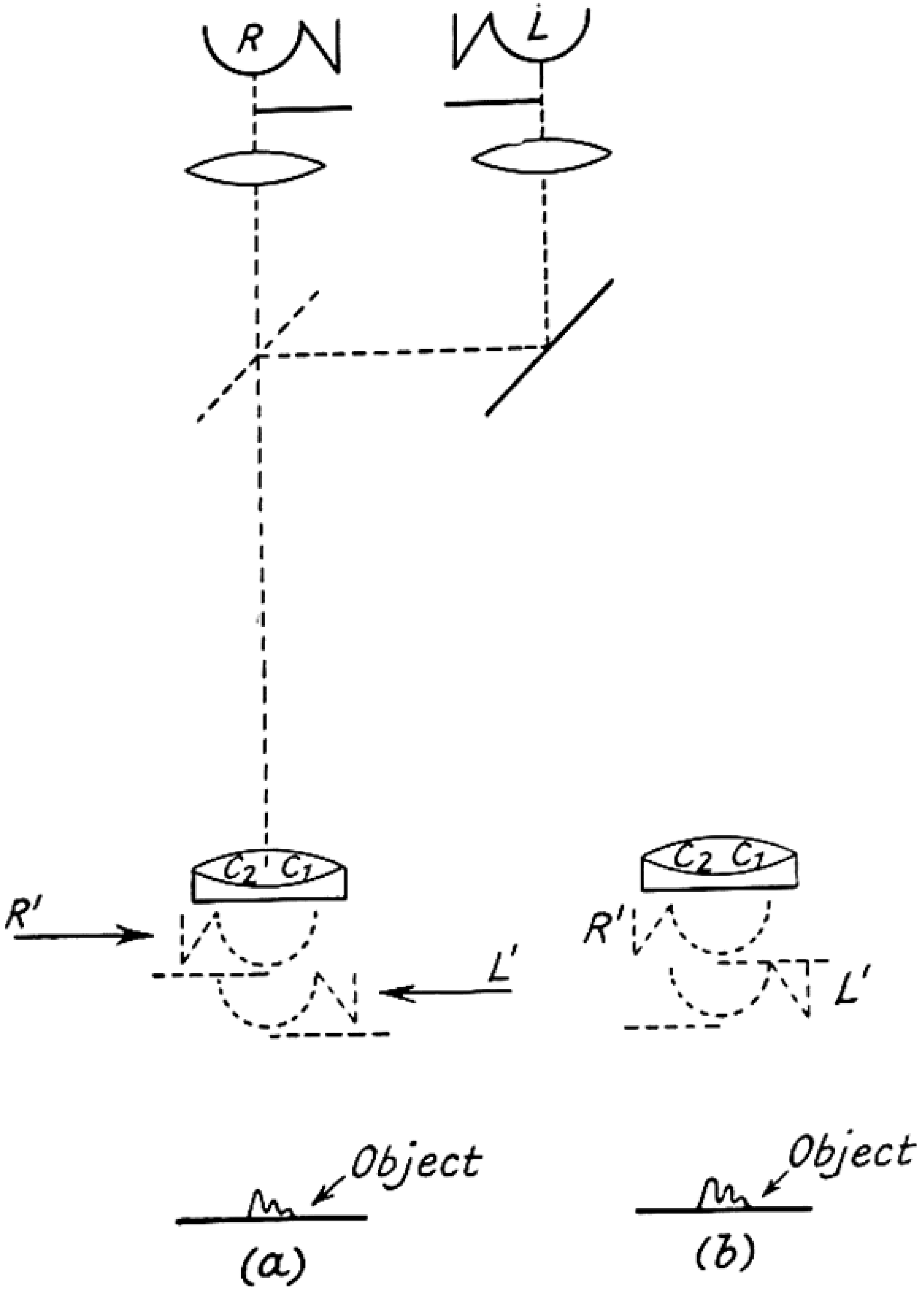

FIG. 141. THEORY OF STEREO EYEPIECE SYSTEM

eyes of the observer, R and L. No erecting systems are included (the roughly parallel mirrors in the one path produce no erection), so that the pupils are projected reversed and superposed into the entrance pupil.

Now our discussions so far have not dealt properly with lenses of finite aperture, and we must consider such a case before going farther. In Fig. 142, the lens L projects an image of the object

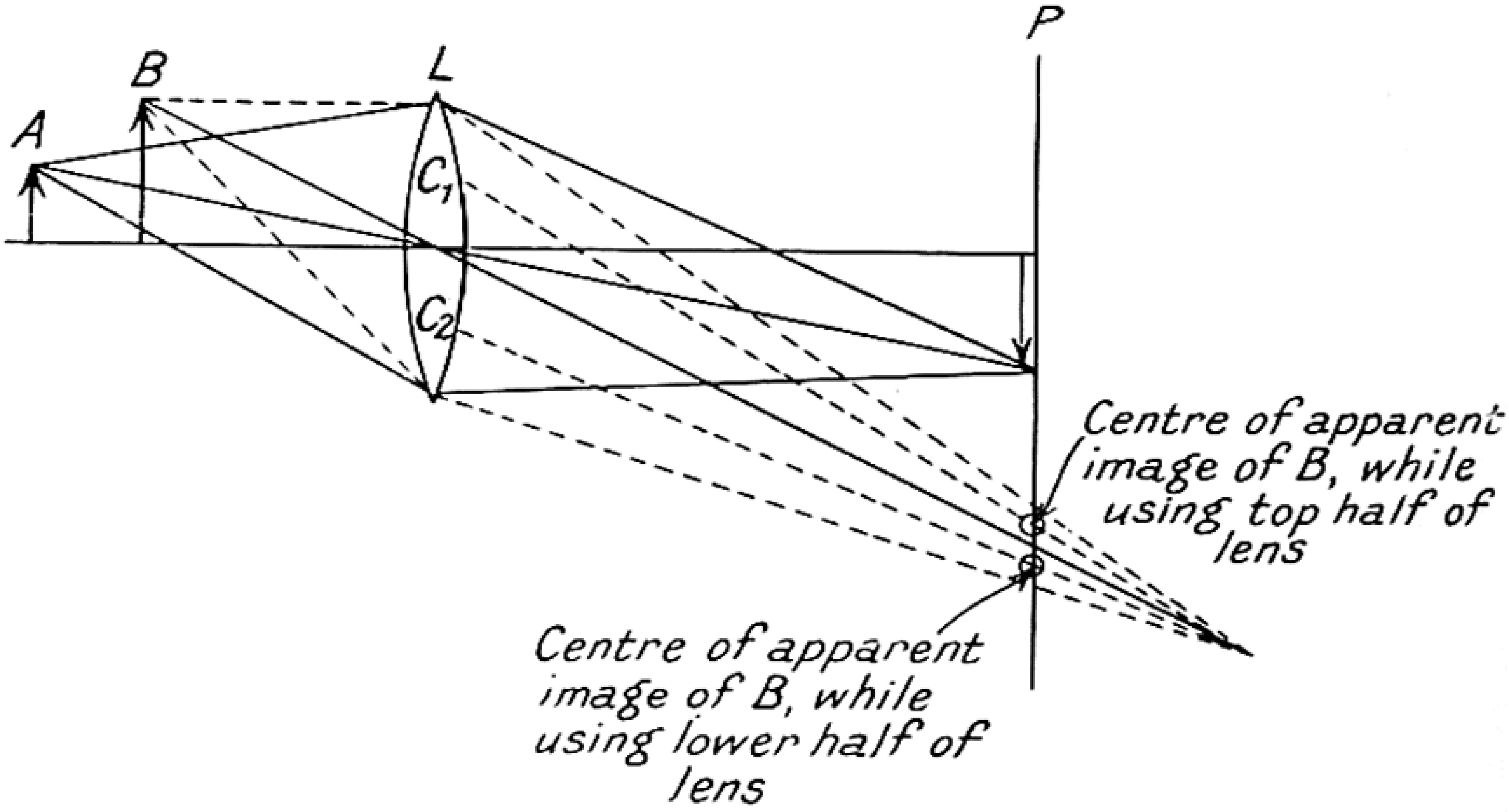
P
B
L
A
C_1
C_2
Centre of apparent image of B, while using top half of lens
Centre of apparent image of B, while using lower half of lens

FIG. 142

points A and B into the plane P. Now the image of A is in focus, while that of B is not in focus, and, consequently, the position of the centre of the image patch on the focal plane P will differ accordingly as the top or bottom halves of the lens L are used. The two pictures are much as we might obtain from two different lenses with centres at C_1 and C_2, which are really the perspective centres. We can look on the top or bottom half of the lens as two thin lenses plus a prism of small angle; thus it happens that these perspectives are superposed.

Let us now go back to the diagram of the optical arrangements in the microscope; let screens be introduced to cover the nasal halves of each pupil; they are shown imaged in front of the nasal parts of projected pupils R′ and L′. Consequently, the perspective centre will be on the C_1 side for R′, and on the C_2 side for L′, where C_1 and C_2 are towards the temporal sides of the projected pupils. A true stereoscopic effect will result.

On the other hand, if the screens had covered the temporal sides of each real pupil we should have the conditions suggested in the smaller figure (Fig. 141 (*b*)); speaking of the projected pupils we now see that the centre for R′ is towards C_2, and that for L′ is towards C_1; it is clear that the effective viewpoints are now interchanged, left for right, and a pseudoscopic effect must result.

Abbe pointed out that it is unnecessary to consider the particular means by which the separation of the centres is made, i.e. whether by a prism dividing the beam from the objective, or whether by screens in the pupils. Nor need the differentiation of the centres be effected except by a screen in *one* pupil, unless the fullest degree of relief is required. If the pupils appear as on the left of Fig. 143, the effect will be stereoscopic; if as on the right, pseudoscopic.

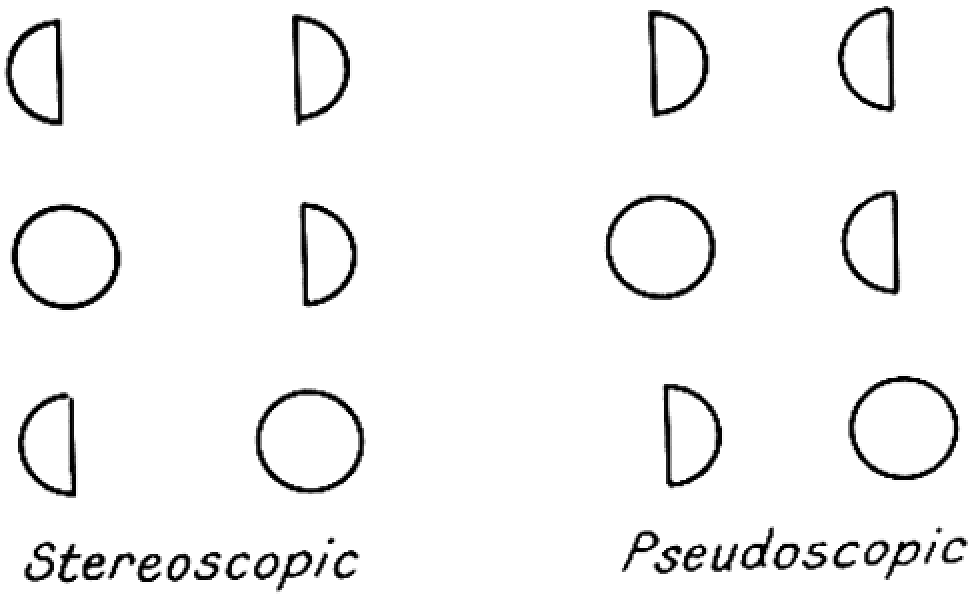

FIG. 143

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CHAPTER V

PHOTOGRAPHIC LENSES

PHOTOGRAPHIC lenses are usually required to project an image on a flat plate or film held perpendicular to the optical axis. Telescope objectives often only cover a semi-field of about 2° or 3°, so that if a high degree of correction is given for spherical and chromatic aberration, and if coma is kept small by attention to the fulfilment of the sine condition, the outstanding defects of astigmatism and curvature of the field can usually be tolerated. As against this, photographic lenses may have to give good definition for image points 25° or even 45° off-axis. The requirements of a large angular field and a large aperture are somewhat antagonistic, and Fig. 144 suggests (*a*) the relative convergence angles for the extreme rays at different "stop numbers," and (*b*) the maximum aperture, usually associated with commercial photographic lenses covering fields of various angular extents. The stop number is the relative aperture of the pupil, and is written as a fraction of the focal length; thus *f*/4 means that the diameter of the pupil is a quarter of the focal length.

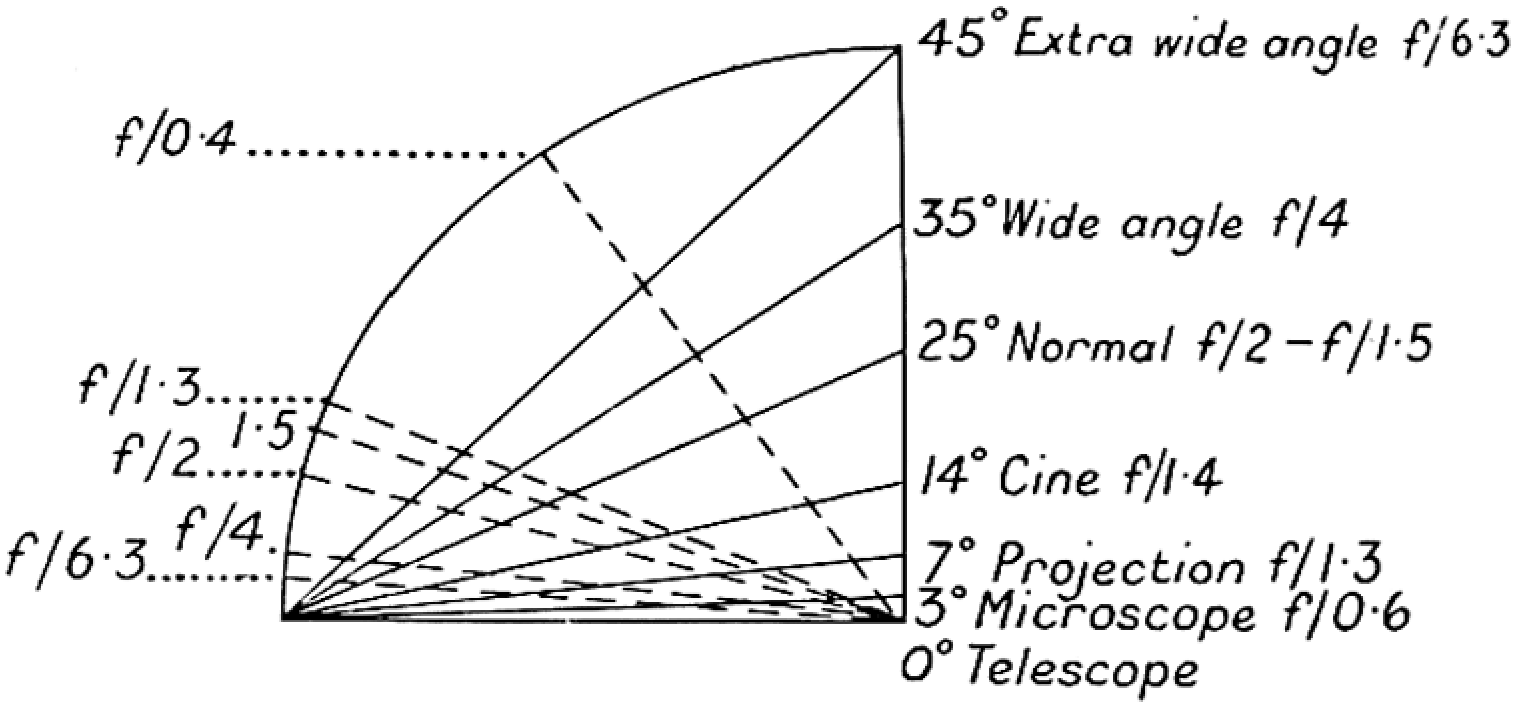

FIG. 144. ANGLE OF FIELD AND RELATIVE APERTURES OF TYPICAL LENSES

The principal types of commercial photographic lenses, including, however, some used for projection rather than photography, are summarized in the table shown on the next page.

The Illumination of the Image. Reference may be made to Chapter VI for the fundamental photometric concepts; it will be convenient

Type	Notes	Approx. Stop Number	Approx. Semi-field
(*a*) Landscape lenses .	Simple lenses or doublets	$f/11$	30°; sometimes up to 50° at $f/16$
(*b*) Normal camera lenses	Various types, some symmetrical	$f/8$–$f/2$	30°–15°
(*c*) Wide-angle lenses .	,,	$f/6{\cdot}3$–$f/4$	47°–35°
(*d*) Miniature camera lenses using 24 × 36 mm. film	,,	Aperture up to $f/1{\cdot}4$	Usually $< 25°$ for focal lengths $\geqq$ 2 in.
(*e*) Kinematograph camera lenses	,,	$f/3$–$f/1{\cdot}4$	20°–14°
(*f*) Telephoto lenses .	Negative back component	$f/5{\cdot}6$–$f/3{\cdot}3$	Usually $< 15°$
(*g*) Enlarging and process lenses	With two short conjugates	$f/10$–$f/8$	30°–15°
(*h*) Projection lenses .	Usually of Petzval type	$f/3$–$f/1{\cdot}3$	10°–5°
(*i*) Radiographic lenses	Photograph the image on a fluorescent screen	$f/1{\cdot}5$–$f/0{\cdot}8$	10°

to deal with the photometric aspects of photographic lens theory immediately. In photographic lenses the diaphragm is within or close to the lens system; the entrance pupil is usually the image of this diaphragm seen from the front, and the exit pupil is the corresponding image seen from the rear. Let us consider a small elementary object of area σ, and of luminance* B perpendicular to the axis in the object space. Fig. 145 will be of help. The corresponding image is of area σ'. If m is the linear magnification, then

$$\sigma' = m^2\sigma$$

Let p be the radius of the entrance pupil, then the amount of light entering the lens from the elementary object area is approximately (if q is the distance of the object from the entrance pupil)

$$\frac{\sigma \mathrm{B} \pi p^2}{q^2}$$

This light suffers partial absorption by the lens, and a fraction k

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* *See* p. 241.

reaches the image, where the amount of light falling on unit area (the *illumination*) is

$$\text{Illumination of the image} = \left(\frac{k\sigma \mathrm{B}\pi p^2}{q^2}\right) \div m^2\sigma$$

$$= \frac{k\mathrm{B}\pi p^2}{m^2q^2}$$

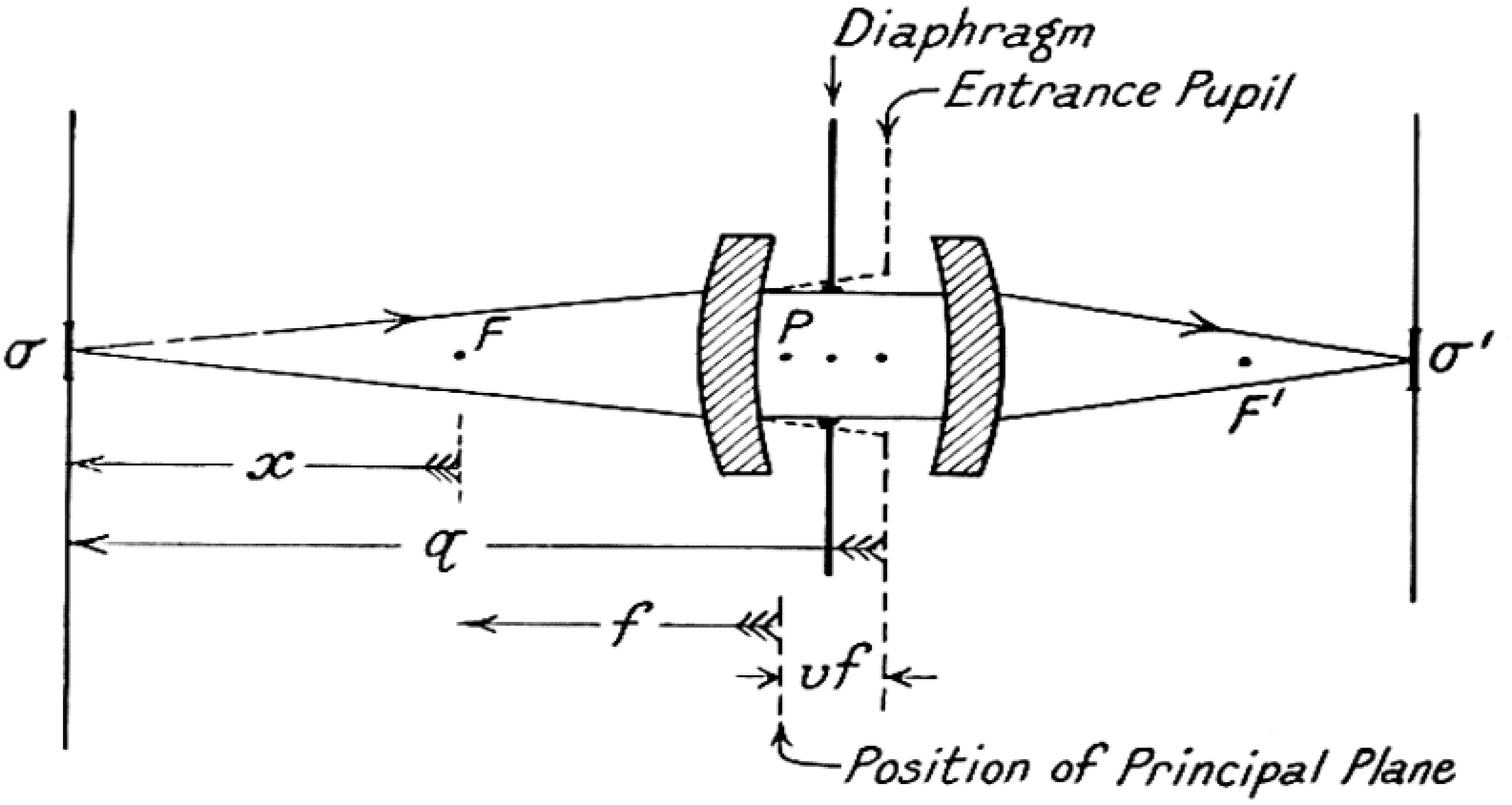


FIG. 145

In accordance with the usual notation, the symbol x denotes the distance of the object from the first focal point of the lens. Then we have

$$q = (f + vf + x), \text{ say,}$$

where vf is the small distance of the first principal point from the entrance pupil; but since $m = \frac{h'}{h} = -\left(\frac{f}{x}\right)$ (see Vol. I, p. 43), we get

$$q = f\left\{(1 + v) - \frac{1}{m}\right\}$$

and
$$mq = f\{(1 + v)m - 1\}$$

The expression for illumination is then

$$\text{Illumination} = \frac{k\mathrm{B}\pi}{\{(1 + v)\, m - 1\}^2}\left(\frac{p}{f}\right)^2 \quad . \quad (5.01)$$

The fraction $\left(\frac{p}{f}\right)$ is half the reciprocal of the stop number, since the stop number was defined above as the quotient of the focal length divided by the diameter of the effective stop or diaphragm.

Note that we measure the *entrance pupil*,* i.e. the image of the diaphragm seen from the front of the lens, and not the diaphragm seen directly. It is also to be noted that m will be numerically negative for an inverted image, and that when the object is infinitely distant its value will be zero. The value of v will be negligible in many practical cases.

Alternatively, if $v'f'$ is the distance of the second principal point from the exit pupil, it may be shown that the illumination will be given with sufficient accuracy for most purposes by the expression

$$\text{Illumination} = \frac{\pi k \mathrm{B}}{(1 - m + v')^2}\left(\frac{p'}{f'}\right)^2 \quad . \quad . \quad . \quad (5.02)$$

where p' is the radius of the exit pupil.

Both the above expressions require modification if the pupil diameter is large in comparison with the conjugate distance. It will be seen in the chapter on the photometry of optical systems that the illumination of the image will be nearly proportional to $\sin^2 \alpha'$, where α' is the convergence angle of the extreme ray, which is not proportional to p'/f'.

It has been urged with some reason, that the "stop number" engraved on lenses could with advantage be made to indicate the effective illumination, taking all the factors of actual conditions such as absorption etc., into account.

The relative illumination of the axial part of the image with lenses of differing stop number is given by the following table, in which α' is the angle of obliquity of the extreme marginal ray. No allowance is made in this table, however, for losses by reflection, etc., in the lenses.

α' (in degrees) . .	1·8	3·6	5·1	7·2	10·2	14·5	19·5	30
Stop number . .	16	8	5·6	4	2·8	2	1·5	1·0
Relative illumination, or speed . .	1	4	8	16	32	64	114	256

The exposure time required with photographic materials follows an approximate "reciprocity relation," i.e. it is roughly inversely proportional to the illumination; thus, if with a given film and subject the correct exposure at $f/16$ were 1/10 sec., a lens working at $f/1$ would only require 1/2560 sec.

Illumination of the Image Away from the Axis. Consider the case when the lens projects the image of an extended uniformly diffusing surface of luminance B. A small area ρ (Fig. 146) sends light to the entrance pupil so that the rays make an angle α with the axis of the lens, and with the normal to the surface at ρ. The amount of light dF radiated by the element into a solid angle $d\omega$ in this direction is (p. 243)

---

* It may be measured in practice with the help of a reading microscope with a fairly long focus objective.

$$dF = \rho B \cos \alpha \, d\omega$$

and the solid angle subtended by the pupil is, very closely,

$$d\omega = \frac{\pi p^2 \cos \alpha}{(q \sec \alpha)^2}$$

where q is the distance of the entrance pupil from the object plane, and p is the radius of the pupil.

Hence $$dF = \pi B \rho p^2 \cos^4 \alpha \quad . \quad . \quad . \quad (5{\cdot}03)$$

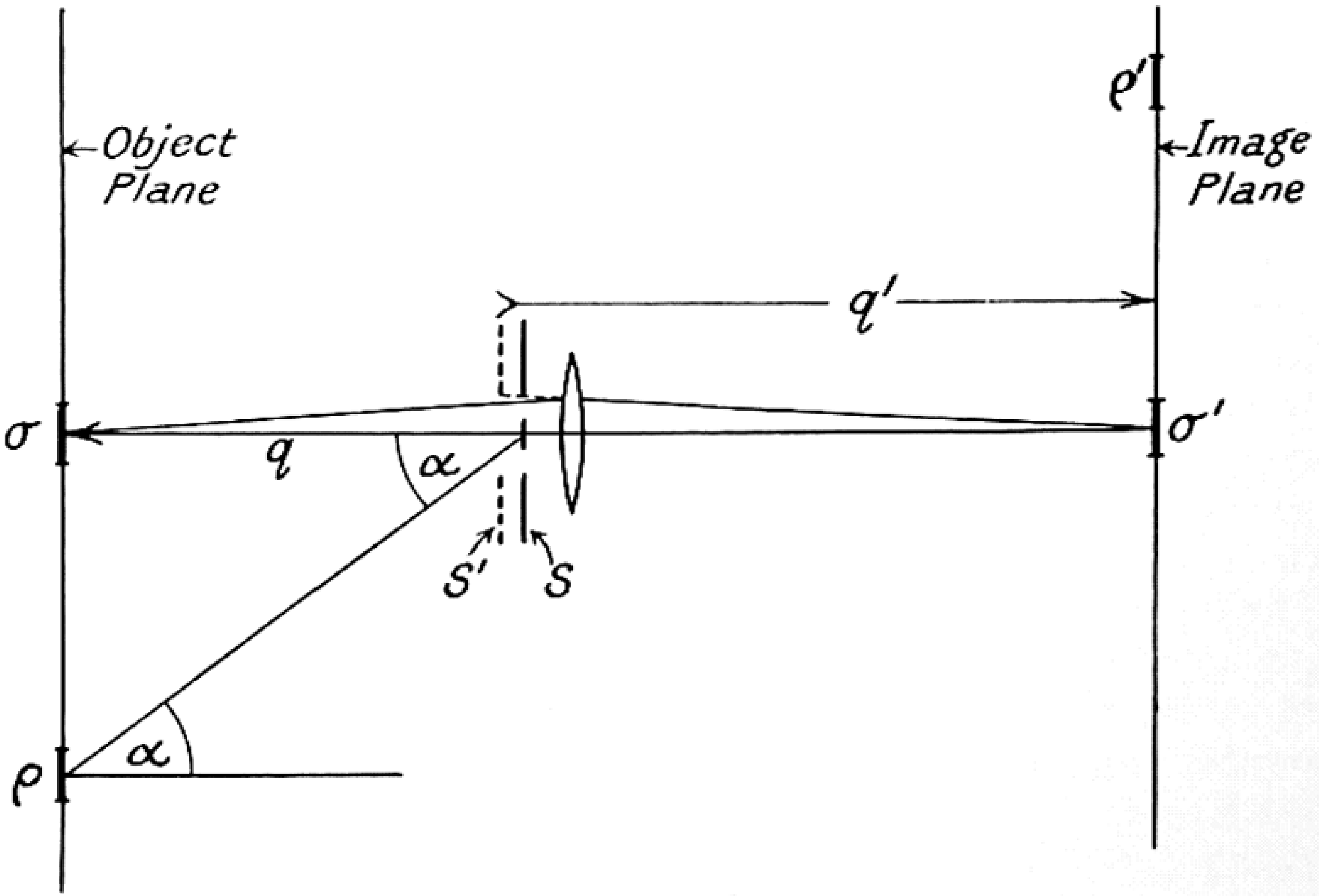


FIG. 146

In cases where there is no distortion, the area of the corresponding image patch will be uniform and independent of α. Hence the illumination of the image plane will vary as $\cos^4 \alpha$, even if vignetting by the diaphragm does not occur (see below). It is easily calculated that $\cos^4 \alpha$ becomes 0·56 at 30°, and 0·25 at 45°.

The effects of vignetting were referred to on p. 99, Vol. I. It may be illustrated by reference to the symmetrical system shown in Fig. 147, where two similar lenses have a stop between them. If the stop is opened out to full aperture so that an incident bundle of rays parallel to the axis is completely transmitted, we see that the actual separation produces such vignetting that, when a bundle of the same diameter is incident at about 30° with the axis, the part of the back lens transmitting light is limited to the lune-shaped area shown in the side projection. The effect of the vignetting in reducing the *relative* illumination of the outer parts of the image can be avoided by using a sufficiently small stop. Wide-angle lenses are used with small stops for this reason.

Ordinary modern lenses will rarely transmit any rays at all at greater angles with the axis than 45° or 50°. The diagonal of the plate may be expected to subtend an angle of 40° to 50°, so that the most oblique rays will not make angles with the axis much over 25°. The loss of light, as well as the optical aberration, sets a limit to the area which can be usefully covered by the lens.

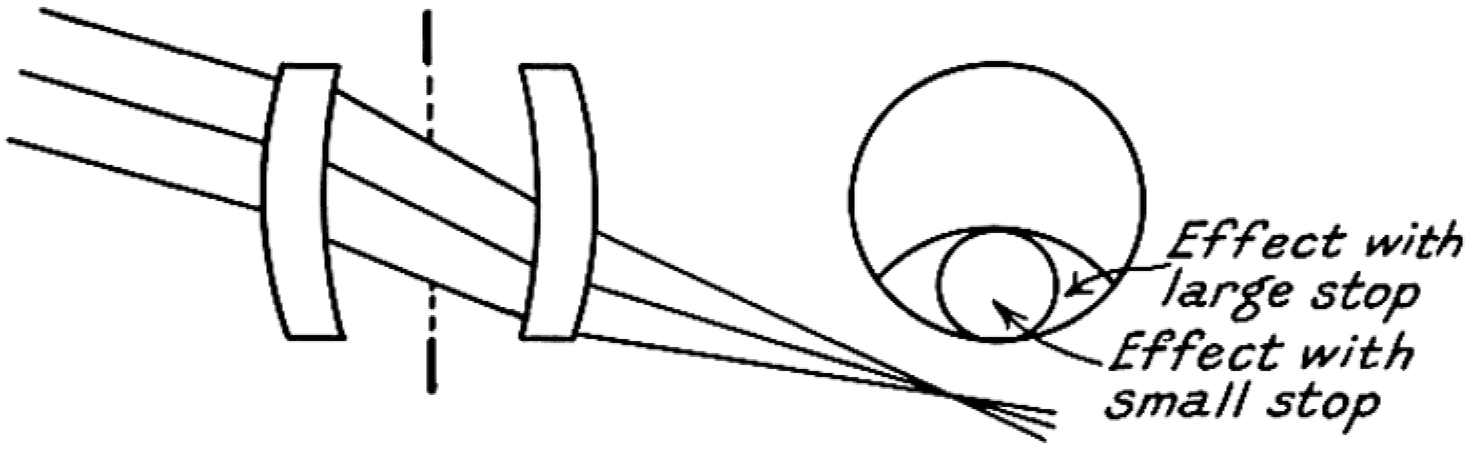


FIG. 147

However, it was pointed out by Slussareff[1] that the $\cos^4$ law is only true if the pupil is free from certain types of aberration. Roussinov has designed a lens (Fig. 179), in which heavy coma of the entrance pupil is produced although without any corresponding aberration of the normal image; it is claimed that the relative variation of the illumination approximates to $\cos^3 \theta$.

Systems of Thin Lenses. A co-axial system of thin lenses intended for a photographic objective should satisfy (*inter alia*) four simple conditions which can be concisely stated. Let $\boldsymbol{F}$ be the power of any one lens, V the constringence, and n the refractive index of the glass of the lens; y is the height, at the lens, of a paraxial ray originally parallel to the axis of the system, while $\bar{y}$ is the height, in the lens, of the principal ray which passes the axis at the pupils of the system. Then see Appendix, p. 380, and Vol. I, p. 138, the conditions are as follows—

1. The system must have a predetermined positive power; call this unity; then the condition is

$$\frac{1}{y_a}\Sigma y\boldsymbol{F} = 1 \quad . \quad . \quad . \quad . \quad (5.04)$$

where y_a is the height of the paraxial ray in the principal Gaussian planes of the system.

2. The Petzval sum should be zero; the condition is

$$\Sigma \frac{\mathbf{F}}{n} = 0 \quad . \quad . \quad . \quad . \quad (5.05)$$

3. The axial focus should be achromatized; the condition is

$$\Sigma \frac{y^2\mathbf{F}}{V} = 0 \quad . \quad . \quad . \quad . \quad (5.06)$$

4. The chromatic difference of magnification should be eliminated; the condition is

$$\Sigma \frac{y^2\mathbf{F}}{V} \frac{\bar{y}}{y} = 0 \quad . \quad . \quad . \quad . \quad (5.07)$$

As already explained, and is clear from equation (5.06) the achromatic system *must* contain some negative components, hence if the lenses are fairly close together so that the y values do not differ very greatly the positive lenses must tend to have higher V values than the negative ones, so that the negative terms do not exceed the positive. Now as the range of refractive indices available in optical glass is not very great (about 1·50 to 1·75) the sum of the individual powers must be small to satisfy (5.05), but the positive balance of power of the whole system can be kept great enough (condition (5.04) if the positive elements have higher values of y than the negative ones. As in the dialyte objective (p. 33), this can be achieved in a system of two lenses by a suitable separation of the positive and negative lenses. However, though such a pair of separated lenses can satisfy conditions (5.04), (5.05), and (5.06) they cannot satisfy equation (5.07) at the same time, though (5.07) would be satisfied if the lenses were in contact. If a stop is in contact with one lens of the system so that $\bar{y}$ is zero, such a lens can add nothing to the sum in (5.07). However, it will be clear that (5.07) can be satisfied by the use of a system symmetrically disposed with respect to a stop, provided that the conjugate distances are numerically equal; in this case the values of $\bar{y}/y$ take equal and opposite signs for the corresponding lenses in the system. Alternatively, if the system is composed of thin close achromatic doublets, the zero sum in (5.06) being satisfied for each doublet, condition (5.07) will be satisfied also. Such conditions have to be modified if the components are not thin; but these conditions furnish useful first ideas.

Requirements for the Formation of Images on a Plane. Fig. 148 shows an optical system which is forming an axial image B′ of an axial object B. The system is free from spherical aberration for these conjugate points; hence all rays from B pass through B′.

If now, in addition, we have another object point B_1, situated in the plane perpendicular to the axis through B, and near to the axis, the condition that the corresponding image B_1' may be sharply defined was worked out in Vol. I, p. 104, and is known as the *sine condition*. Let ω_1, ω_1'; ω_2, ω_2', be angular inclinations of any corresponding rays passing through B and B′; then the sine condition states that

$$\frac{\sin \omega_1'}{\sin \omega_1} = \frac{\sin \omega_2'}{\sin \omega_2} = \text{constant.}$$

This relation must, therefore, be fulfilled if we are to obtain good definition of the images of points surrounding B, wherever these images are situated.

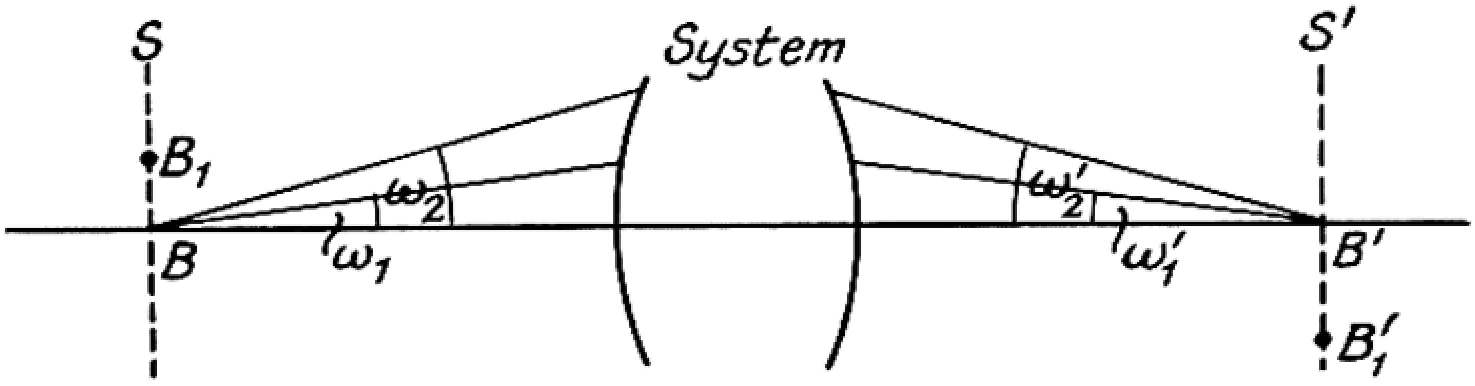


FIG. 148

The sine condition is the necessary criterion for freedom from "coma," or differences of magnification for different zones of the lens, but its fulfilment does not secure freedom from oblique astigmatism, which can be explained in terms of the differential limitation of the perpendicular width of a bundle of rays passing obliquely through a round stop. The foreshortened width in the "tangential" direction is $d \cos \alpha$ (where d is the diameter of the stop, and α is the inclination), as against a width of d in the sagittal direction. If a pencil goes centrally through a thin lens, the same approximate relative retardation is impressed on the central parts, as compared with the marginal parts of each section; since the "tangential" section is the narrower, we can visualize a refracted wave of greater curvature and shorter focal distance than for the sagittal section. In Chapter IX of Vol. I, p. 319, it was shown, however, that the effects of astigmatism, at least for a very narrow bundle, could be removed by the use of an axial stop at a finite distance from the thin lens when the latter was bent into a suitable meniscus form concave to the stop. We may, therefore, understand that there are similar possibilities in connection with more complex lens systems if a suitably disposed stop or diaphragm is allowed.

Provided that astigmatism is eliminated, the sharp images are found in cases of simple lens systems to lie on the so-called Petzval surface. From the equation (435) (Vol. I, p. 138) it follows that the radius R of this surface (assuming a flat object plane as in our case) is given by the equation

$$\frac{1}{R} = -\frac{1}{n_1 f'_1} - \frac{1}{n_2 f'_2} - \text{etc.}$$

where n_1, n_2, are the refractive indices of the glasses of which the successive lenses are composed, and f_1, f_2, are the focal lengths which would be found for the lenses if calculated from their radii and refractive indices while neglecting the thickness. According to the simple first order theory, the sharp image should be flattened, then, by choosing the glasses and radii to make R infinite.

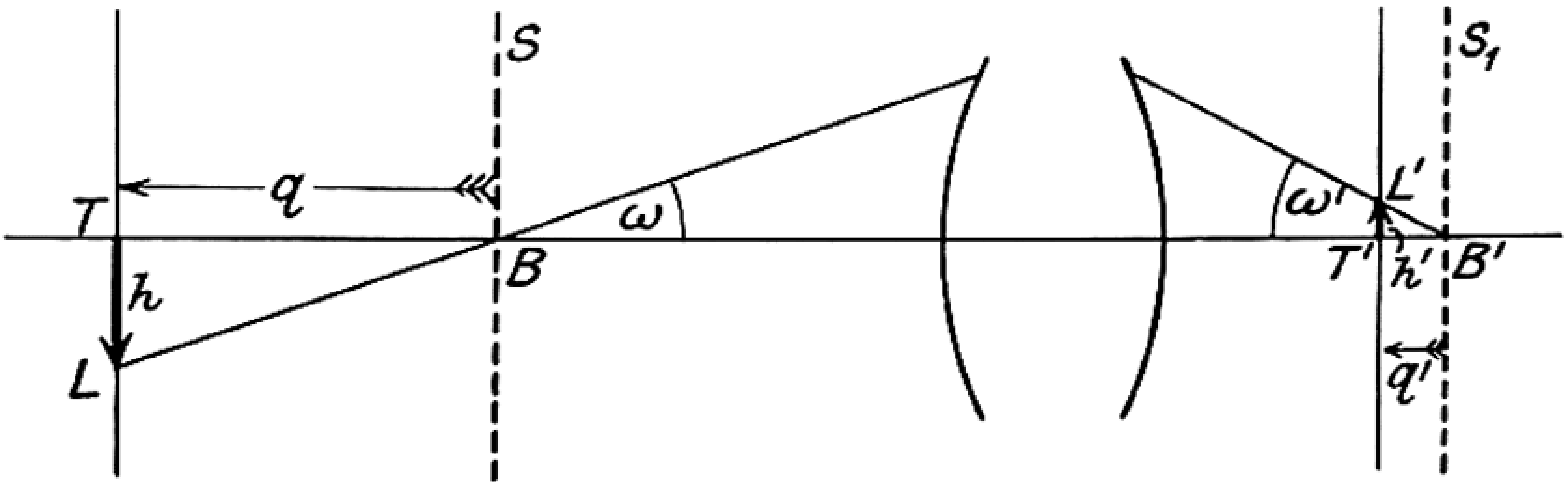


FIG. 149

Distortion. Provided then that we satisfy all the above requirements, we may expect to find a reasonably sharp image in the neighbourhood of the axis, which would fall into focus on a flat plate; the definition might be expected, however, to deteriorate when we exceed a certain distance from the axis, where the angular inclination of the rays with the axis invalidates the simpler theory.

We have, however, still to inquire whether the image in the plane S′ will be geometrically similar to that in S. We know that the scale of the magnification in the plane of B′ is given by $\frac{(\sin \omega)}{(\sin \omega')}$ at least for very small objects, so that there should be geometrical similarity to the object in the close neighbourhood of the axis. Our theory does not allow us to investigate the presence or absence of distortion far from the axial region in these planes S and S′, but as we know that photographic lenses are often required to work at all kinds of relative conjugate distances, it will be of interest to consider a second pair of conjugate planes through T and T′, which we may first assume to fulfil all the requirements of the foregoing paragraphs. These are shown in Fig. 149. Now a ray through B in the object space passes through B′ in the image space, and it passes through the planes through T and T′ in two points L and L′ at distances h and h' from the axis; we will suppose that L′ represents a sharp image point. Then, the ratio h'/h represents the magnification and is given by

$$\frac{h'}{h} = \frac{q' \tan \omega'}{q \tan \omega}$$

where q and q' are the distances from S and S′ to T and T′ respectively. In order that the magnification ratio may be constant for all sizes of object, we must have relations such as

$$\frac{q' \tan \omega'}{q \tan \omega} = \text{constant for any values of } \omega \text{ and } \omega'.$$

But we found above that

$$\frac{\sin \omega'}{\sin \omega} = \text{constant}$$

was the necessary condition for sharp definition in the planes S and S′. Unless $\omega = \omega'$, the above conditions are incompatible. We find in general that, even if distortion is completely corrected for planes containing a pair of *truly* aplanatic points, there *must* be definite distortion in any other pair of planes also giving sharply defined images. This argument only applies to principal ray distortion, however, if B and B′ are the positions of the stops. The conclusion above is justified in practice, but in cases where the extreme angles ω' and ω do not exceed about 4° or 5°, the difference between the sines and tangents will be less than 0·3 in 100, so that the distortion would not necessarily be serious for small aperture systems. If the distortion of a lens is removed by suitable designing for one pair of conjugate planes, the distortion for other planes is not usually prohibitive.

In practice, modern photographic lenses are *not* completely corrected for spherical aberration; this aberration has to be kept below certain limits, but these limits are wide enough to make it much easier to achieve freedom from distortion and other aberrations in a way which is impossible if *strict* aplanatism is called for.

If, then, spherical aberration is to some extent permissible, the position of an image "point" will be fixed by the principal ray through the centre of the diaphragm. We will, therefore, consider a case shown in Fig. 150, where the centre of the diaphragm is situated at the point D. Let us trace a ray through this point in both directions to its intersection with the planes S and S′, which represent the object surface and the photographic plate respectively.

The resulting intersection points V and V′ are now object and "image." If the image is subject to spherical aberration, its position is still marked by the intersection of the principal ray with the plane S′.

Looking into the front of the lens, the axial point of the entrance pupil P would be seen as the image of D. From the rear, the exit pupil would be seen at P′. (Vol. I, p. 98.)

Now it is likely that the axial positions of the entrance and exit pupils will be to some extent dependent on the inclination of the principal ray through D; these images may be subject to the effects of spherical aberration. Let the Gaussian positions of P and P′, calculated for very small inclinations of the principal ray, be denoted by P_o and P'_o. Let ω and ω' be the axial inclinations of the principal ray in the object and image spaces, while x_o and x'_o are the intervals P_oB and P'_oB' from the Gaussian pupils to the conjugate planes respectively; also let $PP_o \equiv \delta$, and $P'P'_o \equiv \delta'$. Thus PB $= x_o + \delta$ and $P'B' = x'_o + \delta'$. Then object and image heights are related by the equation

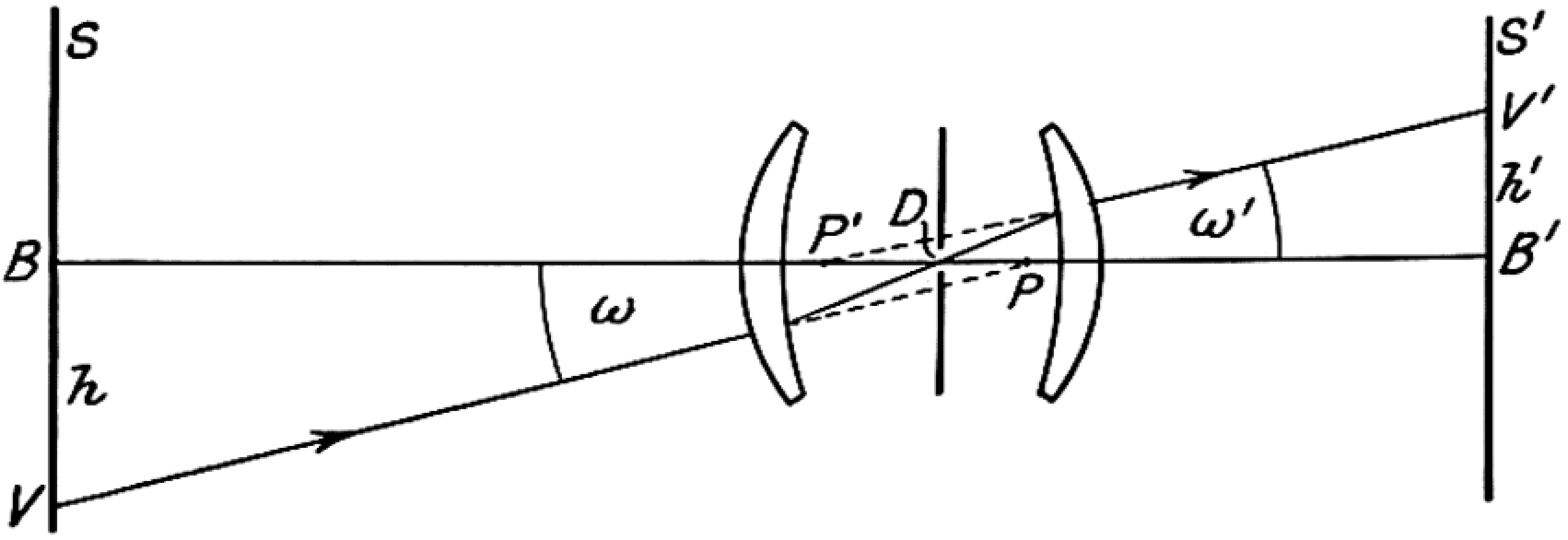


FIG. 150

$$\frac{B'V'}{BV} = \frac{h'}{h} = \frac{(x'_o + \delta') \tan \omega'}{(x_o + \delta) \tan \omega}$$

In the case of an indefinitely small object and image let

$$\underset{(h = o)}{\text{limit}} \; \frac{h'}{h} = m_o$$

Then a convenient specification of the amount of the distortion D is

$$D = \frac{1}{m_o}\left\{\frac{(x'_o + \delta') \tan \omega'}{(x_o + \delta) \tan \omega}\right\} - 1 \quad . \quad . \quad (5.08)$$

This specification is used in Wandersleb's important paper[2] (1907) on distortion, in which the distortions of most of the important photographic lenses of that period are given graphically in terms of D for given angles with the axis.

When the object is at an infinite distance we need to transform the above equation. Remembering from Vol. I, p. 43,

$$m_o = \frac{h'}{h} = -\frac{f}{x}$$

$$= \frac{f'}{x_o}, \text{ for a lens in air, sufficiently nearly}$$

so that

$$m_o\,(x_o + \delta) = f' + m_o\delta$$

$$= f'$$

in the limit since m_o is indefinitely small, and δ is also small. Thus

$$D = \left\{\frac{(x'_o + \delta')\tan\omega'}{f'\tan\omega}\right\} - 1 \quad . \quad . \quad . \quad (5.09)$$

Wandersleb uses a value N in his discussion which is the reciprocal of the magnification m.

Tolerance for Distortion. Distortion may be expressed as above by the percentage

$$100\,D = 100\left(\frac{m}{m_o} - 1\right) = \text{percentage distortion.}$$

In the best symmetrical form lenses the distortion may be kept within 0·1 per cent up to 30° or more with the axis. The value is sometimes greatest for an intermediate inclination of the pencils. With the "Hypergon" (see below) it is only 0·3 per cent at 54° with the axis. It is more difficult to control with the unsymmetrical forms. The curves for an early Cooke lens are given below, but much smaller amounts are found with later lenses. Thus with one form Wandersleb finds the distortion to be zero at 27° with the axis, and reaching a maximum of ± 0·6 per cent. The tolerance for distortion varies enormously with the object of photography. Several units per cent can be tolerated for landscapes, but the requirements for architectural subjects are much more stringent, say, 0·5 per cent. Photographic mapping requires distortion within 0·1 per cent if possible.

The Correction of the Off-axis Image Field. It was mentioned above, p. 195, that if a positive lens of meniscus shape is used in conjunction with a stop on its concave side, the astigmatism can be corrected. The image field then lies in the Petzval surface, having a "round" shape, i.e. concave toward the lens. Any position of the axial stop not in contact with the lens means that the principal ray will intersect the lens eccentrically. By moving the stop we have a new selection of rays with a new grouping around the central one; the aberrations are thus changed, with the exception of the axial spherical and chromatic aberrations and the Petzval curvature.

In Conrady's *Applied Optics and Optical Design*[3] it is shown that the following equations represent the changed values (distinguished by an asterisk) of primary coma and astigmatism of a system, consequent on a shift of the stop

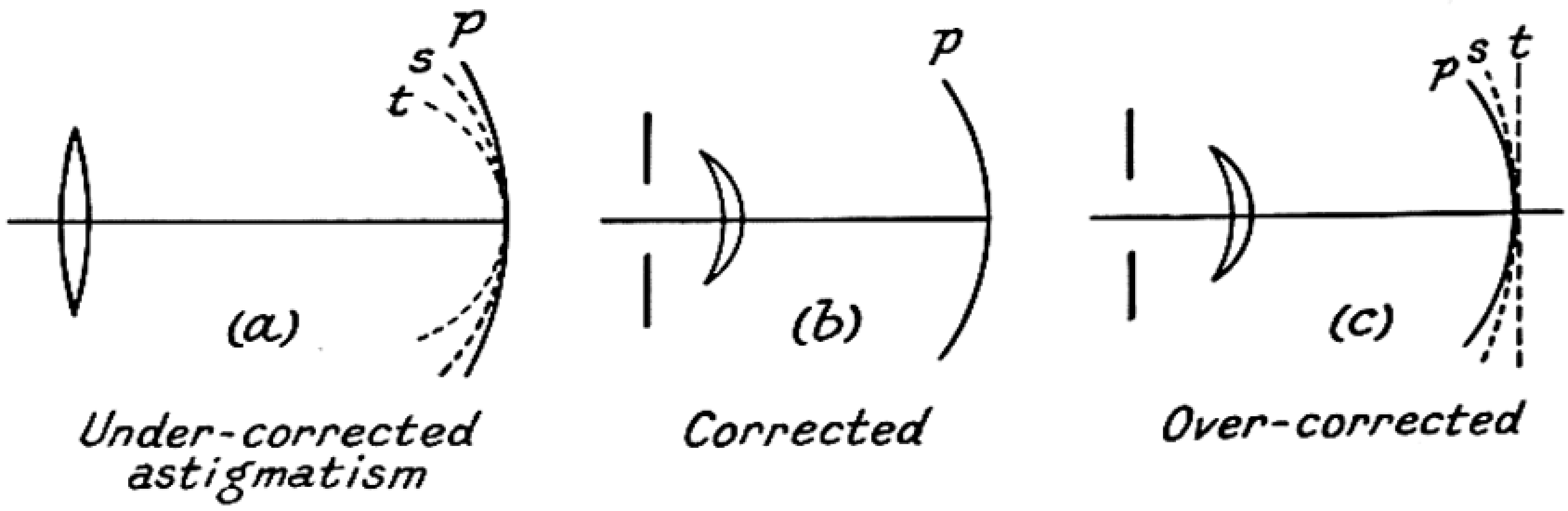


FIG. 151

$$\text{Coma}_s'^* = \text{Coma}_s' + \text{LA}' \,.\, \text{Q} \,.\, u'_k \qquad . \quad . \quad . \qquad (5.10)$$

$$\text{Ast}_s'^* = \text{Ast}_s' + 2\,\text{Coma}_s' \,.\, \text{Q}/u_k' + \text{LA}' \,.\, \text{Q}^2 \qquad . \qquad (5.11)$$

Note that LA′ is the primary axial spherical aberration; Q is a measure of the change of eccentricity of the principal ray; u_k' is the final convergence angle of the marginal ray directed to the axial image point; Coma_s' is the primary sagittal coma, i.e. the perpendicular distance from the intersection point of the extreme sagittal rays to the principal ray; Ast_s' is the sagittal astigmatism, i.e. the distance from the intersection points of the close sagittal rays to the Petzval surface, measured along the principal ray. In shifts of the stop, Q is the only variable on the right-hand side of each of the above equations. Thus on differentiating,

$$\frac{d(\text{Ast}_s'^*)}{d\text{Q}} = 2\,\frac{\text{Coma}_s' + \text{LA}' \,.\, \text{Q} \,.\, u_k'}{u_k'} = 2\,\frac{\text{Coma}_s'^*}{u_k'};$$

and if due attention is paid to the signs of the quantities it is shown that the maximum amount of over-corrected astigmatism will appear when the $\text{Coma}_s'^*$ is zero, and when LA′ is positive, i.e. under-corrected.

In the meniscus lens case, the bending and stop distance may be so chosen that the astigmatism is at its maximum amount when the tangential field is flat, the primary coma then being zero. Fig. 151 suggests how the field curvatures are modified by the stop position. The LA′ of the meniscus lens is positive, so that the final correction of coma is only possible by virtue of the negative value of Coma_s'. The geometrical aspect of these principles can be realized in part from the Figs. 152 (*a*) and (*b*) illustrating the correction of a landscape lens.

Fig. 151 shows the cases of under-corrected, corrected, and over-corrected astigmatism. In the first case, the Petzval surface has a negative radius of curvature, and the tangential and sagittal image surfaces are similarly disposed; it will be remembered that the distance between the tangential and Petzval surfaces is always three times that between the sagittal and Petzval.

In the second case the astigmatism is corrected, and the sharp image lies on the curved Petzval surface.

In the third case the astigmatism is so far over-corrected that the tangential field is flat while the sagittal one is slightly round; this condition represents a very usual compromise in the direction of getting a flat field for the image. It has been tried in some cases to go still further, so that the tangential and sagittal fields are symmetrically disposed on each side of a plane surface, but this is only at the disadvantage of a greater amount of astigmatism which itself causes a severe loss of definition.

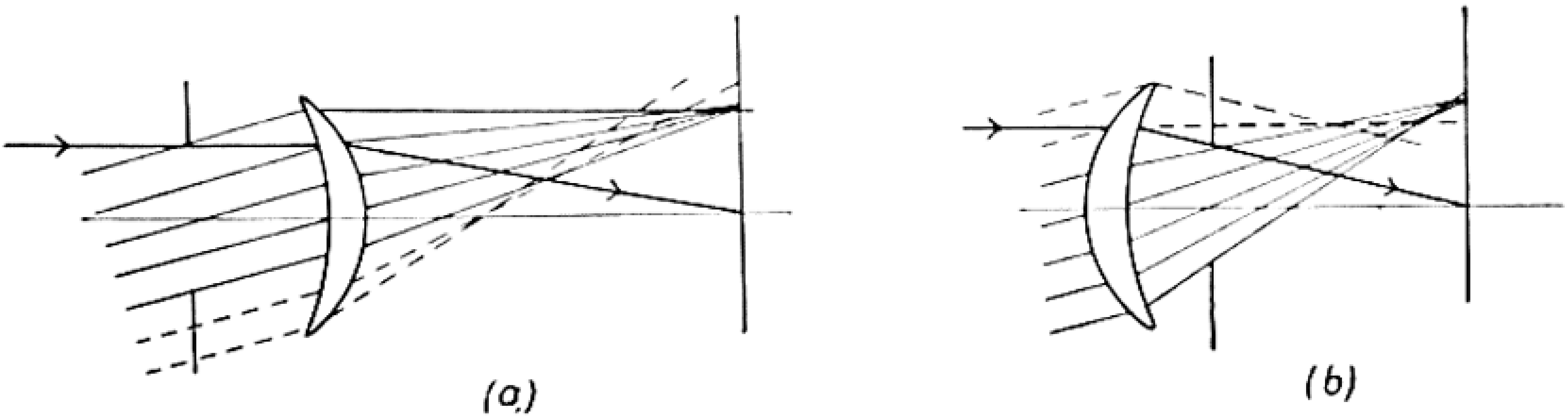

(a) (b)

FIG. 152. MENISCUS LANDSCAPE LENS

The Landscape Lens. The lenses fitted to inexpensive hand cameras are often of simple meniscus form; they are placed at a suitable distance behind a circular stop. There is no chromatic correction; the focus is fixed once for all, reliance being placed on the fact that the spectral region of actinic activity for *ordinary* photographic emulsions is between about $0{\cdot}35\mu$ and $0{\cdot}46\mu$, the short-wave limit being set by the glass absorption.

Fig. 152 (*a*) shows a diagrammatic section of such a lens, traversed by an oblique bundle of parallel rays. Such a lens suffers in the first place from spherical aberration of the ordinary "under-corrected" type which causes the outer rays to reach the focus closer to the lens than the inner ones. In this case, however, *coma* is also present. The upper part of the lens is traversed more or less symmetrically by the rays in the diagram. Now the deviation produced by a prism, it will be remembered, is a minimum where there is such a symmetrical disposition. This then tends to lengthen the focus in the upper part, but the rays in the lower part traverse the lens very unsymmetrically, so that there is relatively great deviation in the sense suggested by the figure. Hence, the coma opposes the spherical aberration in the upper part and supports it in the lower part of the lens under the conditions drawn.

For this reason a stop is placed in front of the lens which limits the oblique pencils, as indicated in the diagram; the spherical aberration and coma thus tend to balance their effects in the oblique pencils. Since the spherical aberration must be tolerated, the allowable amount sets a limit to the useful diameter of the stop; $f/11$ can be attained in practice with ordinary glasses.

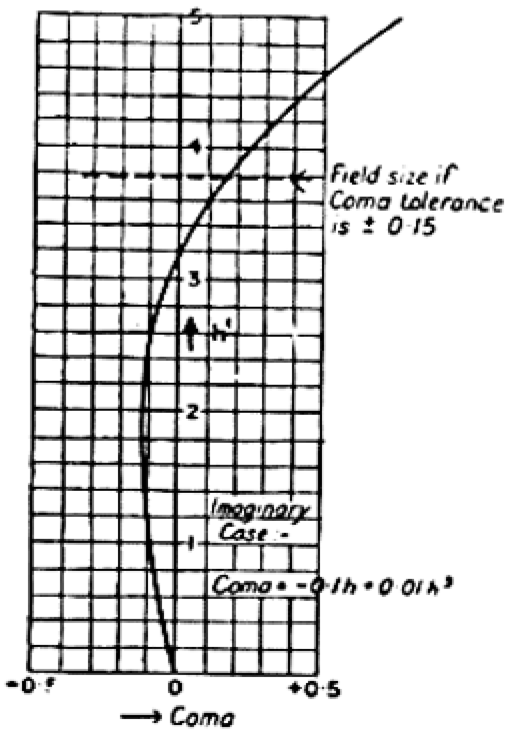


FIG. 153. BALANCE OF ABERRATIONS

Contrast this with the case where the lens is reversed, Fig. 152 (*b*). The tangential coma referred to the principal ray passing through the centre of the lens is now positive and aids the spherical aberration in shortening the focal distance of the outer pairs of rays. If we desire to obtain a correction, the stop must now be brought behind the lens.

Balancing Aberrations. The outstanding defects of the above landscape lens are now—

1. Axial spherical aberration.
2. Residual astigmatism.
3. Distortion.
4. Axial chromatic aberration.
5. Lateral chromatic aberration.

If the primary coma is made to vanish, experience shows that higher order coma will manifest itself in parts of the image away from the axis. The primary coma grows with h', the image height; while the higher coma has components grow with h' and h'^3, and may also have appreciable components growing with h'^5, etc. In the course of the trigonometric design of the lens it will, however, often be the practice to correct the tangential coma for oblique rays at full aperture for an image point distant from the axis by about seven-eighths of the full field. This can generally only be done by balancing a certain amount of low-order coma against the higher components of opposite sign. The effect is suggested in Fig. 153.

The balance between the aberrations prevents the resultant from reaching an undue amount anywhere within the allowed field, but the higher order terms have an overwhelming effect beyond it.

This principle of balance is of the greatest importance in respect of photographic lens design, and it will be seen below how it is also applied to the correction of the other aberrations.

The Higher Aberrations. A short account of the higher aberrations is essential for the comprehension of the methods of correction of photographic lenses. Remembering equations (426) and (427) of Vol. I, we here denote the semi-aperture of the zone of the system by y and the image height by h; then the following secondary or "fifth order" aberrations, expressed in terms of *lateral* aberrations, are distinguished—

Name	Grows with
Secondary spherical aberration	y^5
Secondary circular coma	y^4h
Hour glass aberration (or oblique spherical aberration) .	y^3h^2
Secondary elliptical coma	y^2h^3
Secondary astigmatism	yh^4
Secondary distortion	h^5

There are, in addition, still further tertiary aberrations, etc., which may reach appreciable amounts in the more complex systems. The amounts of these higher aberrations which arise at any surface are dependent on the primary aberrations, but will not in general be eliminated simultaneously with them. Note that the corresponding longitudinal measures of the above aberrations will be obtained approximately by multiplication by l'/y, where l' is the conjugate distance; thus the terms of higher axial spherical aberration depend on y^4, y^6, etc.

Achromatic Landscape Lenses. The longitudinal chromatic aberration and the greater part of the lateral chromatic aberration can be removed by the use of a doublet construction. In lenses intended for visual focusing, the "*d*" and "*g*" foci may be combined, but in a case where the focus will be set mechanically, and panchromatic films will be used with a yellow filter, the region of stationary focusing distance (the minimum of the curve of the secondary spectrum) will be arranged for the yellow-green, so that a "C and F" correction will be preferable. The appropriate constringence value will of course be computed for any case in hand.

Combinations of the older glasses were used by Chevalier (1830) and Grubb (1857); the first has heavy residual primary spherical under-correction, but it will be noticed that the contact surface (of negative power) in the Grubb type has relatively heavy curvature, and this encourages the development of higher negative spherical aberration. (See below, p. 214.) We can thus attempt a balance between the positive primary axial aberration (growing as y^2, where y is the radius of a zone) and the negative higher aberration which depends on y^4 and perhaps higher terms.

Old and New Achromats. When the glasses used in achromatic combinations have a large difference of V values, it is possible to keep fairly shallow curves for the lenses, and thus to avoid zonal spherical aberration. This is a very important matter in telescope objectives, and keeps the "old achromat" crown and flint combination, with its large difference of V values, of outstanding importance.

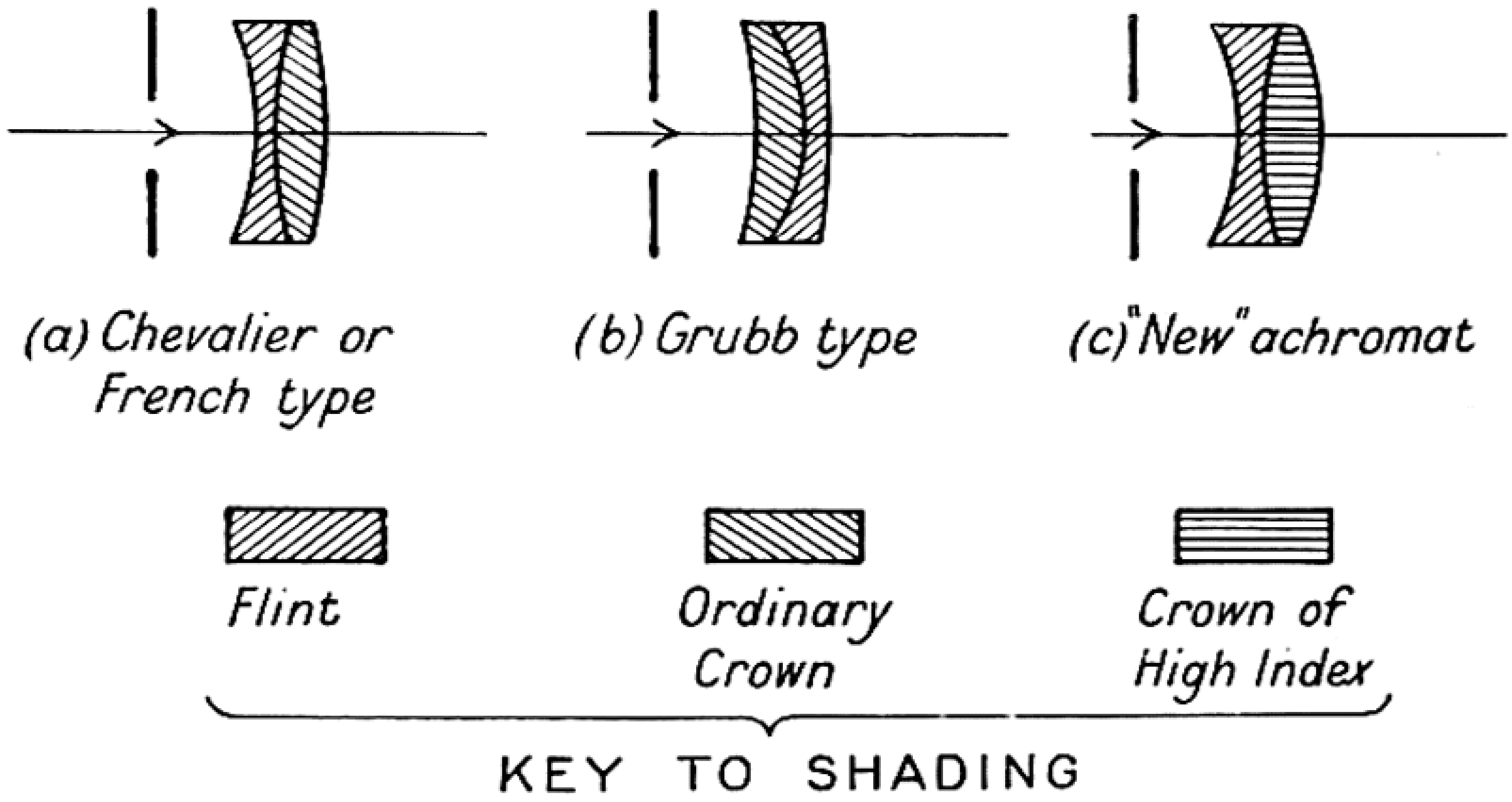

(a) Chevalier or French type
(b) Grubb type
(c) "New" achromat

FIG. 154. LANDSCAPE LENSES

Some of the more recently produced glasses have different optical properties. Let us compare the following combinations—using typical glasses. We will use α, β, and γ to denote the relative partial dispersions for $C - d$, $d - F$, and $F - g$, respectively.

	Glass	n_d	V	α	β	γ
1	Hard crown . . .	1·5186	60·3	0·305	0·695	0·544
	Dense flint . . .	1·6041	37·8	0·295	0·705	0·582
2	Dense barium crown .	1·6089	57·4	0·305	0·695	0·545
	Extra light flint . .	1·5472	45·8	0·300	0·700	0·563

We notice that in the second pair, the glass with the higher refractive index has the higher V value (or lower relative dispersion). Calculating an ordinary achromatic combination from each, we find for unit focal length (a and b standing for crown and flint respectively)—

Combination 1 $\begin{cases} f'_a = 0·373 \\ f'_b = -0·595 \end{cases}$
(Old achromat)

Combination 2 $\begin{cases} f'_a = 0·2021 \\ f'_b = -0·2533 \end{cases}$
(New achromat)

When used in the form of flint-in-front menisci behind the stop, they have the general form of (*a*) and (*c*), Fig. 154. We have much shorter focal lengths and heavier curves in the units of combination 2, a type of "new achromat." Moreover, the contact face has a positive power, and will not give negative higher aberration to counteract the general under-correction. On the other hand, the closer correspondence between the relative partial dispersions of the glass used in the new achromat would ensure a less pronounced secondary spectrum. (See Vol. I, p. 243.) Calculating the radius of the Petzval surface we find R = − 1·38 for the old achromat, and R = − 1·94 for the new. Hence the image field, if corrected for astigmatism, will be much flatter with the new achromat than with the old.

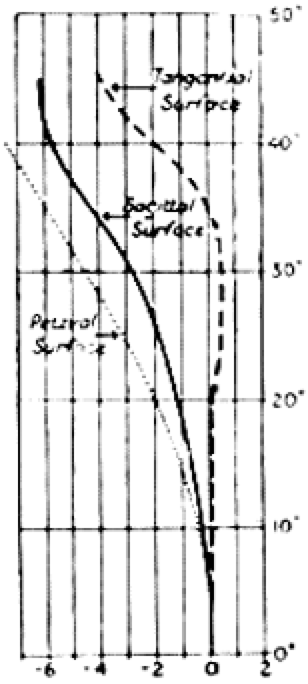


FIG. 155
ASTIGMATIC IMAGE FIELD FOR NEW ACHROMAT

The field corrections of these achromatic landscape lenses follow similar lines to those of the simple lens discussed above. Numerical study of a Chevalier type lens showed that a slightly hollow tangential field can be associated with a satisfactory balance for coma, but the secondary astigmatism is of the same sign as the primary and no balance is possible in this respect.

On the other hand, an achromat made of the newer glasses in which the refractive index of the positive member is the higher can be given a very flat tangential field, up to more than 30° with the axis. The sagittal field is slightly round in the case illustrated in Fig. 155 (which was calculated for a landscape lens having a focal length of 12·8), but it bends back and crosses the tangential field at an angle of just over 50° from the axis.

A lens of this kind is subject to heavy spherical aberration, but if used at an aperture of *f*/16 the residual defects are not great enough to prevent a fairly good performance over a very wide field, i.e. up to 40° or more with the axis. It is useful for landscape photography, where fairly long exposures can be given.

Graphical Representation of the Aberrations of Photographic Lenses. The multiplicity of photographic lenses makes it desirable to have a uniform method of expressing their aberrations. The kind of scheme used by Merté[4], adapted from von Rohr[5], is exemplified in Fig. 156, in which curves are given for a "French" achromatic landscape lens used behind a stop.

In order to make the lenses comparable, the results are given in each case for a lens made on a scale which would give it a focal length of 100 mm. for the D line. Then the ordinates of the spherical aberration curves represent the incidence height in millimetres of the incident ray (parallel to the axis); the abscissae of the broken line curves are the longitudinal aberrations* of the focal length, and those of the full line curves represent the aberrations of the axial intersection distances of the rays.

In the curves showing the astigmatism, the ordinates represent the semi-angle of the field, i.e. the angle between the axis and the principal ray in the image space.† The abscissae of the full line curves give the distances by which the focusing screen must be removed from the axial focus position, to bring the sagittal bundles into focus; the broken line represents the corresponding distance for the tangential bundles.

The *coma* is not plotted directly in these diagrams, but it will now be shown that a measure of the coma is obtained by finding the horizontal distance between the two curves, dotted and full line, in the spherical aberration diagrams. From the relatively small intercepts in the curves, it will appear that the elimination of coma is regarded as one of the most important conditions to be secured in the design of such lenses.

The *distortion* is plotted in terms of the percentage value (see above, p. 199) at various angles in the field. Note, in Fig. 156 the under-corrected primary spherical aberration, the over-corrected tangential field curvature, and the comparatively large distortion.

In spite of the very considerable information which such curves impart, it is, however, only partial. The performance of a lens may be greatly affected by higher oblique spherical aberration which, unlike the ordinary spherical aberration, increases with the angle of the field; moreover, the tangential and sagittal focusing distances for rays at finite aperture may differ considerably from those plotted for very narrow fans; also the usual curves do not specify the chromatic corrections of the lenses, axial or oblique, and give no information on the chromatic variation of the various aberrations. It is therefore not possible to make a *final* judgment of the performance of any lens from such diagrams.

The Sine Relation and Coma. The investigation of the "optical sine relation," Vol. I, p. 104, showed that the dimensions of object and image are governed by the relation

$$nh \sin \alpha = n'h' \sin \alpha'.$$

---

* In these curves, under-correction for both spherical aberration and astigmatism is indicated by a negative sign. This convention conflicts with Conrady's convention, used in this book elsewhere.

† Von Rohr used the "grade" as an angular unit; this divides the right angle into 100 parts. At one time it was expected that this unit of angular measure would supersede the degree, but it has not come into prominence so far.

Consider an axial object point at a very great distance l from the first principal point of the system, then if the incidence height of a ray from this point, on arrival in the system, is y, we have (very nearly)

$$\sin\alpha = y/l,$$

and thus,

$$h' = \frac{h}{l}\left(\frac{n}{n'}\right)\frac{y}{\sin\alpha'}$$

$$= -\frac{y}{\sin\alpha'}\cdot\frac{n}{n'}\cdot\tan\omega$$

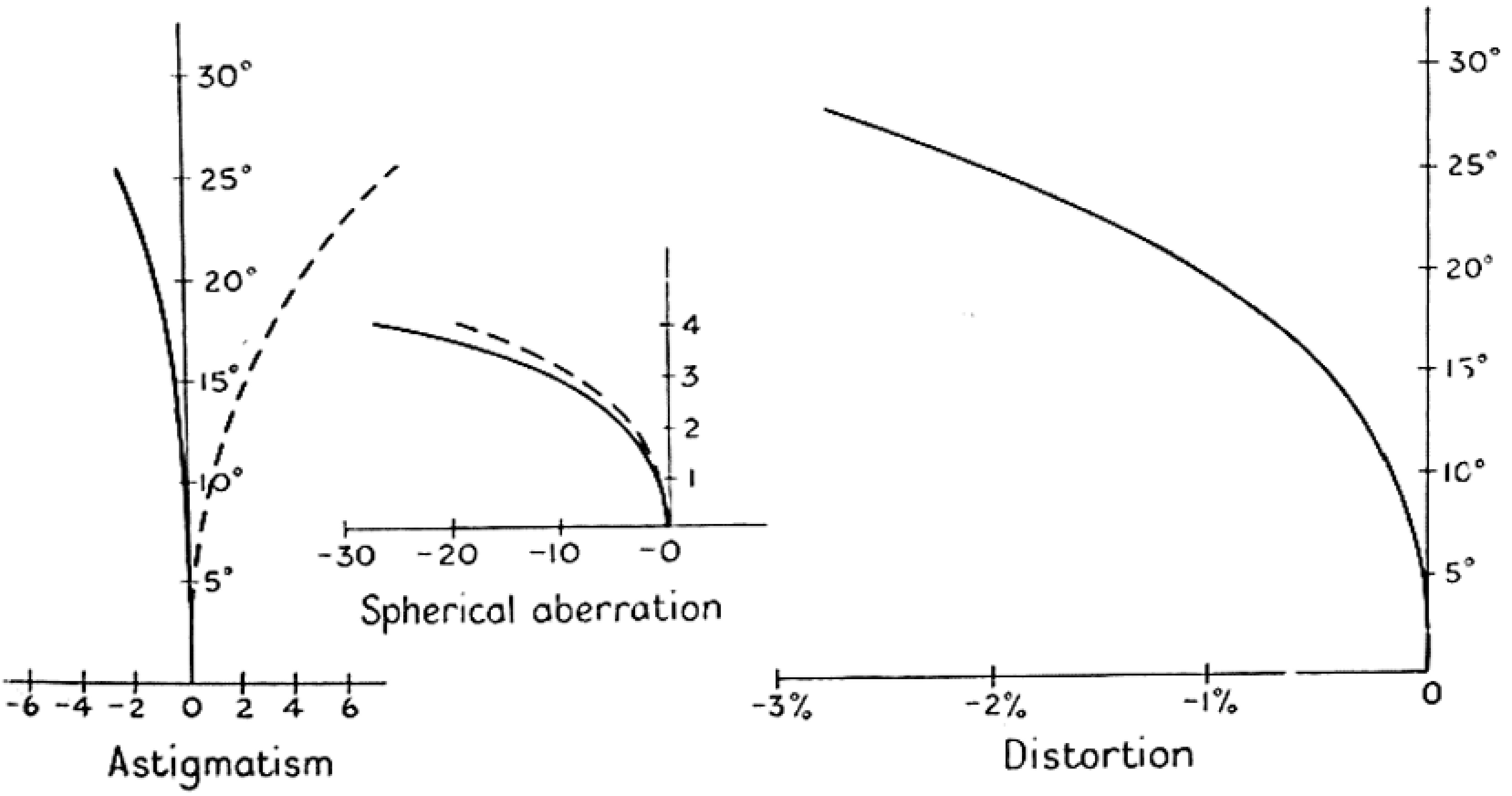


FIG. 156. TYPICAL ABERRATION CURVES FOR A LANDSCAPE LENS (CHEVALIER TYPE)

where ω is the angular distance of the object point from the axis.

This shows that for a given value of h, the different zones will produce an image of the same size if $\frac{y}{\sin\alpha'}$ is constant. Since by our ordinary conceptions of "focal length" we have

$$h' = f\tan\omega \quad \text{(Vol. I, p. 50)}$$

$$= -f'\left(\frac{n}{n'}\right)\tan\omega$$

we may therefore interpret the quantity $\left(\frac{y}{\sin\alpha'}\right)$ as the equivalent focal length of the zone under consideration, and the condition that images formed by successive zones of the system shall be of the same size is that $\left(\frac{y}{\sin\alpha'}\right)$ be constant. Provided the system is free from spherical aberration, this condition is used to secure the absence* of *coma* in the regions near the axis.

If coma is present, let h'_m be the "height" of an image for a marginal zone, and let h' be the corresponding value for a paraxial zone; then the amount of the coma is reckoned by the fraction—

$$\text{Measure of coma} = \frac{h' - h'_m}{h'}$$

$$= 1 - \frac{h'_m}{h'} = 1 - \frac{y_m \sin \alpha'}{y \sin \alpha'_m}$$

$$= 1 - \frac{f'_m}{f'} \quad . \quad . \quad . \quad . \quad . \quad (5.12)$$

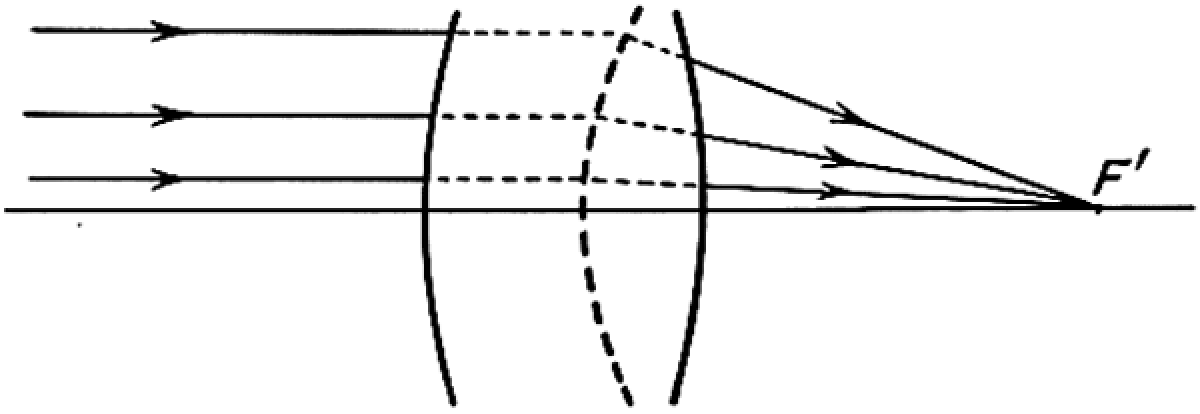


FIG. 157. INTERSECTION OF INITIAL AND FINAL RAY PATHS IN A COMA-FREE LENS

the suffix m being used to denote "marginal zone" values. This measure of coma is also a measure of the "offence against the sine condition."

Still dealing with very distant object points, and taking a number of incident parallel rays into a system at different distances from the axis (Fig. 157), we see that the constancy of $\frac{y}{\sin \alpha'}$ will require that the incident and emergent rays shall intersect each other in a spherical surface. Also if the conjugate distances are finite, constancy of magnification calls for fulfilment of the sine condition,

$$\frac{\sin \alpha}{\sin \alpha'} = \text{constant.}$$

If the initial and final rays when produced intersect each other in a point P, the axial object and image points being B and B′ respectively, then the above condition results in the relation

$$\frac{\text{B}'\text{P}}{\text{BP}} = \text{constant.}$$

---

* Some degree of higher coma may still be present. See Conrady's textbook,[3] p 371.

The locus of P is therefore a circle (Euclid VI, 3), if the system is to be free from coma as well as spherical aberration.

Meaning of the Sine Relation in the Presence of Spherical Aberration. In Fig. 158 we represent a lens system with the apex A of the last surface, the stop, or exit pupil, with centre R, and the marginal and paraxial image points on the axis by B'_m and B' respectively. The focal surfaces for a marginal and paraxial zone of the exit pupil will be (sufficiently nearly) planes in the neighbourhood of the axis, and we may represent the corresponding extra-axial image points formed by these marginal and paraxial zones by C'_m and C' respectively, where $B'C' = h'$ and $B'_mC'_m = h'_m$.

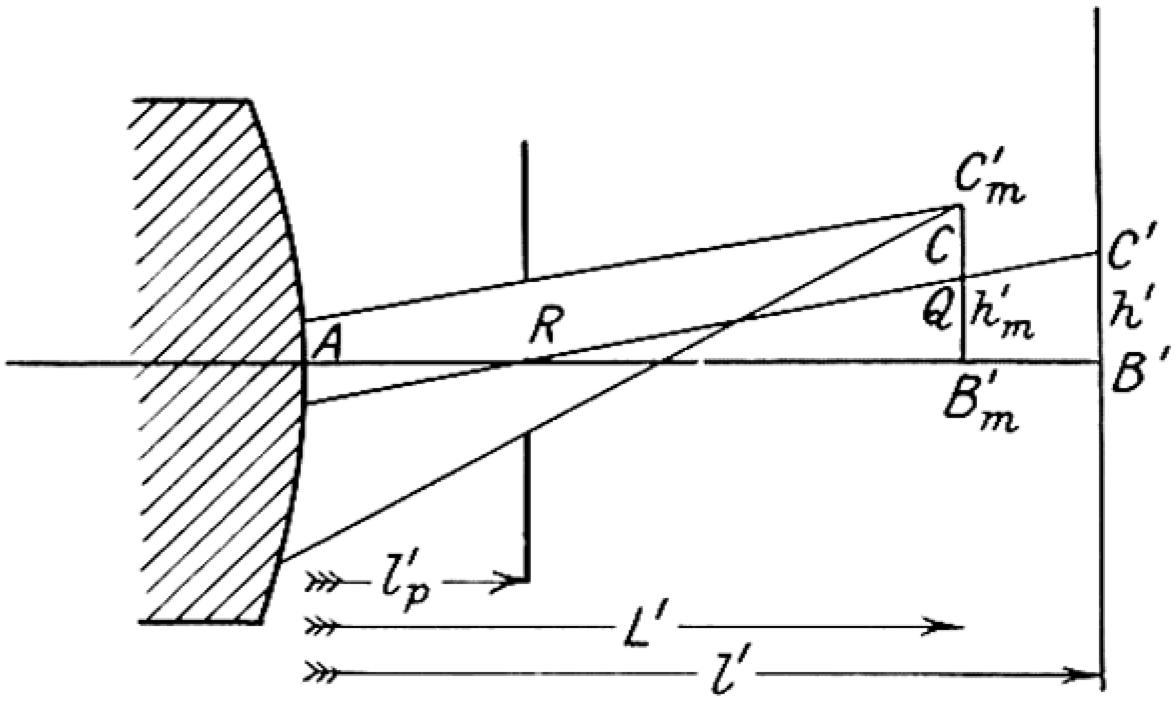


FIG. 158

Now it is clear the amount of the *coma* under the present condition is the vertical intercept $c = QC'_m$ between the line RC' and the point C'_m

Hence in the presence of spherical aberration,

$$\text{Measure of coma} = \frac{c}{h'}$$

But

$$c = h'_m - B'_mQ$$

and from the similar triangles RB'_mQ and $RB'C'$, we find

$$B'_mQ = B'C'\left(\frac{RB'_m}{RB'}\right)$$

Let AB' be denoted by l', and AR (the distance of the exit pupil from the apex) by l'_p; also let the distance AB'_m to the marginal focus be L'; then

$$B'_mQ = h'\left(\frac{L' - l'_p}{l' - l'_p}\right)$$

Thus,

$$\text{Measure of coma} = \frac{c}{h'} = \frac{h'_m}{h'} - \frac{L' - l'_p}{l' - l'_p}$$

Writing
$$L' - l'_p = (l' - l'_p) - (l' - L')$$

we obtain
$$-\frac{c}{h'} = \left(1 - \frac{h'_m}{h'}\right) - \left(\frac{l' - L'}{l' - l'_p}\right)$$
$$= \left(1 - \frac{f'_m}{f'}\right) - \left(\frac{l' - L'}{l' - l'_p}\right)$$
$$= \left(\frac{f' - f'_m}{f'}\right) - \left(\frac{l' - L'}{l' - l'_p}\right)$$

This equation bears a close relation to the actual coma, being dependent on the spherical aberration. Compare equation (5.10).

Now if the stop is close to the second principal surface of the lens, then f' will not be very different from $(l' - l'_p)$, when the object is at infinity. Hence very nearly.

$$\text{Measure of coma} = \frac{(l' - L') - (f' - f'_m)}{f'} \qquad . \qquad . \quad (5.13)$$

$$= \frac{\text{(axial spherical aberration)} - \text{(difference of focal lengths)}}{\text{Focal length}}$$

Thus the measure of the coma is found as mentioned above, by the horizontal intercepts between the "spherical aberration" curves, but only under the limitations stated.

Anastigmatic Correction. The Petzval sum of a lens in air is (Vol. I, p. 137)

$$\frac{1}{R} = -\frac{\mathbf{F}_a + \mathbf{F}_b}{n}$$

where R is the radius of the Petzval surface and $\mathbf{F}_a$ and $\mathbf{F}_b$ are the powers of the two faces. If the lens is infinitely thin all over, then the power is zero,

$$\mathbf{F}_b = -\mathbf{F}_a,$$

and the Petzval sum is zero also. However, by thickening the lens without changing the powers of the surfaces, the lens can be given a positive power without increasing the Petzval curvature; since

$$\mathbf{F} = \mathbf{F}_a + \mathbf{F}_b - \mathbf{F}_a\mathbf{F}_b d$$

where $\mathbf{F}$ is the power of the combination and d is the reduced thickness. Thus

$$\mathbf{F}_b = \frac{\mathbf{F} - \mathbf{F}_a}{1 - \mathbf{F}_a d},$$

and the Petzval sum becomes

$$\frac{1}{R} = \frac{1}{n}\left(\frac{\mathbf{F}_a^2 d - \mathbf{F}}{1 - \mathbf{F}_a d}\right) \qquad . \qquad . \qquad . \quad (5.14)$$

so that the Petzval surface will be flat if

$$\mathbf{F} = \mathbf{F}_a^2 d \quad . \quad . \quad . \quad . \quad . \quad (5{\cdot}15)$$

To take a numerical example let $\mathbf{F} = 1\text{D}$, $\mathbf{F}_a = -10\text{D}$, then d (for a flat field) $= 0{\cdot}01$ m. Thus if the glass has a refractive index of 1·52, the required thickness will be 1·52 cm. For a similar general reason the components of many photographic lenses tend to be in the form of thick menisci with roughly equal curvatures on their outer faces.

A similar flattening of the Petzval surface will be obtained by the separation of the two relatively thin parts of a compound lens. It will be seen at once that a thin lens of power -10D followed at a distance of 1 cm. by a second thin lens of power $+10\text{D}$ would give a combination of power $+1\text{D}$, with a flat Petzval surface. We do in fact find many photographic combinations using this principle, as will be seen below.

The equation for the axial achromatism for a pair of separated thin lenses is obtained in Appendix IV, p. 380; if the separation is small the expression is

$$V_b f_b' + V_a(f_a' - 2d) = 0,$$

whence

$$d = \tfrac{1}{2}\left(f_a' + f_b'\frac{V_b}{V_a}\right) \quad . \quad . \quad . \quad (5{\cdot}16)$$

from which it is easily seen that for a given pair of glasses and a given positive value of f_a', an increasing value of d will call for a shorter (negative) focal length for b. The condition that the Petzval sum may vanish for a pair of two thin lenses is

$$\frac{1}{f_a' n_a} + \frac{1}{f_b' n_b} = 0$$

whence, combining the two latter equations,

$$d = \frac{f_a'}{2}\left(1 - \frac{n_a/V_a}{n_b/V_b}\right) \quad . \quad . \quad . \quad (5{\cdot}17)$$

which indicates that we shall be able only to obtain closely-spaced achromatic combinations ($d \rightarrow 0$) with a low Petzval sum if we can find pairs of glasses for which $n_a/V_a = n_b/V_b$. This is of course far from being the case with the old *hard crown* and *dense flint* combinations. Using the convention for specifying optical glass in which the first three decimal places of the refractive index are followed by three significant digits of the V number, we can compute the separations, *d*, required to give zero Petzval sum in achromatic combinations and make the following table—

Glass pair		d/f'_a
Hard Crown, Dense Flint . .	520600, 620320	0·22
Dense Barium Crown, Light Flint .	612585, 578407	0·14
Special Barium Flint, Light Flint .	744447, 578407	0·0

It is thus possible by using glasses of high refractive index and relatively low dispersion for the "crown" member, to obtain relatively thin achromatic combinations with a zero Petzval sum—or at least a greatly reduced value; if, however, the older glasses are used, the Petzval sum can only be lowered by the use of relatively thick lenses with outward meniscus shape (that is, if a cemented doublet is required) or by well-separated thin lenses. It will be very instructive for the student to examine the possibilities associated with other pairs of glasses. Although, in practice, the Petzval sum of actual photographic lenses is not invariably zero,* it will generally be found to be small; otherwise any attempt to flatten the tangential field will lead, as in the landscape lens, to considerable amounts of primary astigmatism.

Piazzi-Smyth Lens. The Petzval portrait objective suffers from a considerable roundness of field in the sense of under-correction. In 1874 Charles Piazzi-Smyth[6] used a simple dispersing lens, placed comparatively near the photographic plate, to improve the definition. In such a position it had comparatively little effect in introducing other aberrations. The combination constituted the first high-intensity anastigmat system, but it was nearly seventeen years later that the first closely spaced anastigmats appeared. Negative field flattening lenses are occasionally used in modern systems.

Relations between the Primary Aberrations. It is shown in Conrady's book that the primary coma contribution (CC′) given by any surface to the final total coma is given by

$$CC' = SC'\left(\frac{i_{pr}'}{i'}\right)u_k' \quad . \quad . \quad . \quad (5.18)$$

where SC′ is the contribution to the axial spherical aberration, i_{pr}' and i' are the angles of refraction for the principal ray and the paraxial type ray traced for the full nominal aperture; and u_k' is the final convergence angle of this latter ray.

Again the primary astigmatic contribution (AC′) given by any surface to the total final astigmatism is given by

$$AC' = SC'\left(\frac{i_{pr}'}{i'}\right)^2 . \quad . \quad . \quad . \quad . \quad (5.19)$$

---

* It is often considered (mainly on empirical grounds) that a negative Petzval radius numerically about ten times the focal length should be allowed.

From these equations it is seen at once that both the primary coma and astigmatisn arising at a surface grow with the spherical aberration, but they are also controlled by the relative angle of incidence of the principal ray. The astigmatism always has the same sign as the primary spherical aberration, but this is not necessarily the same for the coma.

A formula for the primary spherical aberration contribution SC′ due to a single surface (j) of a system is as follows—

$$SC' = \tfrac{1}{2} l_k'^2 \frac{y_j^2}{y_k^2} \frac{N_j'}{N_k'} \left\{ \frac{(1 - r_j/l_j')}{r_j} (i_j' - u_j)(i_j - i_j') \right\} \quad . \qquad (5.20)$$

The causes of the growth of the primary aberration can easily be studied. Note that when l_j' approaches equality with r_j the angles of incidence and refraction i_j and i_j' both become small; moreover, when i' approaches equality with u we are near the case of aplanatic refraction when not only the spherical aberration but also the coma and astigmatism will vanish.

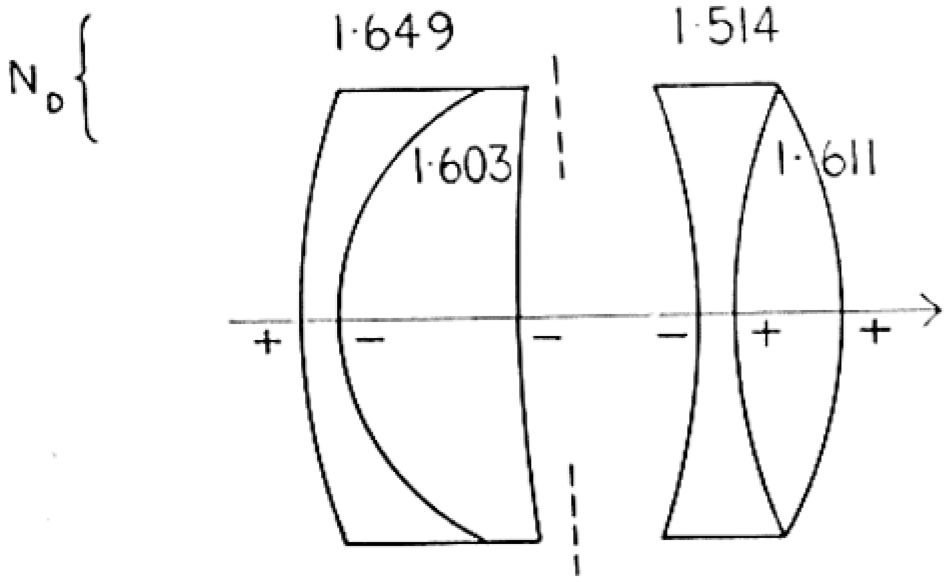


FIG. 159. EARLY PROTAR LENS $f/12{\cdot}5$ (RUDOLPH)

It can be shown further, that the distortion contribution of a surface depends on the Petzval sum, as well as on the astigmatism, coma, and spherical aberration, but the discussion of the relation would require more space than can be given to it here. We only require the foregoing equations to indicate that the primary aberration contributions can easily be calculated from the results of paraxial ray tracing, while the total aberration contributions can be obtained from the results of tracing rays at finite apertures. Thus we can assess the higher aberrations arising at a surface by finding the differences.

The Protar Lens. Designed by P. Rudolph of Messrs. Carl Zeiss, and first made about 1891, this lens is mainly of historical interest as illustrating the approach to the problem of making an anastigmat lens of higher aperture. In the Protar the first doublet, Fig. 159, has an "extra dense flint + dense flint" combination with a contact face of negative power capable of acting in a manner similar to that in the Grubb lens; the second doublet brings a "crown of high dispersion" into contact with a "dense barium crown," the contact having positive power. The total power of the first doublet is negative; it acts as a correcting member to the second. If we calculate for a lens of focal length 100 mm. the contributions* to the primary spherical aberration characteristic of a ray at an aperture of about *f*/16 we shall obtain the following results, expressed in wave-lengths of optical path difference with reference to the paraxial path.

SPHERICAL ABERRATION (PRIMARY) *f*/16

Surface	Contribution	Surface	Contribution
1	0·905	4	− 1·50
2	− 2·06	5	0·89
3	− 0·50	6	2·63
Net totals	− 1·655		+ 2·02

Final = 0·365 wave-lengths (under-corrected).

This amount of aberration is not excessive for a photographic lens; however, calculation of the higher aberration contributions show that the under-correction is a little reduced by higher negative aberration arising chiefly at surface 2. If now to exaggerate the tendency we trace a hypothetical ray at *f*/8 (double the above aperture) and calculate the contributions we find the following results—

SPHERICAL ABERRATION CONTRIBUTIONS *f*/8

Surface	Primary	Total*	Surface	Primary	Total*
1	+ 14·4	+ 15·7	4	− 24·0	− 19·1
2	− 33·0	− 113·9	5	+ 14·2	+ 23·7
3	− 8·0	− 12·3	6	+ 42·2	+ 44·6
	− 26·6	− 110·5		+ 32·4	+ 49·2

Final totals: Primary 5·8λ; Total = − 61·3λ.

* A. E. Conrady (*Dictionary of Applied Physics IV*, p. 215) has given formulae by which it is possible to calculate the difference "axial path *minus* zonal path" for an axial ray track and the track of a disturbance incident at a given zonal radius, from the same object point. The result applies to the path differences at the point where the zonal ray crosses the axis after passing through the surface or system. The differences arising surface by surface are calculated and added together.

By using the nominal angles, etc., characteristic of the paraxial path it is possible (with the above formula suitably modified) to calculate the amount of the *primary* spherical aberration present *at the paraxial focus*.

If the only aberration present is *primary* aberration the two results have the same magnitude, but opposite signs (owing to the different reference points). Hence the difference between the numerical magnitudes of the "primary" and "total" aberration is some indication of the amount of higher aberration arising at each stage, though further calculation would be necessary in order to obtain a satisfactory measure of the higher aberration at the paraxial or other focus.

The higher aberrations now overwhelm the primary contributions. Remembering that the numerical differences are the measure of the higher aberration we notice that the higher aberration arises mainly at the contact faces, and the over-correction is predominant.

When the aberration is expressed in terms of axial intercepts and plotted, the resulting curve is similar to Fig. 160 (*b*). The modest aberration corresponding to *f*/16 is shown, and it will be clear that the over-correction would be enormous at *f*/8. In fact the aperture allowed to this lens was limited to *f*/12·5.

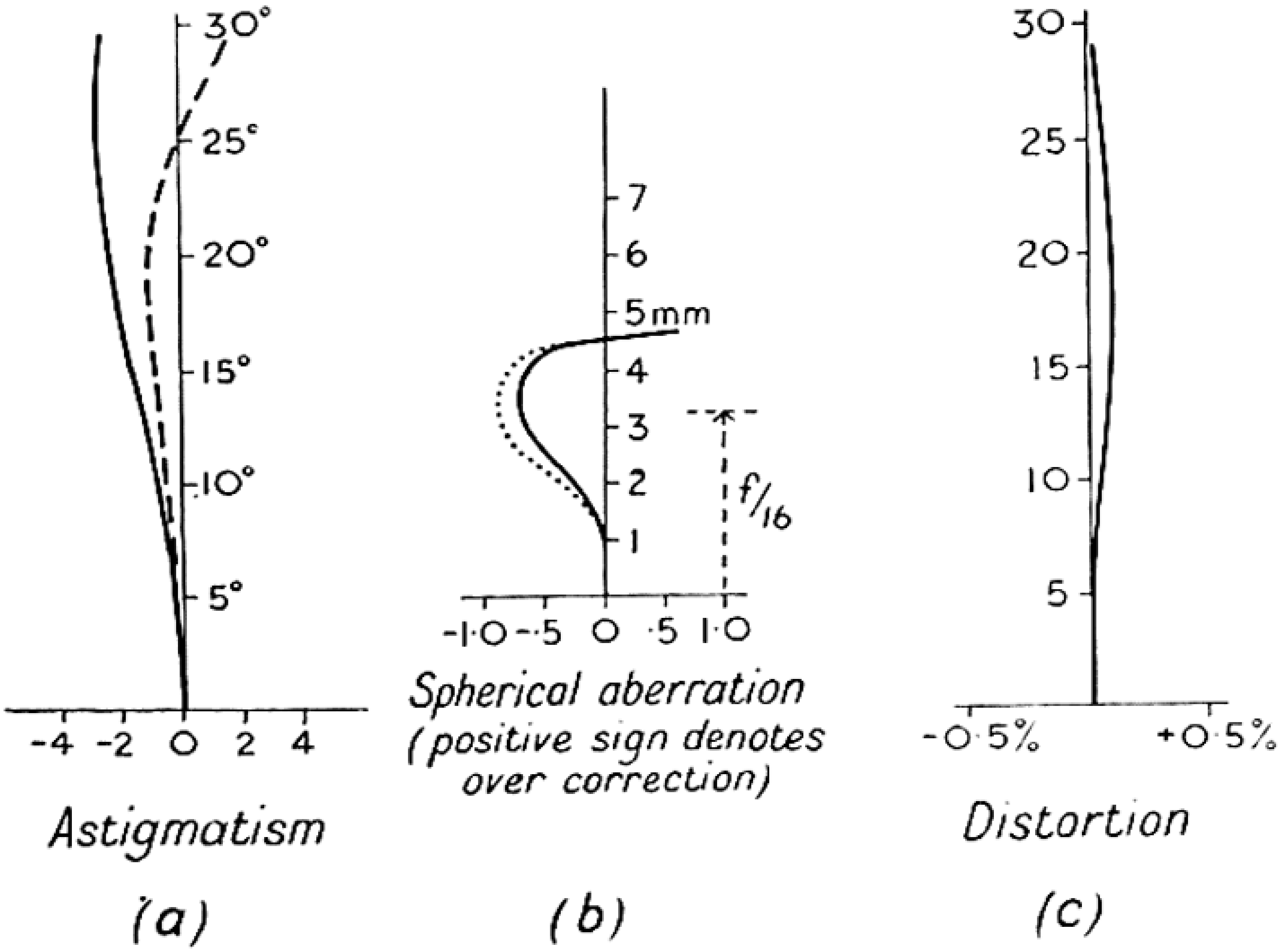


FIG. 160. ABERRATION CURVES FOR EARLY PROTAR

The curves of Fig. 160 (*a*) show that the primary astigmatism, both sagittal and tangential, is subject to the same principle of tendency to balance. However, it is a fairly general rule that the possibility of obtaining such a balance is subject to making the primary aberrations reasonably small to begin with. The horizontal intercept between the curves of Fig. 160 (*b*) disappears at a height of about 4·5, indicating that the coma is also balanced. (See p. 210.)

Note that if the axial path is greater than the marginal path at the paraxial focus we have "under-correction," in the case of primary spherical aberration, and vice versa; but in the case of the total aberration no such simple criterion can be applied to cover all cases. However, in order to prevent the tables in the text from giving a misleading first impression the "Total aberration" will be entered with reversed sign so that the difference may convey at once the general idea of the magnitude of the higher aberration.

Later, about 1894, Rudolph used similar principles of correction in a four-component cemented lens *f*/12·5 which would work on one side of a diaphragm, Fig. 161; or two such lenses could be used as a symmetrical combination with good results (see below). These lenses were also called "Protars."

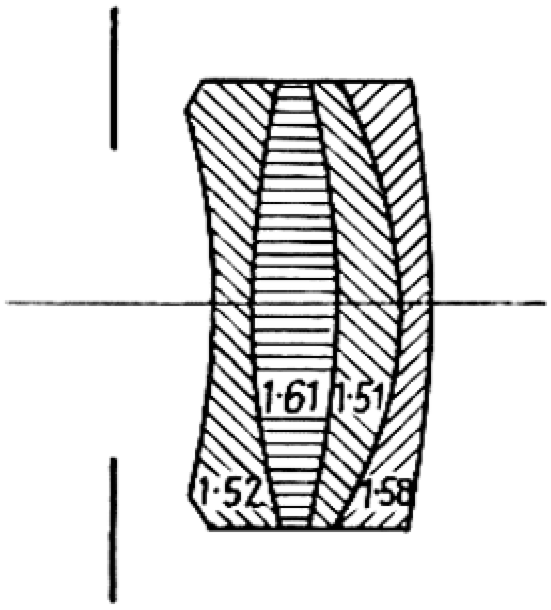


Fig. 161
Later Protar Lens

Symmetrical Lenses. Important advantages are obtainable from the employment of systems which are built symmetrically with a central stop. In Fig. 162, let X and Y be two lenses of such a system; the central stop is at R. Consider two parallel rays, AB and CD, between the lenses, symmetrically situated with regard to R. If these are traced through the lenses their crossing points B_1 and B_1' in the object and image region must also be symmetrical with regard to the system. Also, there must be some ray through the centre of the stop, which will pass through both B_1 and B_1'. Hence the system must be free from coma for one zone at least, when the image size is exactly equal to the object size as must now be the case, further, the exact equality of object and image dimensions must result in the complete elimination of distortion.

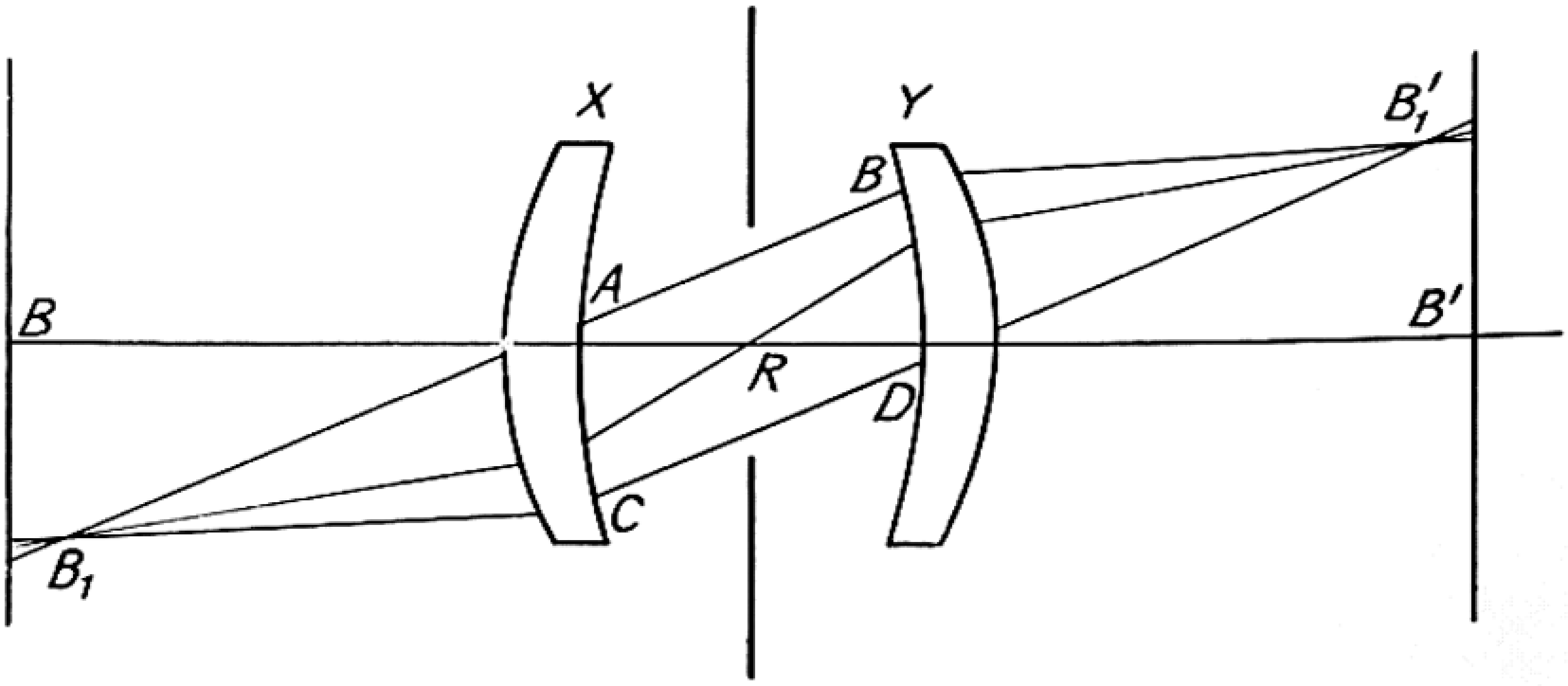


Fig. 162. Symmetrical System

Also, since the above reasoning would apply to rays of differing wave-length, the image must now be free from chromatic difference of magnification.

The symmetry of the system does nothing to remove the effects of axial chromatic aberration, spherical aberration, astigmatism or curvature of the field, but the freedom from coma, distortion, and chromatic difference of magnification is a very important advantage.

Let us now consider the conditions when the object and image are situated at different distances from the system. In the first place the parts of the principal ray outside the lenses will be parallel to each other, so that $\omega = \omega'$ (see equation above). The absence of distortion then requires $\delta = \delta'$, or a constancy of the positions of the images of the centre point of the stop for all inclinations of the principal ray. A similar argument applies to differences of magnification due to colour; in order to avoid these, the position of the entrance and exit pupil should be independent of wave-length. Further, the argument which established the freedom from coma is no longer valid.

In all these cases, however, the symmetry of the system is a very great advantage. If the separate systems are moderately well corrected in themselves they can be combined in the symmetrical manner without fear of introducing serious coma, and with the reasonable assurance that the distortion and any chromatic difference of magnification will be made practically negligible. Moreover, the shortening of the focal length due to the combination produces a lens working satisfactorily, in certain respects at least, at double the relative aperture, or more.

The Hypergon lens of *Goerz* (Fig. 163) consists of two deep menisci, made of one glass, of which the outer surfaces very nearly form a sphere. Owing to the absence of chromatic correction it must be used with a small stop, but the symmetry of the system removes the coma, while the astigmatism, as discussed above, p. 199, can be removed in such a system by a suitable position of the stop and bending of the lenses. The smallness of the stop produces a sufficient depth of focus to allow satisfactory definition on a flat plate, and the lens has a total field of 135° free from coma, astigmatism, and distortion. It is, therefore, suitable for photographic surveying if a long exposure is acceptable.

A "Rapid Rectilinear" lens usually consists of a pair of lenses of the cemented landscape type mounted symmetrically with a stop between them. The system is, however, not so important as it used to be, owing to the round field and lack of the anastigmatic correction which can be attained by lenses of simpler construction as will be seen below.

Hemi-symmetrical Systems. Somewhat better results may in some cases be obtained by altering the scale of the two components, while their separation from the stop is adjusted proportionally. The two lenses are still independently corrected; thus the user of such a system has the choice of three focal lengths, i.e. the combination and either the front or the back component.

The Double Anastigmat. The "Dagor" lens, of which the prototype was designed by von Höegh for the firm of Goerz about 1892, furnishes a good example of a symmetrical anastigmat. It consists of two similar parts in each of which a diverging lens of mean refractive index is cemented between lenses of lower and higher indices respectively; see Fig. 164. Either part can be used separately if desired, or in combination.

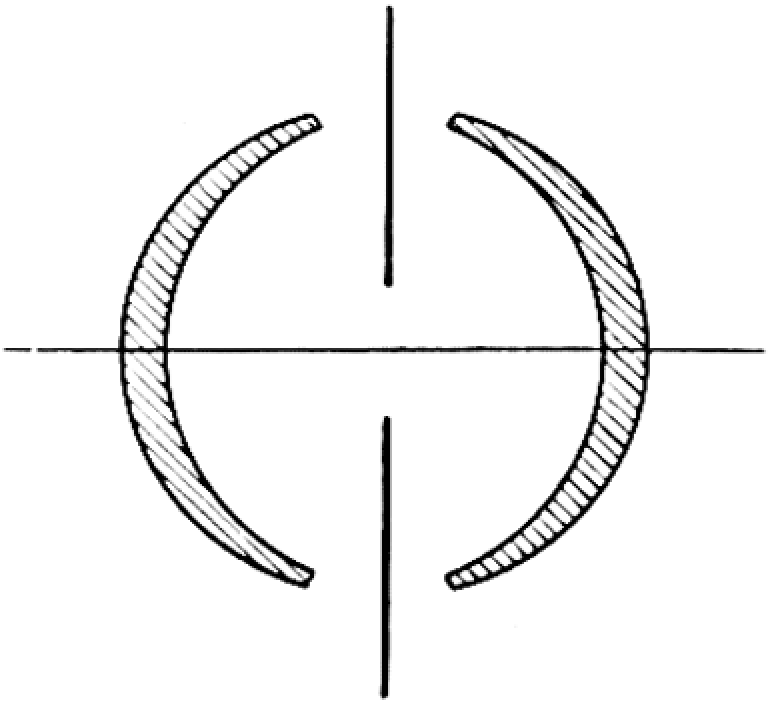


FIG. 163. THE HYPERGON
(Diagrammatic only)

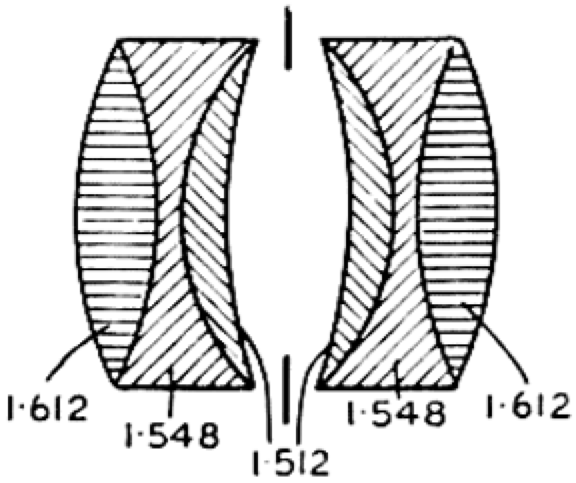


FIG. 164. THE DAGOR LENS

In each part there is a collective contact surface with a fairly large radius and considerable angle of incidence for the principal ray; and there is also a diverging surface with a short radius and modest angle of incidence. Analysis of the contributions made by each surface to the spherical aberration of the final image shows the following results—

SPHERICAL ABERRATION (OPTICAL PATH), ZONE *f*/8

Surface	Contributions		Surface	Contributions	
	Primary	Total*		Primary	Total
1	8·48	9·00	5	− 7·23	− 7·66
2	2·82	3·28	6	− 7·04	− 10·38
3	− 6·22	− 8·81	7	1·21	1·43
4	− 2·24	− 2·27	8	14·52	15·81
Totals .	2·84	1·20		1·46	− 0·8

* Net totals: Primary = 4·30λ, Total = 0·4λ.

Note the relatively large amount of higher negative spherical aberration arising at the deeply-curved negative contact surface. This grows more rapidly with increase of zonal radius than the primary, and causes over-correction for outer zones.

Calculation of the sagittal astigmatism, i.e. the intercept on the principal ray between the Petzval surface and the sagittal focus for narrow aperture, shows the following contributions to the primary aberration for a ray incident at 15° with the axis.

SAGITTAL ASTIGMATISM (PRIMARY)

Surface	Contribution	Surface	Contribution
1	1·817	5	− 4·631
2	1·605	6	− 0·047
3	− 0·043	7	1·099
4	− 2·367	8	2·445
Totals .	1·012		− 1·134

Net total: − 0·122 (over-correction).

While the primary astigmatism is over-corrected, calculation of the total sagittal astigmatism shows a small under-correction due to the "higher" positive contributions of surfaces 1, 2, 7, and 8, which outweigh negative contributions from 4 and 5. The contributions of 3 and 6 are insignificant.

We can further calculate the tangential coma for a principal ray at the same angle with the axis and for an aperture *f*/8. The results are as follows—

TANGENTIAL COMA (PRIMARY)

Surface	Contribution	Surface	Contribution
1	0·520	5	0·768
2	− 0·282	6	0·076
3	− 0·069	7	0·153
4	− 0·306	8	− 0·790
Totals .	− 0·137		+ 0·207

Net total 0·070.

Ray tracing gives a tangential coma 0·067, showing that the higher contributions have almost balanced each other; it will further be noticed how the symmetry of the design gives contributions of opposite sign, and at least of the same order of magnitude, from corresponding surfaces. For this reason coma presents less of a problem to the designer in symmetrical or hemi-symmetrical systems than the other aberrations, especially in cases where the conjugates are equal or in due proportion respectively. Even in cases such as the above where the conjugates are unequal, there is a valuable tendency towards a balance, even for the higher contributions. The general aberrations curves for a lens of the Dagor type are suggested in Fig. 165.

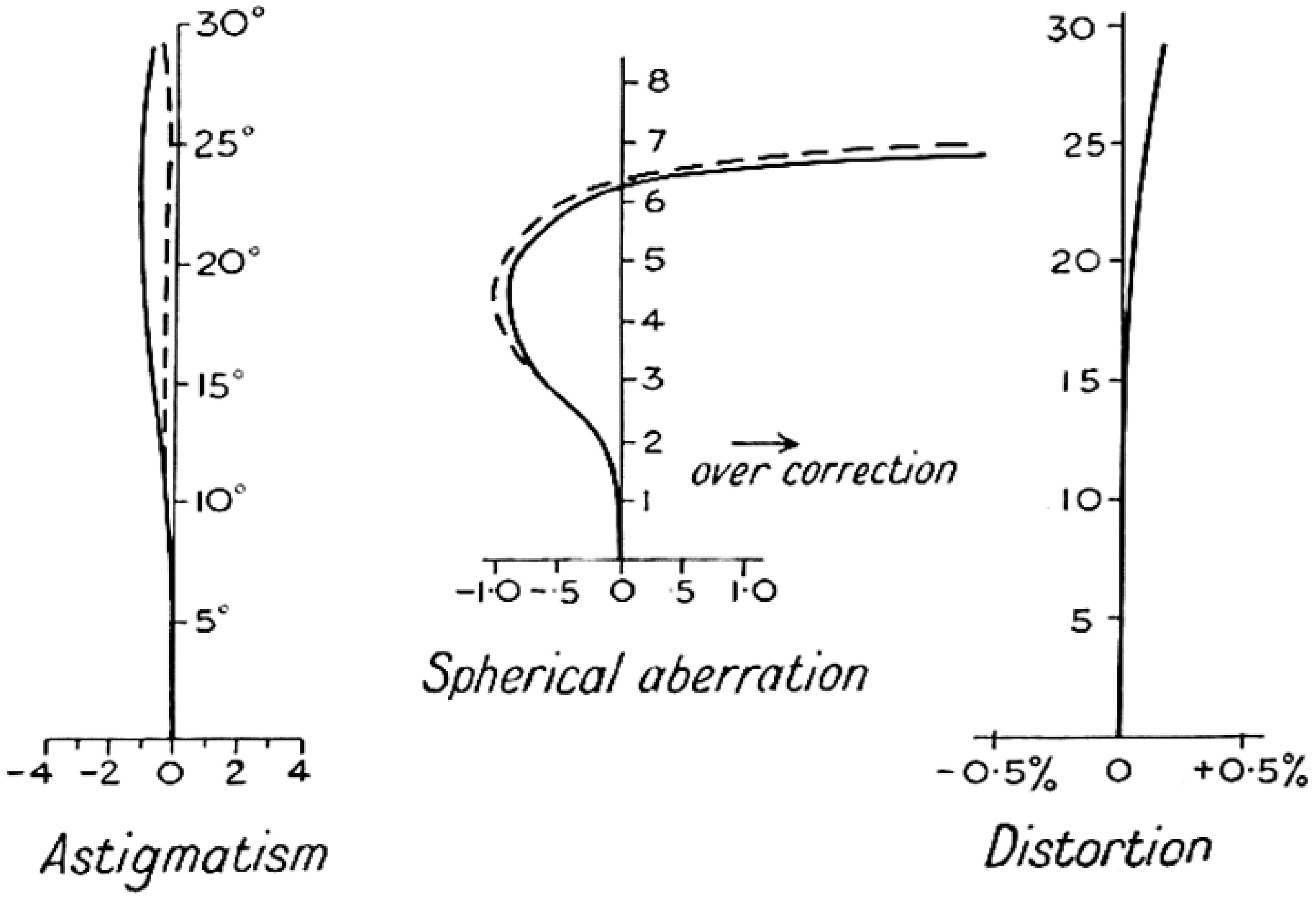


FIG. 165. ABERRATION CURVES FOR A LENS OF THE "DAGOR" TYPE, $f/6{\cdot}8$

Four Component Air-spaced Lenses. The separation of the relatively thin components of an achromatic pair helps to lower the Petzval sum. The "Celor" lens, $f/5$, also designed by von Höegh (in 1899) used double-convex and double-concave lenses, and a number of related designs have been adapted to use later glasses. The Aviar lens, Fig. 166, designed about 1917 by Warmisham (Messrs. Taylor, Taylor & Hobson, Ltd.) has a similar form.

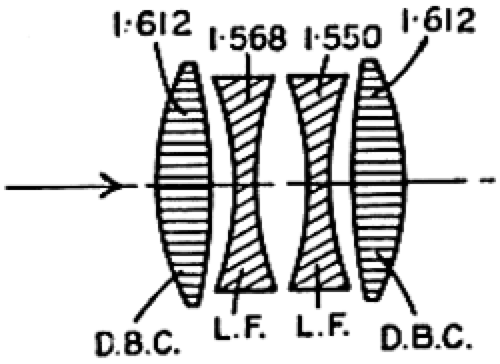
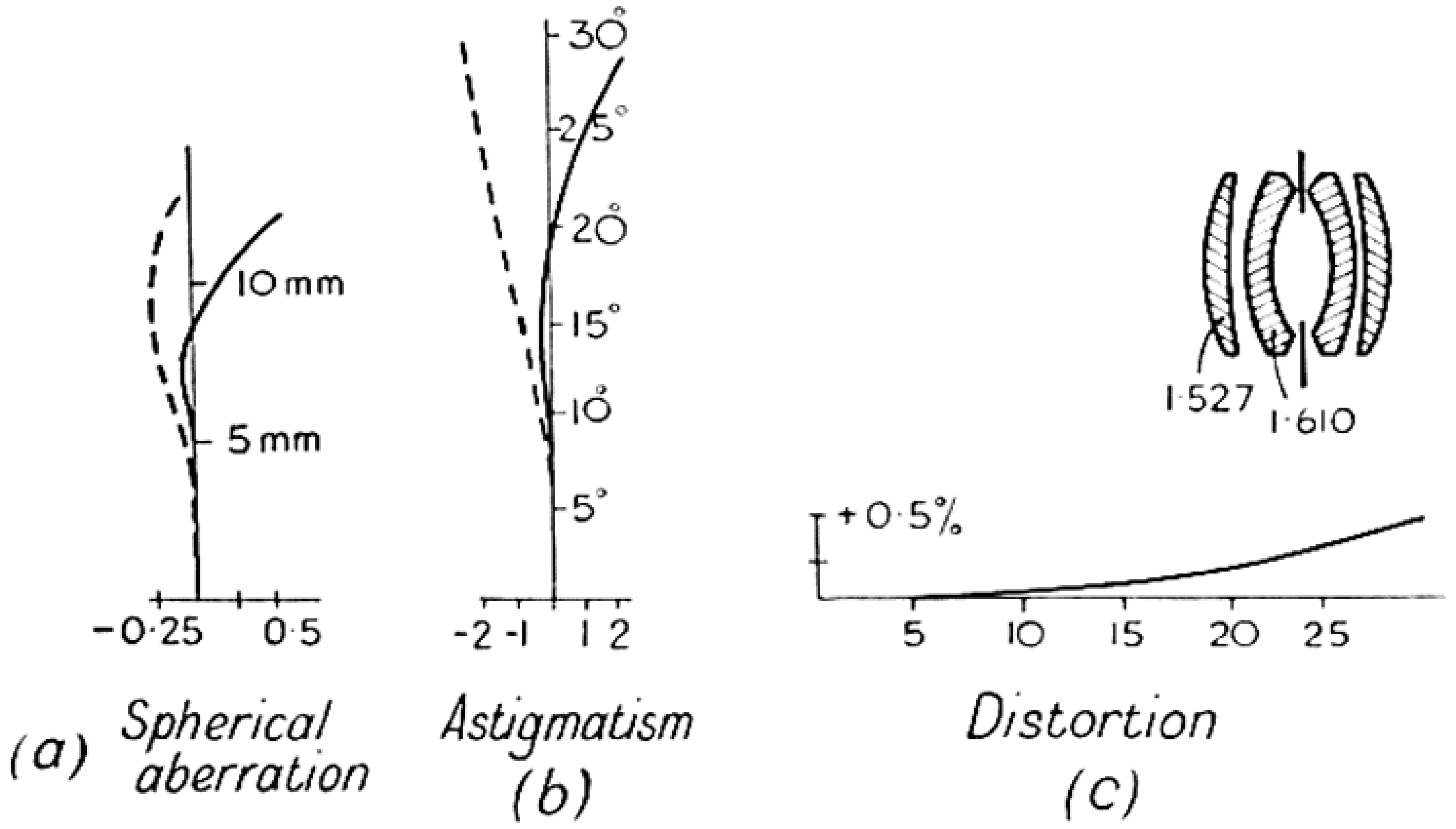


FIG. 166. AVIAR $f/4{\cdot}5$

A second type of four component construction, originated by Alvan Clark about 1899, used two opposed telescope objectives to make a portrait lens. The Gauss telescope objective, which employs a positive and a negative meniscus suitably separated, can be corrected for spherical aberration in two wave-lengths; thus photographic lenses built in this way can be given very similar aberration curves for all colours, and are valuable for process work (the photo-mechanical reproduction of coloured pictures).

A similar general construction is exemplified in the "Omnar" (K. Martin, 1901), the "Aristostigmat," and the "Homocentric" (Messrs. Ross, Ltd.). Data are given in the Appendix for a similar design patented by Rodenstock in 1905; it will be noted that the glasses in this lens are normal crowns and flints; it is *not always* necessary to use crowns of high index for anastigmatic correction. The contributions to the spherical aberration calculated for a ray at $f/10$ are for a focal length of about 100 mm.



(a) Spherical aberration Astigmatism (b) Distortion (c)

FIG. 167. FOUR COMPONENT AIR-SPACED LENS AND TYPICAL ABERRATION CURVES

SPHERICAL ABERRATION (RODENSTOCK, $f/10$)

Surface	Contributions		Surface	Contributions	
	Primary	Total*		Primary	Total*
1	+ 1·59	+ 1·63	5	− 3·90	− 4·03
2	+ 0·01	+ 0·01	6	+ 0·56	+ 0·55
3	− 0·04	− 0·05	7	− 0·01	− 0·01
4	− 1·26	− 1·28	8	+ 3·12	+ 3·23
Totals .	+ 0·30	+0·31		− 0·23	− 0·26

Net totals: Primary = + 0·07λ; Total = + 0·05λ.

The general corrections possible with this type are shown in Fig. 167. The spherical aberration remains very small up to an aperture of $f/5$.

The Cooke Three-component Lens. The original triple lens designed by H. Dennis Taylor in 1894 for the (then separate) firm of Cooke & Sons, York, consisted of three separately achromatized elements. It was later realized by Taylor that the required corrections could be obtained by the use of three separated single lenses.

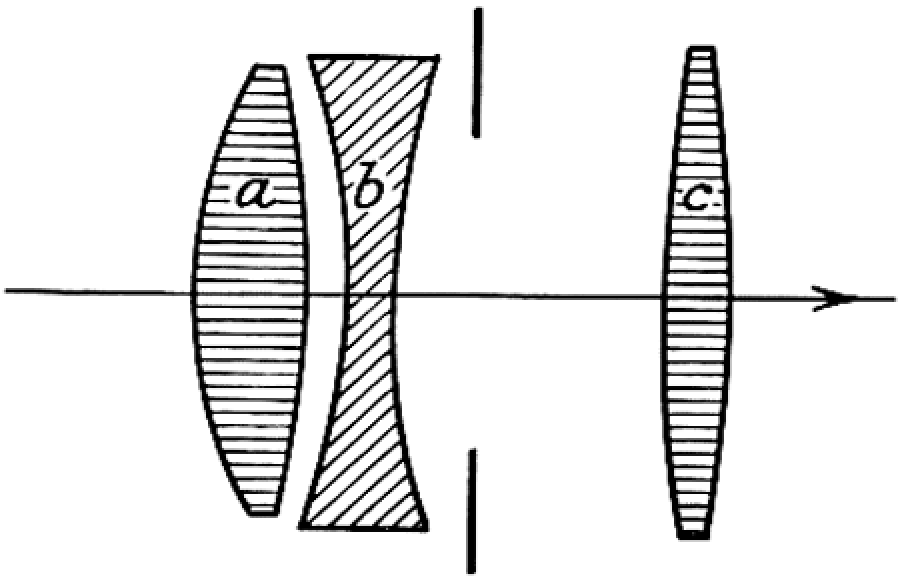


Fig. 168. Cooke Lens

a } Dense barium
c } crown
b Light silicate flint

In its modern forms it can be given an aperture of *f*/8 to cover a fairly wide angle, while for portraiture the aperture can be raised to *f*/3·5. The contributions to the spherical aberration for a typical design, Fig. 168, Appendix V, focal length = 100 mm., are as follows—

Contributions to Spherical Aberration (*f*/8 Zone)

Surface	Primary	Total*	Surface	Primary	Total*	Surface	Primary	Total*
1	+ 10·44	+ 11·18	3	− 17·67	− 20·46	5	− 0·00	− 0·0
2	+ 12·96	+ 14·97	4	− 4·66	− 4·94	6	+ 1·33	+ 1·38
Totals .	23·40	26·15		− 22·33	− 25·40		1·33	1·38

Net Totals: Primary = $2·40\lambda$; Total* = $2·13\lambda$.

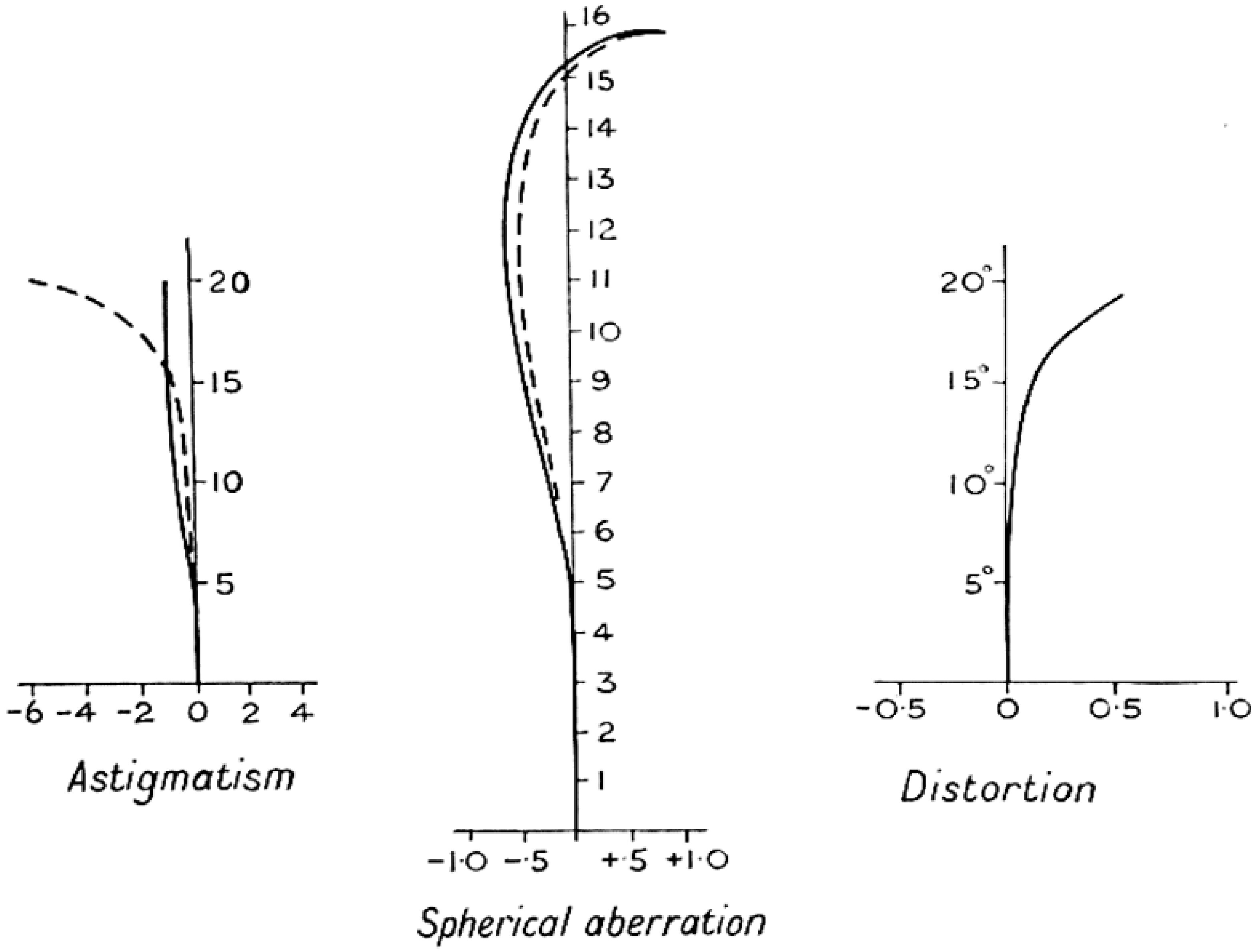
Typical aberration curves for a triplet construction are shown in Fig. 169. The triplet lens illustrates the influence of the two chromatic corrections (for axial achromatism and achromatism of magnification) on the design. Given the equality of final axial focusing positions for two coincident rays of differing colour derived from the same object point, we recall the expression for magnification, i.e.

$$m = \frac{n \sin \alpha}{n' \sin \alpha'}$$

---

* For the significance of the "total" aberration, see p. 214.

If the initial and final media are both air, then $n = n' = 1$ for all colours; therefore for identity of magnifications for two colours of an incident ray we must have the same value of α' for each of them; i.e. the *final* tracks of these two colours must be coincident also; they emerge from the second lens at the same point. Thus the separation in the coloured rays, which is a maximum at the middle lens of the triplet, must be zero when the last surface is reached. The effect is suggested with some exaggeration in Fig. 170. This requirement is common to all photographic systems which are to be free from lateral and axial chromatism.



Astigmatism

Distortion

Spherical aberration

FIG. 169. ABERRATION CURVES FOR TRIPLET LENS (B.P.155640)

Enlarging and Process Lenses. These lenses are made to various designs, though the four-component air-spaced construction is often employed for reasons mentioned above. Strict correction for the two chromatic errors is essential, since the lens must reproduce coloured pictures to the same scale in three differing colours. On the other hand, short exposures need not be contemplated, and a relative aperture of $f/10$ is usually adequate. This should allow of the use of glasses which favour the reduction of the secondary spectrum through the similarity of their relative partial dispersions.

The use of a symmetrical or hemi-symmetrical design will tend to avoid lateral colour.

Portrait and Projection Lenses. The *Petzval* portrait objective (Fig. 171), designed by Petzval as long ago as 1840, secures a large relative aperture, *f*/3, and satisfactory correction of chromatic and spherical aberrations, as well as chromatic difference of magnification. The coma also is small. With an aperture of this magnitude, great pains had to be taken to secure freedom from zonal spherical aberration.

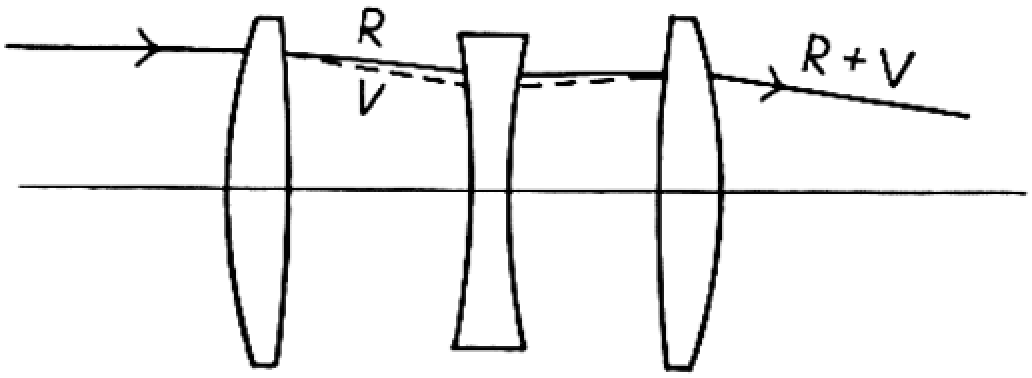


FIG. 170. COOKE TRIPLET LENS SHOWING FINAL COINCIDENCE OF R AND V RAYS AFTER SEPARATION

Since the axial region of the image is well corrected, while the marginal points suffer from astigmatism and curvature of field, the lens is suitable for portraiture, where good definition is usually only necessary over a limited region to include the features of the sitter.

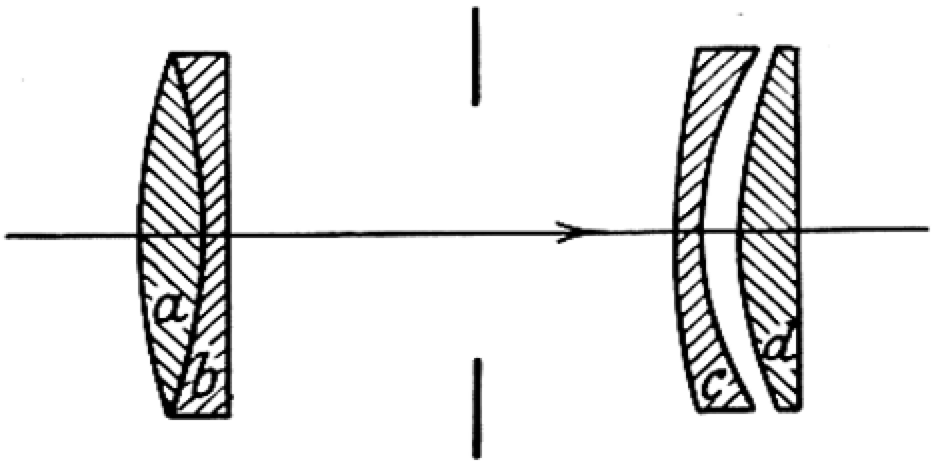


FIG. 171. PETZVAL PORTRAIT LENS

	n_D		
a	1·5181	*c*	1·5783
b	1·5783	*d*	1·5152

Owing to the large separation of the components, there is considerable loss of light away from the axis, due to the restriction of the effective aperture for oblique pencils.

The lens is still in wide use, and is often employed as a projection lens in projection lanterns, and for enlarging.

Modern Lenses. The limitations of the simpler designs, both in aperture and angular field, can usually be overcome only at the expense of some increase in their complexity; though it is possibly true that even the simpler types might give better results with the most suitable choice of materials together with the best possible curves, etc. Such an optimum choice, however, entails a prodigious amount of work which is not always possible.

The choice of a higher refractive index for a lens will usually allow of the use of shallower curves for a given focal length, and will give reduced aberrations. Further, when a single lens is giving undue aberration matters can often be eased by two or more thinner lenses to share the refractive task of the single one. An example of this is seen in the Cooke lens Series X, Fig. 172, which works at *f*/2·5; the rear lens of a Taylor triplet has been separated into two parts.

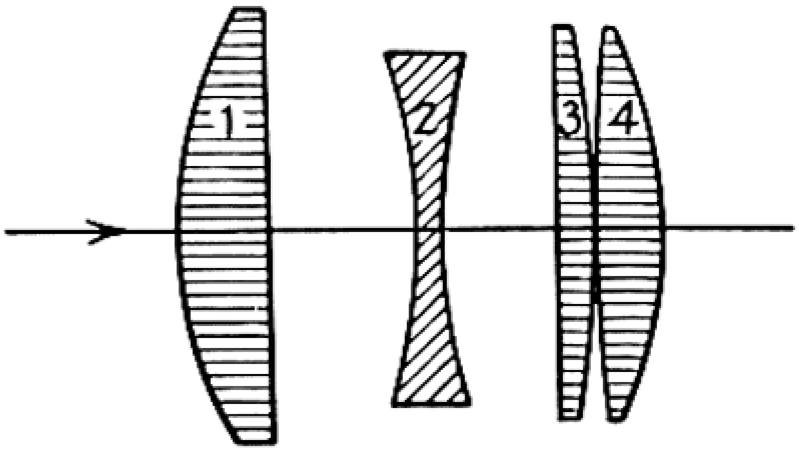


FIG. 172. COOKE LENS, SERIES X, *f*/2·5

	n_D	V		n_D	V
1.	1·613	56·5	3.	1·613	58·5
2.	1·651	33·7	4.	1·613	58·5

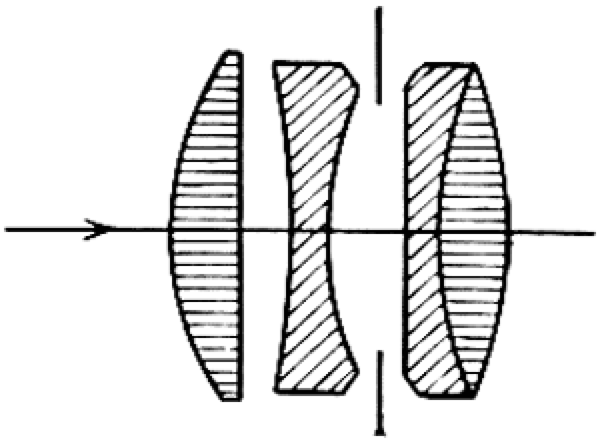


FIG. 173. TESSAR LENS *f*/4·5

Another method is to convert the single lens into a cemented doublet, introducing a difference of refractive index across the contact face. (The colour correction of a lens can also be changed if the two components of the compound lens are given the same refractive indices but differing dispersions.) The Zeiss *Tessar*, *f*/4·5, Fig. 173, may be regarded as related to the Cooke Triplet in this manner, though its designer (Rudolph) described its evolution in another way; the contact surface in the rear element is of positive power. The Tessar has a flatter field than the Triplet, and the oblique spherical aberration is said to be better corrected.

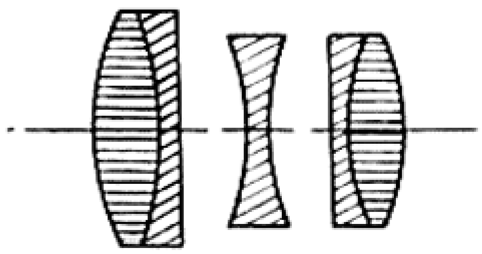


FIG. 174. PENTAC LENS *f*/2·9

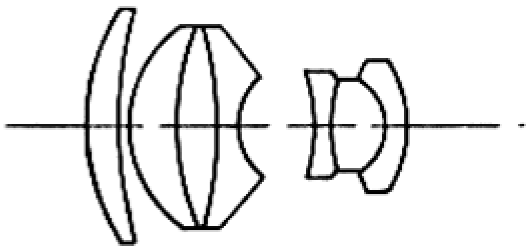


FIG. 175. SONNAR LENS *f*/1·5

The *Pentac* lens, Fig. 174, designed by L. B. Booth for Messrs.

Dallmeyer, uses a cemented doublet for both outer components of the triplet and attains an aperture of $f/2{\cdot}9$.

Further designs have been obtained through elaboration of the triplet construction. In the *Sonnar* (Bertéle) $f/1{\cdot}5$, Fig. 175, we may regard the first component of the triplet as divided into two menisci, and the rear lens becomes a cemented triplet; the negative lens, in the form of a thick cemented meniscus is in contact with the rear face of the first member. Lenses of this type ($f' = 2$ in.) have been useful for miniature cameras using 36 × 24 mm. film.

The four component air-spaced lens was further developed by Lee in the *Opic* $f/2$; a similar design is now produced by Messrs. Taylor, Taylor, & Hobson, Ltd., under the name "*Speed Panchro,*" Fig. 176, and is used for kine-photography; in these lenses the inner negative components have been thickened and transformed into cemented doublets. Other modern lenses, which seem to be related, are the Zeiss *Biotar* $f/2$, the Kodak *Ektar* $f/1{\cdot}9$, the Dallmeyer *Super*-6 $f/1{\cdot}9$, the Ross *Xpres* $f/1{\cdot}9$, the Leitz *Summar* $f/2$, the Bausch and Lomb *Raytar* $f/2$, the Wray *Unilite* $f/2$, together with many others. The extreme apertures now obtainable have made it possible to make kinematograph pictures in difficult lighting conditions. Further developments of the basic "Double Gauss" system have allowed improved field corrections and apertures up to $f/1{\cdot}4$.

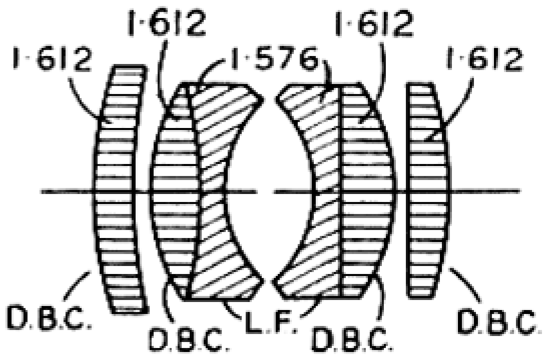


FIG. 176. SPEED PANCHRO LENS $f/2$; SIMILAR TO THE "OPIC"
(Design founded on the Gauss objective)

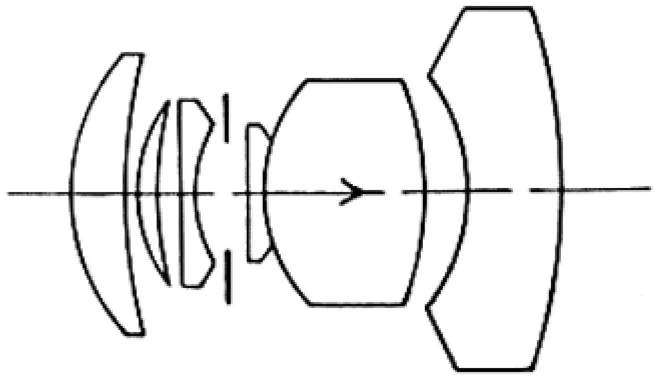


FIG. 177. DESIGN BY BERTÉLE $f/2{\cdot}7$
(Wide angle) suitable for miniature cameras
The back lens nearly touches the film

Even more stringent demands for aperture are made in connection with lenses designed to register the images, on the fluorescent screen, obtained in "mass radiography" and kine-radiography; though the types useful for ordinary kinematography are sometimes employed, there is scope for lenses built somewhat after the plan of a reversed microscope objective. It should be remembered in looking at diagrams of short focus lenses, that the working distance measured from the back lens may be quite small in comparison with the thickness of the lens itself. Generally speaking (as was pointed out at the beginning of this chapter), an increase of the relative aperture beyond a certain limit entails a reduction of the size of field over which correction can be obtained. Fig. 177 shows a design of a lens with a short working distance.

Wide-angle Lenses. It is understandable that wide-angle lenses will have to be used at a modest aperture. Figs. 178 (*b*) and (*c*) illustrate the Ross *f*/4·5 wide-angle lens (this covers a semi-field of 35°), and the Ross *f*/5·5 wide-angle survey lens, which covers a semi-field of 47·5°. This is made in focal lengths from 3¼ in. to 6 in., but the technical difficulties in the production of the deep menisci rapidly increase with their diameters.

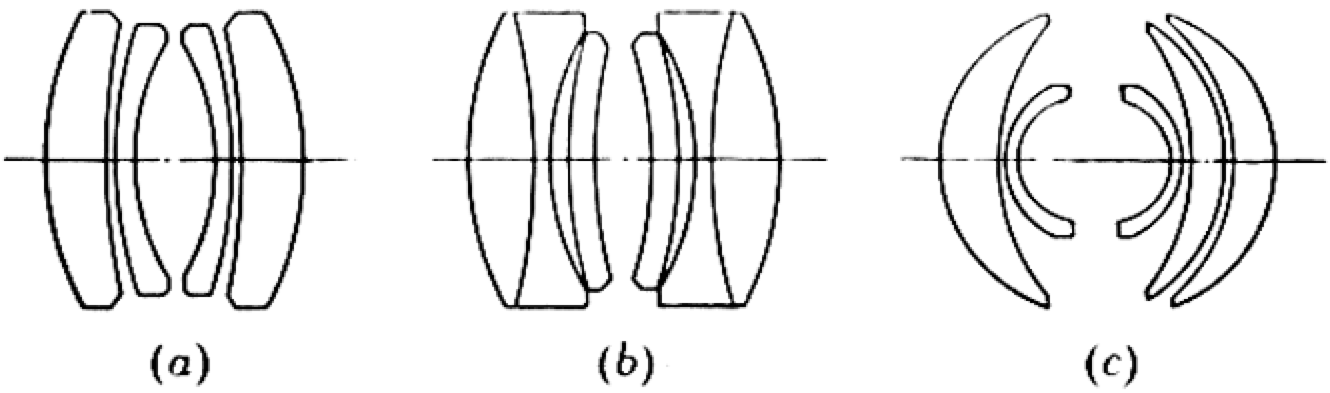


FIG. 178. WIDE-ANGLE LENSES

(*a*) Four component wide-angle lens
(*b*) Ross *f*/4·5 wide-angle lens
(*c*) Ross *f*/5·5 survey wide-angle lens

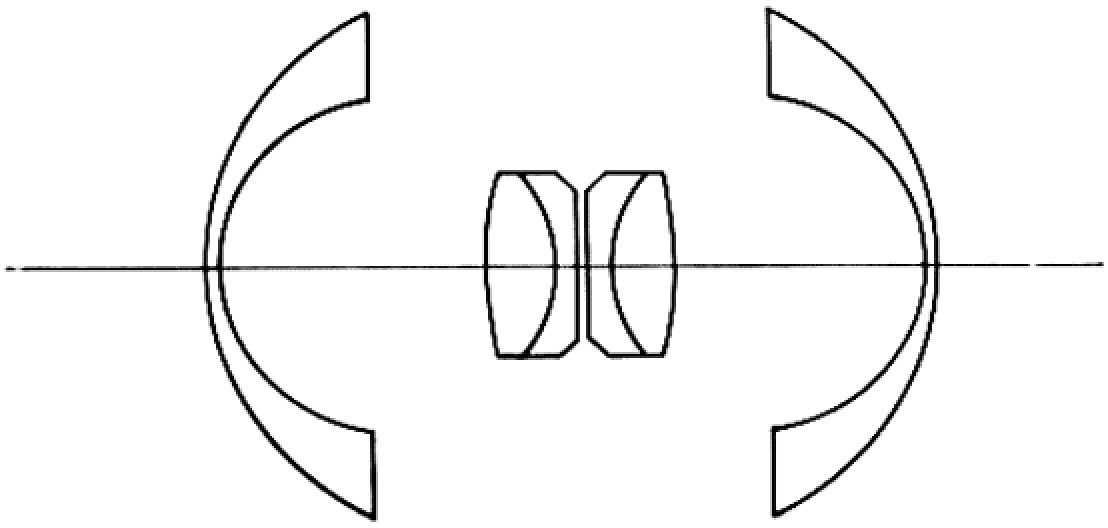


FIG. 179. LENS WITH IMPROVED MARGINAL FIELD ILLUMINATION (Roussinov)

Wynne[7], using an air space between surfaces with a small difference of curvature in place of the cemented surface of the Tessar has designed an *f*/4·5 lens covering a 65° field. A lens by Bertéle using a Gauss front combination followed by a convergent and a divergent system in that order covers a field of 62° at an aperture of *f*/2·8.

At lower apertures, *f*/8 lenses have been described by Roussinov to cover 133° with a flat field (Fig. 179). This construction employs the Slussareff principle (p. 193 above) in which the $\cos^4$ loss is reduced; also the principal ray through the centre of the stop is less steeply inclined than in outer space; a degree of correction is thus secured which allows of the vignetting of the oblique aperture to be reduced so that the illumination of the remote parts of the field can be maintained. Wild and Bertéle have described somewhat similar systems.

Lenses for Aerial Reconnaissance. Special difficulties are encountered in the construction of the lenses of long focal length, 20 in. to 36 in. or more, which are necessary for aerial reconnaissance. Telephoto lenses are often employed and are dealt with below. Of the "straight" lenses, some of the best known are the 20 in. T.T.H. Aviar *f*/5·6, covering 20°; the Ross 25 in. Xpres E.M.I.* *f*/6·3, covering 20°, Fig. 180; the Ross *f*/6·3 Aero Lens (Astro type), covering 10° semi-field, Fig. 181. In Great Britain, the picture size for military reconnaissance cameras was standardized for convenience in processing, at 9 in. × 9 in. Thus the required angular field for a long focal length has been fairly small, and this has been a feature of some importance in the design. German cameras have often used much larger film sizes.

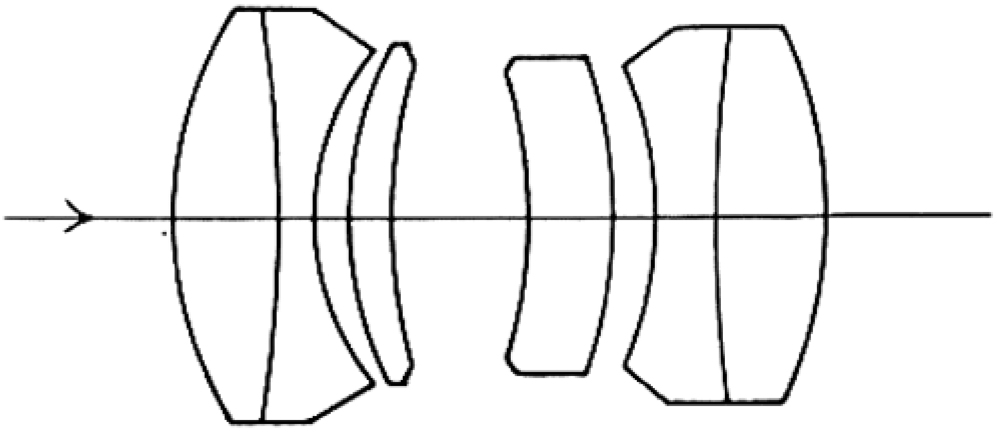


FIG. 180. ROSS AERO LENS 25 IN. XPRES E.M.I. *f*/6·3

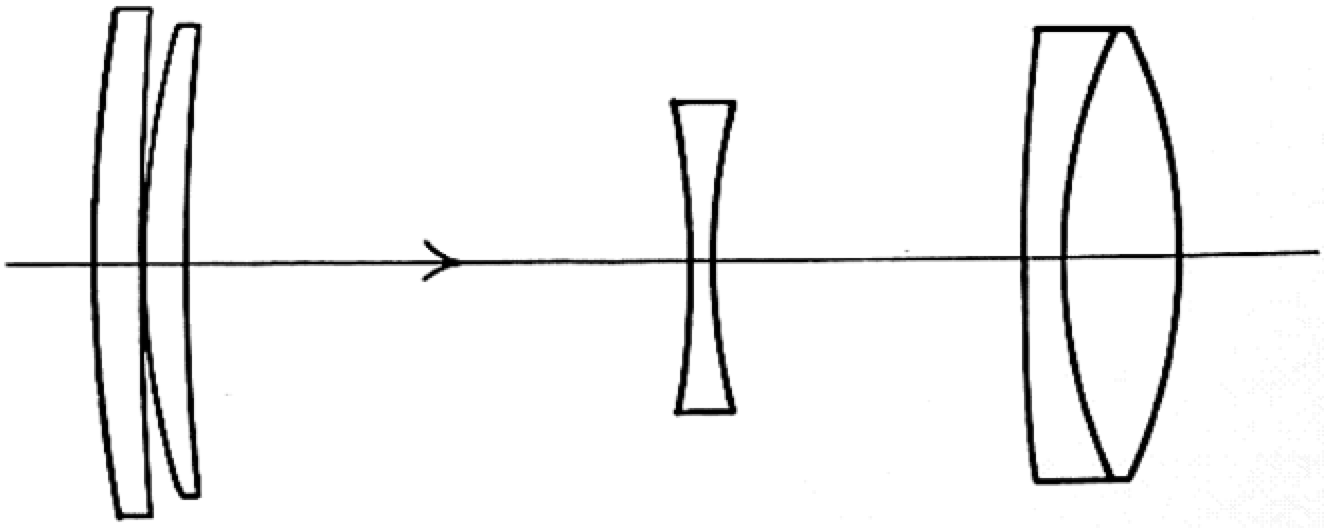


FIG. 181. ROSS AERO LENS 30 IN. ASTRO *f*/6·3

The Ross 25 in. *Xpres* can be looked upon as a modification of the *Dagor*, obtained by separating the inner two lenses and bending them to a somewhat weaker curve. Designed by Hasselkus and Richmond about 1930, some types of the lens can be made to cover almost 35° at $f/4$, but it was carefully re-designed during 1943–45 so as to produce the most exact definition over the 9 in. × 9 in. field. The Ross 30 in. Aero lens is clearly a modification of the Taylor Triplet.

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* The letters E.M.I. denote "extra-marginal illumination." The lens has outer components of diameter greater than the inner components so as to avoid vignetting.

Difficulties in the production and adjustment of such lenses are greatly accentuated with focal lengths above 36 in. or so. The focus of the camera will be to some extent dependent on the temperature and pressure of the air, and also on the temperature history of the camera. Automatic means of keeping the focus correct have been studied.

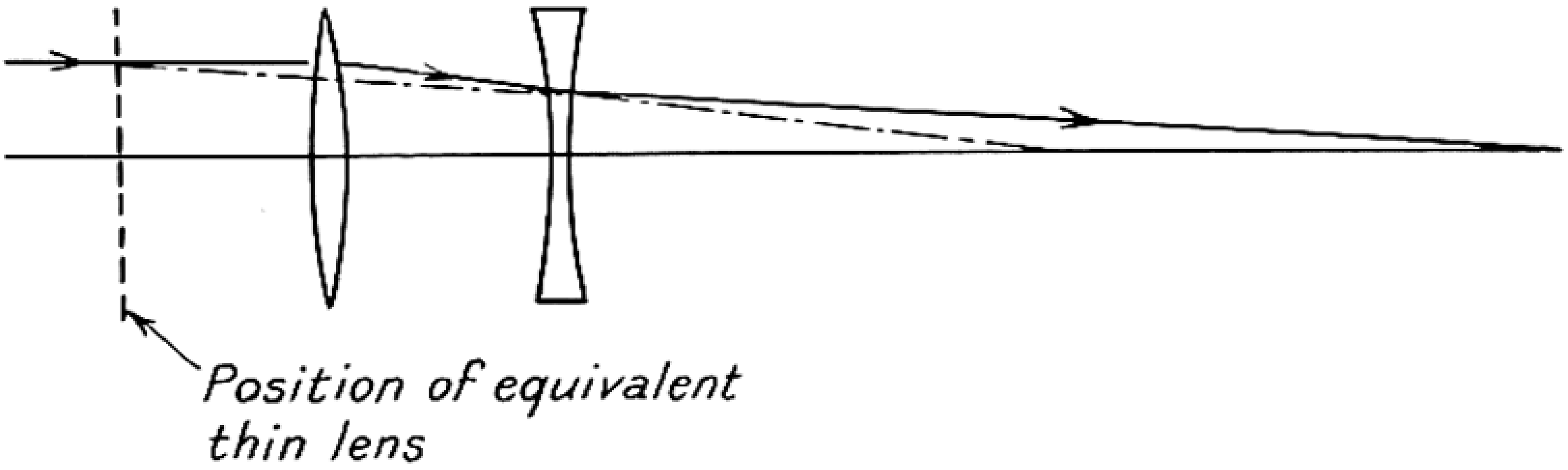

FIG. 182. TELEPHOTO COMBINATION

The Telephoto Lens. Consider a combination of two lenses of positive and negative powers numerically equal. Referring to pages 56 and 57 of Vol. I, we find the power of a combination given by

$$\mathbf{F} = \mathbf{F}_a + \mathbf{F}_b - d\mathbf{F}_a \,.\, \mathbf{F}_b$$

and the distance of the second principal plane from the second lens is

$$\mathrm{P}_b\mathrm{P}' = -\frac{\mathbf{F}_a d}{\mathbf{F}}$$

Take, for example, the case where $\mathbf{F}_a = 10\mathrm{D}$, $\mathbf{F}_b = -10\mathrm{D}$, and the separation $d = 4$ cm. ($= 0{\cdot}04$ metres), then

$$\mathbf{F} = 10 - 10 - 0{\cdot}04\,(10)\,(-10)$$
$$= 4{\cdot}0$$

The focal length is therefore 25 cm.

The equation for $\mathrm{P}_b\mathrm{P}'$ gives this length as $\dfrac{-10\,(0{\cdot}04)}{4} = -\dfrac{1}{10}$ metre $= -10$ cm.

Hence the second principal surface lies -10 cm. in front of the second lens.

The effect of the combination is, therefore, that of a lens of power 4·0D placed 6 cm. in front of the first component as suggested in Fig. 182. The back focusing distance of the combination is clearly 15 cm., whereas the actual focal length is 25 cm. Hence the images of distant objects will be larger in the proportion $\frac{25}{15} = \frac{5}{3}$ than those of a single lens with the same back focal distance.

Modern telephoto lenses employ two separated systems of doublet or triplet type; the first having a positive power, and the second a negative power. Each component must be separately corrected for chromatic aberration owing to the large distance between them,

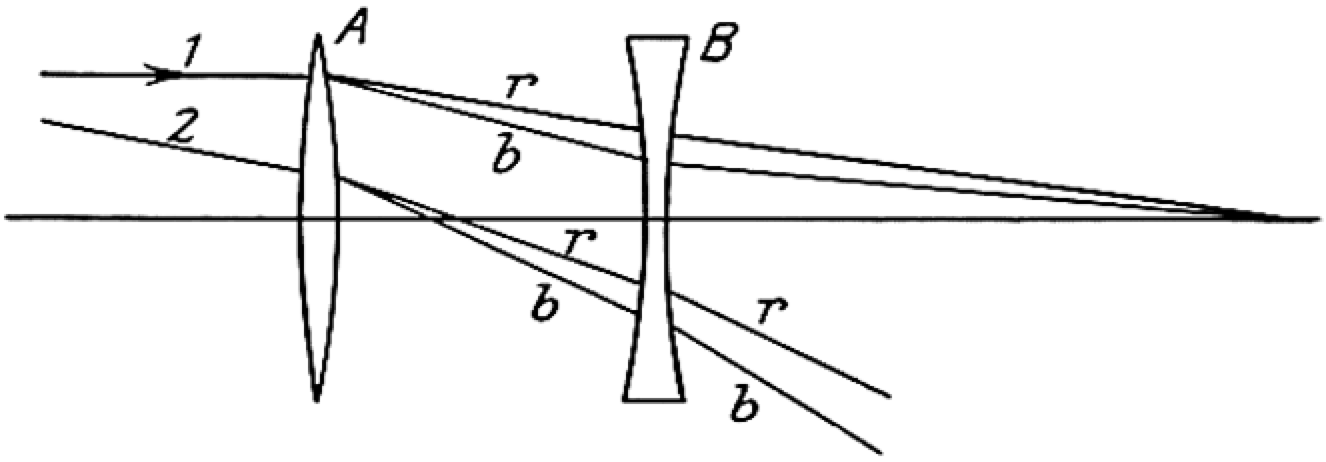

FIG. 183. CONDITION OF ACHROMATISM

for suppose that we consider rays 1 and 2 in Fig. 183; we may imagine that ray 1 is subject to dispersion by lens A, and that the under-correction, by which the blue ray is deviated more than the red ray, is corrected by lens B, the blue now suffering more deviation towards the margin of the negative lens than the red. If correction were thus given for image points very near the axis, consider ray 2 which suffers the same type of aberration in lens A, but meets lens B on the other side of the axis. The aberration is evidently exaggerated. Hence each component must be separately achromatized.

The early telephoto lenses mainly used a negative system behind a photographic lens system of some ordinary type; the power could be varied by varying the separation. The "magnifying effect" m of the telephoto attachment is given by

$$m = \frac{\text{Dimensions of image with telephoto attachment}}{\text{Dimensions of image with positive lens only}}$$

But for an infinitely distant object subtending an angle α, the dimension of the image is $f \tan \alpha$; also for a combination of two lens systems placed with their focal points so that $F'_a F_b = g$, we had

$$f = \frac{f_a f_b}{g}$$

Therefore $$m = \frac{f \tan \alpha}{f_a \tan \alpha} = \frac{f_b}{g}$$

When opticians supply a negative telephoto attachment with a variable separation, a scale on the mount is usually made to indicate values of m or g. The use of such attachments has, however, largely been abandoned in favour of the modern fixed-focus telephoto lens.

As distinct from the magnifying effect, the expression "telephoto effect" is sometimes used—

$$\text{Telephoto effect} = \frac{\text{Focal length of combination}}{\text{Back focal length}}$$

the "back focal length" being measured from the last surface of the lens system, and representing the approximate focal length of an ordinary lens used at the same camera extension.

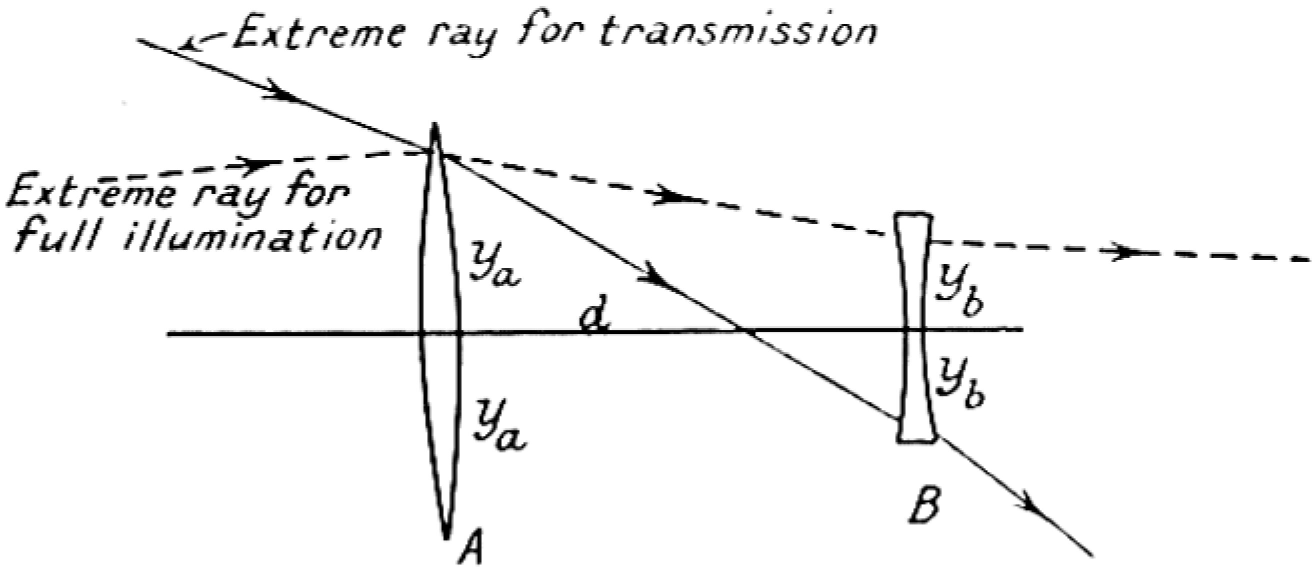
Extreme ray for transmission
Extreme ray for full illumination
y_a
y_a
d
y_b
y_b
A
B

FIG. 184

It is easily shown that this is equivalent to

$$\text{Telephoto effect} = \frac{f'_a}{(f'_a - d)}$$

where d is the separation of the lenses.

Effective Aperture. Consider a lens system A used first alone, then with a negative attachment B. The entrance pupil will be assumed to be the same as that of the system A; the diameter of B is sufficient to allow of this. Then

$$\text{Stop number} = \frac{\text{Focal length}}{\text{Diameter of entrance pupil}}$$

Hence, $$\frac{\text{Stop number with telephoto lens}}{\text{Stop number without telephoto}} = \frac{f}{f_a} = m$$

The exposure must be approximately proportional to m^2.

Field of View. In Fig. 184 let A and B be the positive and negative lenses of a thin lens telephoto system, and let y_a and y_b be the radii of the diaphragms which limit them. The tangent of the angle with the axis made by the most oblique ray which can pass between the lenses is clearly

$$\frac{y_a + y_b}{d}$$

If f'_a is the focal length of the thin lens A, the deviation in a ray produced by a transmission through it at a distance y_a from the axis is approximately $\frac{y_a}{f'_a}$. Hence, tracking this most oblique ray backwards through A, we find the inclination to the axis for the most oblique ray which can enter lens A and be transmitted by the system. Provided that we are dealing with angles small enough to take the tangent of an angle as its numerical value, the field of view is given by

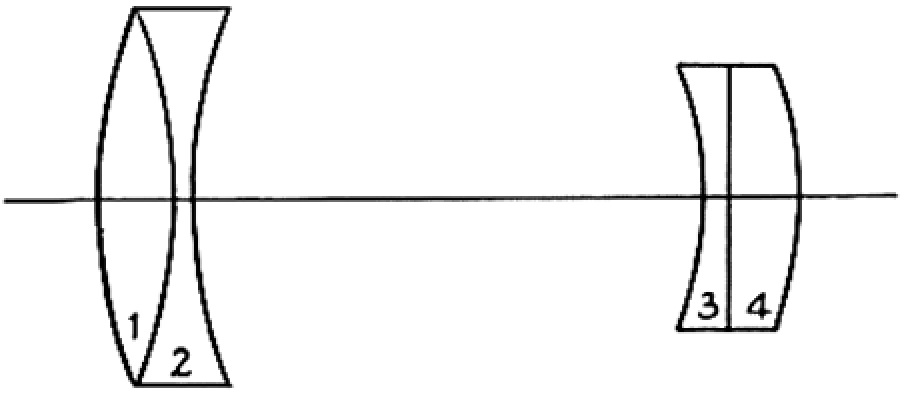

FIG. 185. DALLON LENS ($f/5{\cdot}6$)

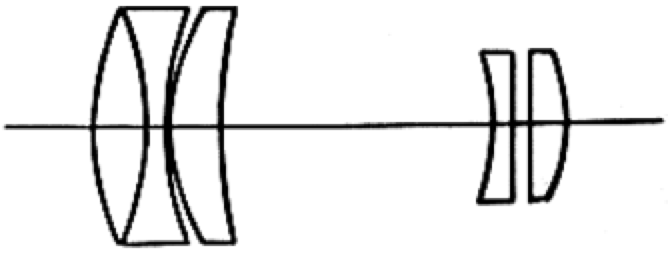

FIG. 186. DISTORTIONLESS TELEPHOTO LENS (LEE)
(*Messrs. Taylor, Taylor & Hobson*)

$$\frac{y_a + y_b}{d} - \frac{y_a}{f'_a}.$$

This represents the extreme limit of the field. It will be seen that the limit of the fully illuminated field is (see broken ray, Fig. 184)

$$\frac{y_a - y_b}{d} - \frac{y_a}{f'_a}.$$

These equations are, however, not strictly accurate with the thick lenses encountered in practice.

Modern Telephoto Lenses. The modern telephoto lenses are mostly fixed focus combinations with anastigmatic correction. The telephoto effect is low, being only two to three, but this suffices for a great number of purposes, more especially as several advantages are obtained, viz. high relative aperture (the aperture of some telephoto lenses has been increased to $f/3{\cdot}5$) anastigmatic correction giving sharp images capable of enlargement, and reasonable economy in size. Fig. 185 shows the Dallon (of Messrs. Dallmeyer, Ltd.)

designed by L. B. Booth, who was a pioneer in the construction of such systems. Typical glasses in lenses of this kind are—

1. Dense barium crown.
2. Dense flint.
3. Light flint.
4. Medium barium crown.

A general review of the development of modern telephoto objectives has been given by Lee,[8] who was successful in producing a distortion-free telephoto lens. He says: "If we consider the construction of the telephoto, a positive lens placed in front of the diaphragm, which will possess pin-cushion distortion, and a negative lens behind the diaphragm which tends to produce the same kind of distortion, it is not surprising that telephotos are afflicted with much pin-cushion distortion, and some designers have considered it inevitable. . . . By separating the components of the negative lens, it was possible to utilize the astigmatism in these surfaces, which is fairly large, to correct the pin-cushion distortion. . . . We are then left with a residuum of under-corrected astigmatism, which is neutralized in the front positive lens by dividing it up into two menisci, one of which is a doublet, for the purpose of achromatism." These words give a brief picture of some of the main stages in the design of a new system. Fig. 186 shows the distortion-free lens, B.P. 222 709.

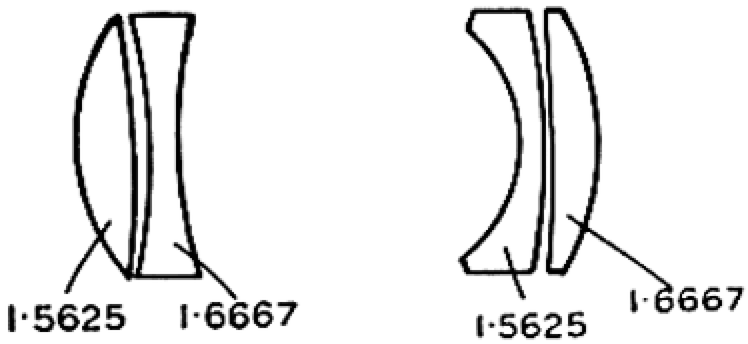

FIG. 187. DISTORTION-FREE TELEPHOTO TYPE LENS

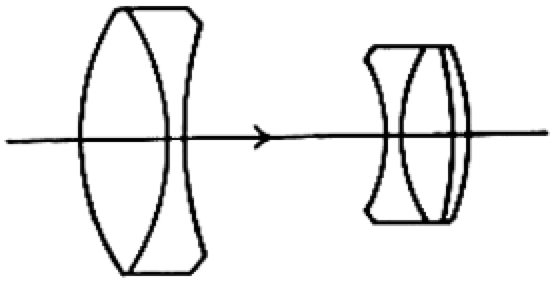

FIG. 188. WRAY TELEPHOTO *f*/6·3 OR *f*/4·5

The aberrations in these systems are not large, and it is possible to make a close approach to useful designs by solving the analytical equations for the elimination of the primary aberrations including distortion. A solution obtained in this way by the present writer is represented in Fig. 187. The *primary* aberrations are small, but there is a considerable amount of higher spherical aberration so that the design would have to be modified for production. The lens is not dissimilar in general form from the "Telikon" designed by Richter.[9] A more complex negative component is used in the "Plustrar" telephoto designed by Wynne (Messrs. Wray, Ltd.; *f*/6·3 or *f*/4·5; see Fig. 188). Improvements effected by various changes of shape are described by Wynne[7] in his review, and apertures up to *f*/4 are mentioned.

Depth of Focus ; Photographic Definition. The "depth of focus" in the image may be defined as that total displacement of the plate on each side of the true focus which is possible without producing an appreciable spreading or loss of definition in the image. In Vol. I, p. 140, the matter was discussed in terms of optical path differences between marginal and paraxial disturbances, and the equation for the shift on one side of the focus is

$$\delta f = \frac{2\,\delta p}{n'}\left(\frac{f}{y}\right)^2$$

where δp is the allowable difference of optical path. Remembering that the total range will be approximately double the above, and that the stop number is $f/2y$, we get for a medium where $n' = 1$

$$\textit{depth of focus} = 16\,(\textit{allowable path difference})\,(\textit{stop number})^2$$

The allowable path difference is, however, not easy to specify very exactly; the requirements for photographic recording are usually less severe than for direct visual observation of optical images where the Rayleigh limit of $\lambda/4$ may be necessary. If in photography the limit were taken as $\lambda/2$, say, then with a lens working at *f*/8 and wave-length $= 0{\cdot}5\mu$, the depth of focus would be 256μ, or about a quarter of a millimetre. On the other hand the question could be discussed on the basis of geometrical optics.

The tolerance for the loss of definition of the image is usually such that a photograph held at the least distance of distinct vision should appear reasonably sharp to the eye, and it is usually estimated that the photographic disc or patch representing an image point may subtend an angle of, perhaps, two minutes of arc.

If d is the diameter of the patch in millimetres, the limit would thus be given by the equation

$$\frac{d}{250} = 2 \text{ min. in angular measure}$$

$$d = \frac{250}{1710}$$

so that d will not exceed about one-seventh of a millimetre, or about $\frac{1}{178}$ in. This limit is on the severe side, and a patch of $\frac{1}{100}$ in., or $0{\cdot}25$ mm., may be tolerated in some cases.

If the diameter d of the patch were determined simply by the diameter of a cone of rays passing through a point in the focus, taken at a distance δf therefrom, we should have

$$\frac{\delta f}{d} = \text{stop number}$$

and if d were 0·25 mm., then $\delta f = 2{\cdot}0$ mm. for $f/8$, and the *total depth of focus* would be of the order of 4·0 mm. It is clear that the usual criteria of optical path are far too severe for this case; on the other hand, the distribution of light is not well represented by the diameter of the supposed "cone of rays," which gives very misleading results near the focus.

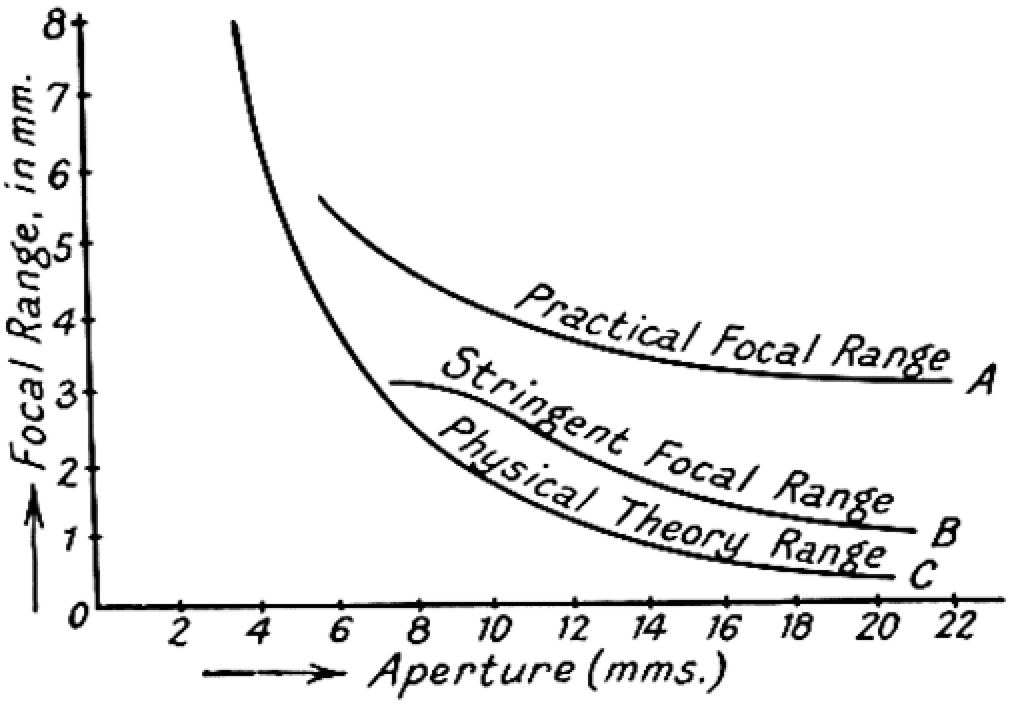

FIG. 189. FOCAL RANGE FOR AN ANASTIGMAT LANDSCAPE LENS

It is thus difficult to give a satisfactory theoretical discussion on any simple lines. Considerable light was thrown on the matter by the experimental work of Miss H. G. Conrady,[10] who used an anastigmat landscape lens ($f/7$ approx.) suffering from a residual spherical aberration of known amount, and investigated the *position* of the best focus, and the focal range for various apertures. The results are shown in Fig. 189, in which curve A represents the focal range giving fairly good definition for practical purposes; curve B shows the range over which no loss of definition is at all perceptible; curve C shows the range predicted by physical theory for a perfect lens, using the "Rayleigh limit" of $\lambda/4$ for allowable path differences. It appears that the range is greatly increased by the presence of slight spherical aberration, and that it may be anything from one to eight times the "Rayleigh limit" range, depending on the amount of spherical aberration, which is naturally increasing in the diagram with increasing aperture. On the whole, it is clear that neither the physical theory nor the geometrical discussion of discs of confusion have a precise significance in this problem. Failing more exact knowledge, however, we usually find that it is very rarely with these photographic lenses that the diameter of the effective patch of light exceeds the limits of the calculated geometrical disc of confusion of the rays, and, consequently, it is the general experience that useful tolerances can be obtained by this theory. See also investigations of "contrast transmission," p. 379.

Depth of Field for a Camera Lens. In the earlier days of photography the picture was usually a straightforward "print" from the negative, and this print would usually be viewed at a normal "reading" distance, say about 250 mm. The disc of confusion in the negative could be regarded as identical with that in the print. The discussions on pp. 16 to 18 will apply to such a case; if the tolerable disc of confusion in the image were supposed to have a diameter of 0·01 in. for soft definition, or 0·0025 in. for sharp definition the resulting depth of field could readily be calculated. The aberration of most camera lenses were so large in comparison with the "Airy disc" that the "physical depth" of focus possessed little relevance.

In modern times the "print" is usually replaced by an enlargement, or will be projected by a suitable projector for convenience of viewing. The guiding principle for satisfactory perspective in viewing is that the enlarged picture should be viewed under an angle equal to that subtended at the camera lens by the corresponding object field. In the majority of cases the aberrations of the lenses are still large in comparison with the Airy disc, and the geometrical discussion is adequate. Hence the relevant factor in determining the depth of field is the limiting visual angle under which the disc of confusion may be seen. The actual disc in the original picture must not exceed this angular subtense at the nodal point. The discussion on p. 46 now applies. A useful generalization puts the limiting angle α for the disc of confusion at from 0·001 to 0·0004 rad. at the camera lens, these corresponding to visual angles of $3\frac{1}{2}'$ and $1\frac{1}{2}'$, respectively.

When very short focus lenses are in question, the granularity of the film will constitute a factor; high speed emulsions have coagulated grain sizes up to 0·04 mm. and fine-grain films about 0·01 mm.; the limits of resolution may be of the order of from 16 to 42 lines/mm., respectively on the film, i.e. for a 40 mm. focal length the limiting interval is from 0·0015 to 0·00068 rad. For such a lens the limiting angle for the disc of confusion may still be put at about 0·001 rad.; but for still shorter focal lengths the granularity will play an increasing role.

In modern times other means of registration of the original image are of increasing importance, such as photo-electric image converters, various scanning systems, and so on; these may produce effects which control the depth of field within which the final image definition does not fall below some given standard; it must not be assumed that the visual limits are operative in such cases.

It should be remembered that the depth of field in the object space only depends on the diameter of the aperture of the entrance pupil. Hence, snapshot cameras using lenses with small entrance pupils have a greater depth of focus in the object space as compared with larger lenses, and have the advantage that the smaller lenses are much cheaper.

Zoom Lenses. A short account of some afocal systems designed to give variable magnification was given on pp. 57–59. A system like that of Fig. 33 can be housed in a dust and weather-proof mount in front of a lens for projection or cinematography; it will contain the appropriate cams, etc., for the change of magnification.

Note that the outer lenses have the function of projecting the image into (or from) certain points A and A′, Fig. 33 (*b*), with respect to the whole system. If now the object conjugate is not infinite a slight appropriate movement of the first outer lens will restore the focus to A (and similarly for the image side); the image produced by the whole system will then remain precisely in focus during the operation of the cams.

The discussion of the correction of such systems is largely outside the scope of this book, but the notes already given with regard to systems of thin lenses will throw some light on the subject. The diagram, Fig. 33, showed the system for magnification m, 1 and $1/m$. The system can be designed to give freedom from spherical aberration, lateral and axial chromatic aberration, and coma, at magnification m; it will then be similarly automatically corrected at $1/m$, and from the symmetry of the system it will be free from lateral colour and coma at $m = 1$; also it will be free from distortion in the mean position. A suitable choice of glasses enables the astigmatism and field curvature to be kept low.

If the spherical aberration, coma, tangential field curvature, and axial chromatic aberration of a system are fully corrected, these aberrations will remain corrected even if the position of the stop is altered. If a corrected afocal system is used in front of a fixed focus camera lens of which the aperture constitutes the effective stop for the whole system the symmetry of the system is lost, but the aberrations remain low; some higher order aberrations may arise, but they do not prevent the use of such simple systems for magnification changes in the ratio of about 3 : 1 when used with television lenses with apertures up to 2 in. and angles of field of 30°.

When zoom ratios up to 5 : 1 are required, the single lenses are made separately achromatic, and the mutual movements of the inner negative power doublets have to be controlled so that correction of the coma is maintained. In Fig. 190 which shows a "zoom" lens designed by Hopkins[11], the lenses 1, 2, 3 and 4 constitute the afocal part of the system, and the doublet 5 represents the "camera lens." The stop in the system, which is mounted behind lens 3, is controlled by a suitable mechanical linkage. The aperture obtainable in the lens designed by Hopkins goes up to *f*/3 for the longer focal lengths. Other "zoom" lenses are described in a survey by Kingslake[12], see for example the system pictured in Fig. 191, the "Pan Cinor" of Berthiot.

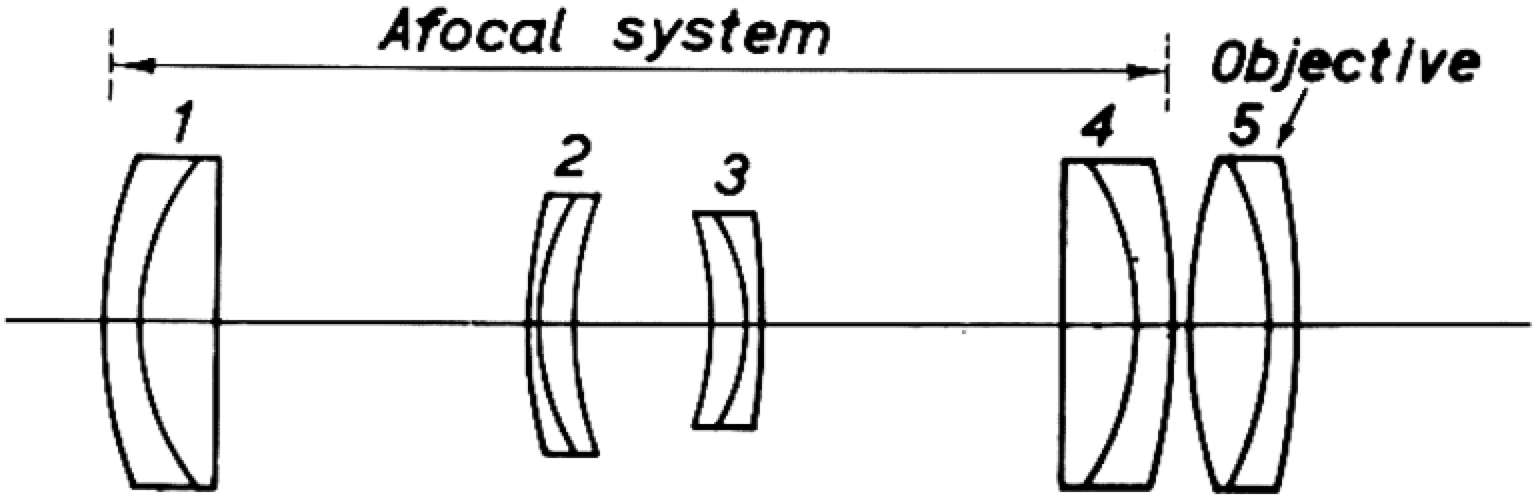

FIG. 190. ZOOM LENS TO GIVE MAGNIFICATION VARIATION 5 : 1

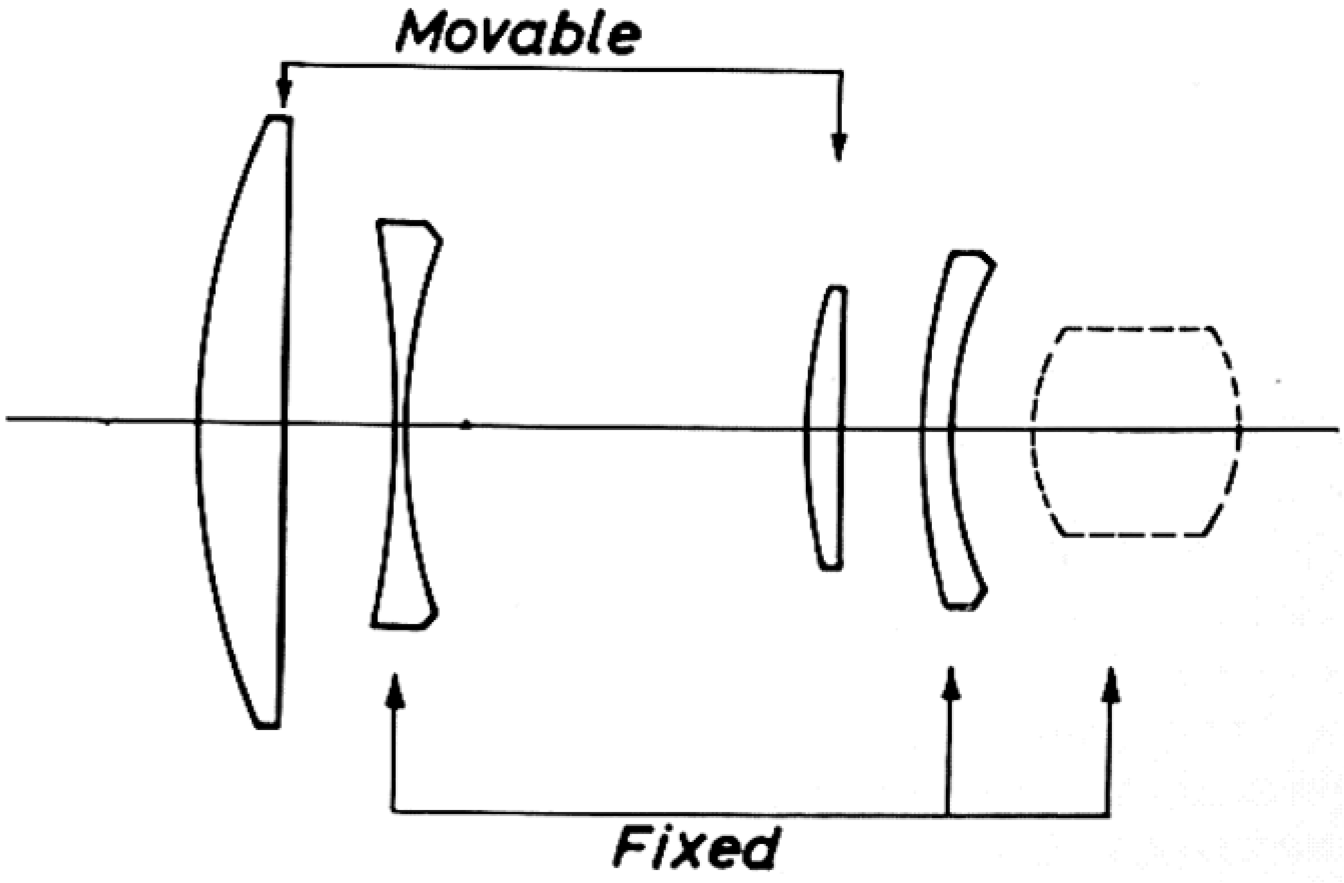

FIG. 191. THE PAN CINOR LENS

Recent Progress. Useful general surveys have been given by Kingslake[12] and Wynne.[7] The use of rare-earth oxides has led to important progress in the manufacture of new optical glasses of high refractive indices above 1·66; moreover, closer control of manufacture has reduced the absorption (usually heavy at the violet end of the spectrum) of the older dense flint glasses, thereby considerably extending their range of usefulness. The use of synthetic crystals has been largely for special applications, e.g. spectroscopy, where their transmission of the non-visible parts of the spectrum is important. Plastic materials have not so far been produced with the necessary stability for commercial optical systems.

As previously mentioned some reduction of aberrations can be attained at the expense of increased complexity. Wynne considers that the improvement in the treatment of lenses by non-reflecting deposits (blooming) has rendered possible the use of systems which would otherwise have been ruled out through the effect of internal reflections and stray light.

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CHAPTER VI

THE PHOTOMETRY OF OPTICAL SYSTEMS AND THE PROJECTION OF IMAGES

In the foregoing discussion of the principles of the telescope and microscope, no attention has been given to the question of the brightness of the image; this aspect of the subject is, however, of the first importance, and must now be considered. A discussion of the sensitiveness of the human eye to light has been given in Vol. I, Chapter V.

For the purposes of elementary discussion, it is assumed that radiant energy would spread out from an elementary source of infinitesimal size along the paths represented by the "rays." The amount of energy per unit time passing any cross-section of a tube whose walls were made up of such rays would, therefore, be constant.

Rays are straight in a homogeneous medium; hence if we consider a conical tube representing a very small solid angle $d\omega$ with its apex in the elementary source C (Fig. 192), the normal cross-sectional areas at distances r_1 and r_2 would be $r_1^2 d\omega$ and $r_2^2 d\omega$. Let the amount of energy passing in unit time be $d\mathrm{F}$; then the energy *per unit area* at these sections will be

$$\frac{1}{r_1^2}\left(\frac{d\mathrm{F}}{d\omega}\right) \text{ and } \frac{1}{r_2^2}\left(\frac{d\mathrm{F}}{d\omega}\right)$$

and the quantities must be equal in the absence of absorption, i.e. (putting the result into words) the amount of energy per unit area falling on an elementary area held normal to the incident light is inversely proportional to the square of its distance from the source. This is the "inverse square" law; but it only has an exact meaning in regard to an imaginary source of infinitesimal size, and, therefore, may only hold approximately in practical cases.

When the energy is evaluated *according to the luminous sensation produced*, the symbol F represents an amount of "light," and $\frac{d\mathrm{F}}{d\omega}$ represents, for some particular direction, "the amount of light per unit solid angle," which is the "candle-power" of the source for that direction. The candle-power is usually denoted by J. Thus

$$\textit{Candle-power} \text{ or } \textit{Illuminating power } \mathrm{J} = \frac{d\mathrm{F}}{d\omega} \quad . \quad (6.01)$$

The amount of light falling per unit area of a surface represents the "illumination" (usually denoted by E), so that

$$\textit{Illumination}\ \mathrm{E} = \frac{d\mathrm{F}}{ds} \quad . \quad . \quad . \qquad (6.02)$$

where *ds* represents an elementary area of the surface.

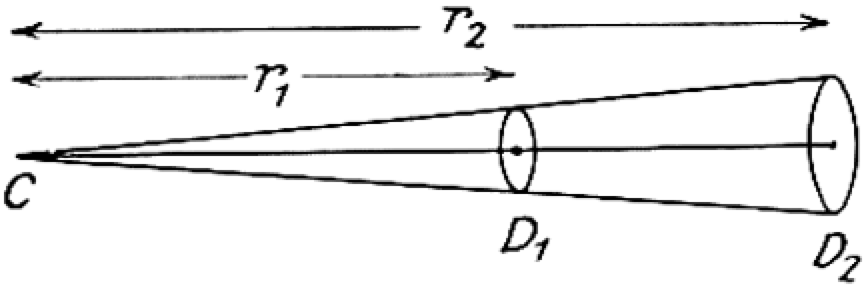

FIG. 192

Lastly, the *luminance* or "brightness" (B) of a surface is defined as the candle-power per unit projected area in a plane perpendicular to the direction under consideration.

Luminance independent of direction is expressible in terms of the "cosine law" below.

The Cosine Law. Careful consideration should be given to the concept of the candle power of a finite area of an emitting surface, as distinct from the point source mentioned above. If the source has a small finite area ds_1 the flux dF is supposed to be measured

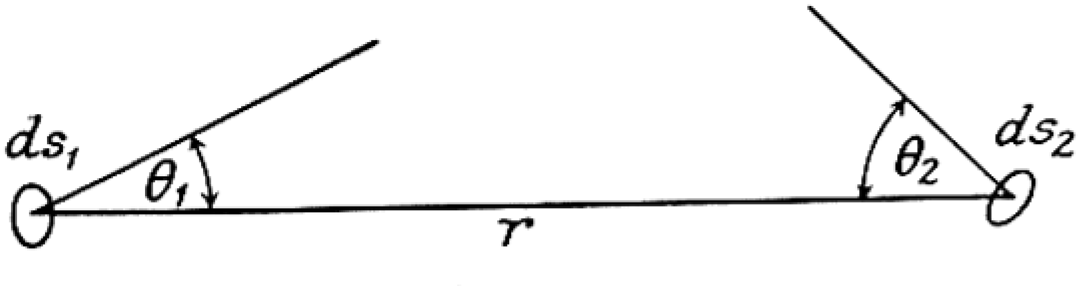

FIG. 193

over a small area, ds_2, say, of a surrounding surface so very distant from the source that this area ds_2 subtends substantially the same solid angle $d\omega$ at any one point of the source; the concept of candle power (the differential coefficient $d\mathrm{F}/d\omega$) follows as before. Notice, too, that luminance is not the "candle power per unit area" of the source, but is the "candle power per unit of projected area." This concept should be more clear from the following considerations.

In Fig. 193 let ds_1 and ds_2 be the elements of the source and receiving surfaces mentioned above; the normals to these elements make angles θ_1 and θ_2, respectively with any line joining them; thus, when r is exceedingly great in comparison with the linear dimensions of either element, the variation of θ_1 and θ_2 will be inappreciable no matter which individual points in ds_1 and ds_2 are connected, and the distance r between such points will only vary inappreciably in the relative sense.

The solid angle $d\omega_2$ subtended by ds_2 at ds_1 is thus

$$d\omega_2 = \frac{ds_2 \cos \theta_2}{r^2}$$

If the luminance of ds_1, in the direction θ_1 is B_1,

where
$$B_1 = \frac{(dF/d\omega_2)_{\theta_1}}{ds_1 \cos \theta_1} \quad . \quad . \quad . \quad . \quad (6.03)$$

since the projected area is $ds_1 \cos \theta_1$, the candle power in the direction of ds_2 is

$$dJ = \left(\frac{dF}{d\omega_2}\right)_{\theta_1} = B_1\, ds_1 \cos \theta_1$$

and the element of flux dF sent to ds_2 is therefore

$$dF = \frac{B_1\, ds_1 \cos \theta_1 \,.\, ds_2 \cos \theta_2}{r^2} \quad . \quad . \quad (6.04)$$

Since the solid angle $d\omega_1$ subtended by the source at the receiving element is

$$d\omega_1 = \frac{ds_1 \cos \theta_1}{r^2}$$

we may write for the illumination, dE, of ds_2

$$dE = \frac{dF}{ds_2} = B_1 d\omega_1 \cos \theta_2$$

We see that the illumination from a small source is proportional to the cosine of the angle of incidence. The illumination obtained from an extended source is found by integration over the whole solid angle which is subtends; thus,

$$E = \int_s (B_1 \cos \theta_2)\, d\omega_1 \quad . \quad . \quad . \quad (6.05)$$

We shall see below that if a surface obeys Lambert's "cosine law" B_1 will be constant; if then the variation of θ_2 over the whole area of the source is such that $\cos \theta_2$ does not suffer appreciable variation we have the relation

$$E = B_1\, \omega_1 \cos \theta_2$$

In the case of illumination by a more or less symmetrical cone of rays where θ_2 has, at most, small positive or negative values so that $\cos\theta_2$ does not change much from unity

$$E \approx B_1 \omega_1$$

This is an important general relation to remember as a guiding principle. In seeking to enhance the illumination of any surface the key quantities are: (1) the luminance of the illuminating source, and (2) its solid-angular subtense.

Lambert's Cosine Law. A self-luminous surface having no observable regular structure may be expected to show a type of emission which is symmetrical with regard to the direction of the normal; in this case the luminance is simply a function of θ_1

$$B_1 = f(\theta_1)$$

A pioneer student of photometry, J. H. Lambert, who produced his book *Photometria* in the year 1760, noticed in effect that many "matt" emitting surfaces have a luminance B which is almost independent of direction; the candle power dJ of an element ds_1 is then (as above) very nearly,

$$dJ = B\, ds_1 \cos\theta_1$$

so that the candle power of the element is then proportional to the cosine of the angle of emission. This relation is usually called the "cosine law" or "Lambert's law," but it is not a law of nature and must not be confused with the exact cosine relation which controls the illumination of a surface element by a small source. Lambert's law has a physical basis in the following kind of consideration. Suppose that an emitting "surface" consists of an assembly of small luminous opaque spheres in mutual contact, of which the centres lie in one plane. Any one sphere can radiate into the direction perpendicular to this plane with light from a complete hemisphere, but, in general, light emitted in an oblique direction will be partly obstructed by the neighbouring spheres, and this obstruction steadily increases with the obliquity; this effect would obviously give a candle-power obliquity relation of the right kind, but the exact function may be expected to depend on the structure of the surface. It is remarkable that many emitting surfaces yield a close approach to Lambert's law.

It is found that smooth *reflecting* surfaces vary much more in the directional properties of their emission. If the illumination of such a surface by a parallel beam of rays is confined to a certain direction characterized by the angle of incidence θ_2, the line corresponding to the projection of an incident ray on the plane of the surface fixes a datum, ϕ_2, of azimuth; hence any particular direction of emission can be characterized by (1) the angle θ' between it and the normal, and (2) the azimuth ϕ' taken with regard to the datum line; then in general the emission luminance is a function of both θ' and ϕ'

$$B = f(\theta', \phi')$$

Specular reflecting surfaces will have, in general, sharply defined reflection when $\theta' = \theta_2$ and $\phi' = -\phi_2$, but many white "matt" surfaces, such for example as chalk, blotting paper, etc., nevertheless show a fair approximation to independence of luminance from direction of emission. Certain specially prepared surfaces may come very close to perfection in this respect, but others have an infinite variety of properties ranging from "perfect diffusion" to specular reflection.

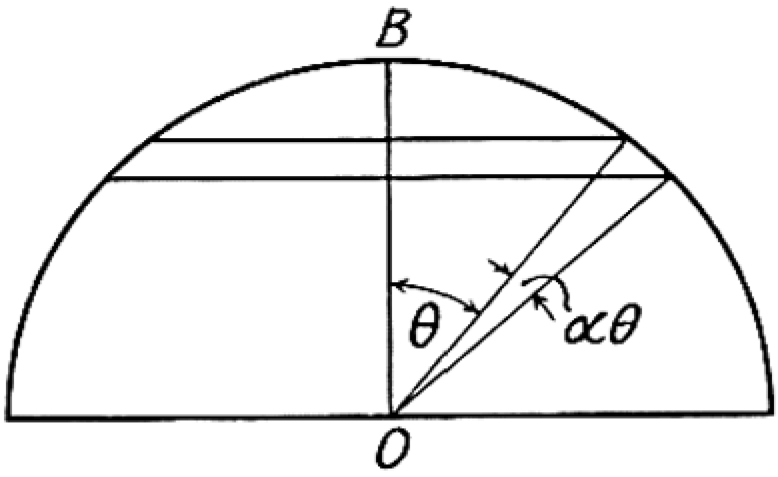

FIG. 194

Any surface obeying Lambert's law is known as a "perfectly diffusing surface"; note, however, that a surface which is almost perfectly diffusing when emitting because of its high temperature may not show good diffusing properties for reflected light.

Total Light Radiated by a Self-luminous Surface. Imagine a small element, of area ds, of self-luminous surface at O, Fig. 194; assuming that radiation takes place in accordance with the cosine law, it is possible to calculate the total light radiated by the element into the space above it. Imagining a hemisphere described above the element, we may calculate the light radiated to a circular strip limited by the angles θ and $\overline{\theta + d\theta}$ between the radii drawn from O and the normal OB. If r is the radius of the hemisphere, the area of the strip is clearly $rd\theta \,.\, 2\pi r \sin\theta$, and the solid angle $d\omega$ subtended at O is therefore

$$d\omega = 2\pi \sin\theta d\theta$$

If B is the normal luminance, the candle-power of the element in the direction θ is

$$Bds \cos\theta$$

Hence the light radiated to the annular strip is

$$dF = Bds \,.\, 2\pi \sin\theta \cos\theta \, d\theta \qquad . \quad . \quad (6.06)$$

To find the total light radiated into a symmetrical cone of angle θ we integrate thus

$$F = \int_0^{\theta} B ds \,.\, 2\pi \sin\theta \cos\theta \, d\theta$$

$$= \pi B ds \sin^2\theta \quad . \quad . \quad . \quad . \quad . \quad (6.07)$$

When the angle θ is $\frac{\pi}{2}$, so that we consider the whole of the space above the element, the result is $\pi B ds$.

Luminance of Optical Images. The first case considered is that in which the image is presented directly to the eye, without being projected on a diffusing screen. The second case will be that of the photographic image or projection lantern image where the light falls on a projection plate or screen. It will be assumed that the object radiates in accordance with the cosine law.

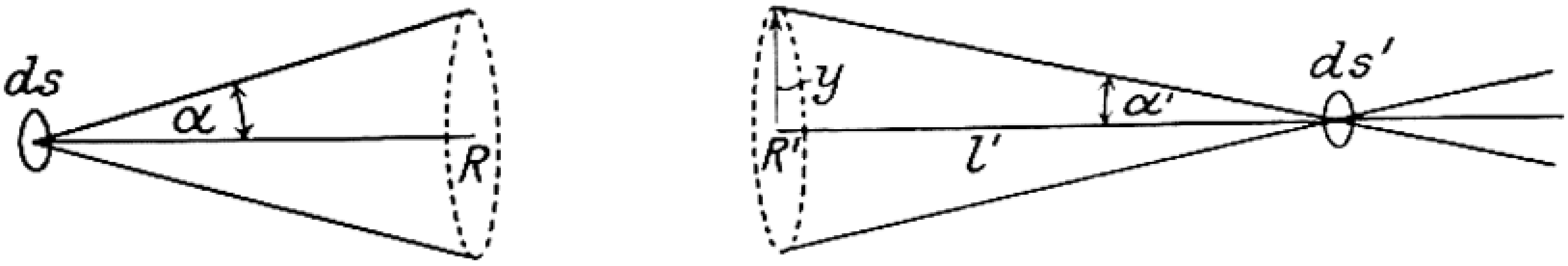

FIG. 195

The simplest case to be considered is that of a symmetrical instrument, Fig. 195, of which the centres of the relevant entrance and exit pupils are at R and R′. Let ds be a small element of self-luminous surface with the optical axis as its normal, and let ds' be the corresponding image. The angular divergence of the rays entering the entrance pupil is α, and the corresponding angle made by the rays with the axis in the image is α'.

Let the normal luminance of the object be B; then the amount of light F entering the instrument is

$$F = \pi B ds \sin^2\alpha_m$$

where α_m is the limiting angular divergence between the extreme rays and the axis in the object space.

The luminous energy flows through the system by various paths which are represented by the ray tracks. If the system fulfils the "sine condition," then when we write the optical sine relation

$$nh \sin\alpha = n'h' \sin\alpha'$$

we know that the ratio of sin α to sin α' will be the same for all zones of the system. The energy radiated from the object element into the entrance pupil of the instrument will be subject to some losses by reflection and absorption. If the effective transmission factor is K, the amount of light reaching the image will be

$$\mathrm{KF} = \mathrm{K}\pi \mathrm{B}ds \,.\, \sin^2 \alpha_m$$

Assuming now that the corresponding image area, ds', has a normal luminance B′, and that it radiates in accordance with the cosine law, the amount of light passing through the aerial image will also be expressed as

$$\mathrm{KF} = \pi \mathrm{B}'ds' \sin^2 \alpha_m{}',$$

where $\alpha_m{}'$ is the limiting angle between rays and axis in the image space. Hence,

$$\mathrm{K}\pi \mathrm{B}ds \,.\, \sin^2 \alpha_m = \pi \mathrm{B}'ds' \,.\, \sin^2 \alpha_m{}' \qquad . \qquad . \quad (6.08)$$

By squaring the sine relation we obtain

$$n^2 ds \,.\, \sin^2 \alpha_m = n'^2 ds' \,.\, \sin^2 \alpha_m{}'$$

and, dividing into the previous equation

$$\frac{\mathrm{KB}}{n^2} = \frac{\mathrm{B}'}{n'^2}$$

or

$$\mathrm{B}' = \mathrm{KB}\left(\frac{n'}{n}\right)^2 \qquad . \qquad . \qquad . \quad (6.09)$$

Returning to equation (6.08), let us differentiate it with regard to α and α', obtaining

$$2\mathrm{K}\pi \mathrm{B}ds \sin \alpha \cos \alpha \, d\alpha = 2\pi \mathrm{B}'ds' \sin \alpha' \cos \alpha' \, d\alpha'$$

Referring back to the investigation above, we easily see that the assumption that the image radiates in accordance with the cosine law is founded on the supposition that the light radiated by the object into an annular cone of angles α and $\alpha + d\alpha$ passes through the image in a corresponding annular cone of angles α' and $\alpha' + d\alpha'$ which can be calculated by the assumption of the sine condition. Differentiating the squared form of the sine relation we obtain

$$2n^2 ds \sin \alpha \cos \alpha \, d\alpha = 2n'^2 ds' \sin \alpha' \cos \alpha' \, d\alpha'$$

which, dividing into the above equation, gives the same result for B′. The investigation is, therefore, in strict accord with the fundamental assumption that the quantity of light remains constant for all cross-sections of a cone whose walls are made up of rays of light. The condition that the image may radiate in accordance with the cosine law is that the transmission factor of the instrument may be the same for all zones. In practice, this is not likely to hold exactly, owing to the different obliquities of the rays at the different surfaces which become much greater towards the margins, thereby greatly increasing the reflection losses.

Before going further, it will be well to re-state the result and draw certain inferences. Few optical theorems have been the subject of so much misunderstanding as this.

Firstly, although the image ds' is radiating in a homogeneous medium, it is unlike the object inasmuch as it can only radiate within definite angular limits; the extreme rays cannot exceed an angle α_m' with the axis.

Provided the eye is so placed that the pupil lies entirely within this angular limit, the image when observed will have the effective luminance $B' = KB\left(\frac{n'^2}{n^2}\right)$; but if the eye is outside the cone of radiation, the image will not be seen. Now, if the image space medium is air, as is usually the case in visual observation, then $\left(\frac{n'^2}{n^2}\right)$ is unity, and the effective luminance of the image is that of the object multiplied by a transmission factor K which is always less than unity. The formal proof has only been given for the most simple case, but it may be taken as a general rule that *the effective luminance of the image of a luminous surface formed by an optical system cannot exceed the effective luminance of the object surface observed directly*, provided that the apparent size of both object and image is not very small.

The Stiles-Crawford Effect. In 1933, Stiles and Crawford[1] discovered that light which enters the eye by the margin of the pupil and falls on the fovea is less effective in producing the sensation of luminosity than the same amount of flux entering through the centre of the pupil. This effect has been measured by means of a flicker photometer so designed that the light illuminating the field passes alternately by a peripheral and a marginal part of the eye pupil, with the result, for example, that when the light is transmitted by an element of the aperture 4 mm. from the centre its quantity has to be increased by a factor of about five times in order to maintain the same sensation of luminosity. Thus the effect is one of major importance in the discussion of the photometry of optical systems. There is some eccentricity in the effect in some eyes, the region of maximum efficiency being displaced from the centre of the pupil. This is thought to depend on the exact direction of the foveal cones, the electrical response of which will be a maximum when their direction is coincident with the incident "rays." We shall, however, neglect this eccentricity in giving an approximate quantitative account. A curve of the relative effectiveness η of light as a function of the radius x (in mm.) of the element from the centre of the pupil is given in Fig. 196; it fits the known experimental results well enough to be used in the present discussion, and is expressed by

$$\eta = \exp(-0{\cdot}105x^2).$$

The total light efficiency S_x of a circular aperture of the pupil of radius x is therefore expressed by

$$S_x = \frac{2\pi\int_0^x \eta x dx}{\pi x^2} = \frac{[-\exp(-0{\cdot}105x^2)]_0^x}{0{\cdot}105x^2}$$

$$= \frac{1-\exp(-0{\cdot}105x^2)}{0{\cdot}105x^2} \quad . \quad . \quad . \quad . \quad (6.10)$$

The following table gives numerical values—

x	1	1·5	2·0	2·5	3·0	3·5	4·0
S_x	0·95	0·892	0·817	0·733	0·655	0·563	0·484

Hence, if we represent the photometric luminosity of an image as a function of the pupil area, we should multiply the latter by the appropriate factor S_x, which we will here call the Stiles efficiency factor. This is also plotted in Fig. 196. A discussion of the quantitative aspects of these phenomena has been given by Moon and Spencer.[2]

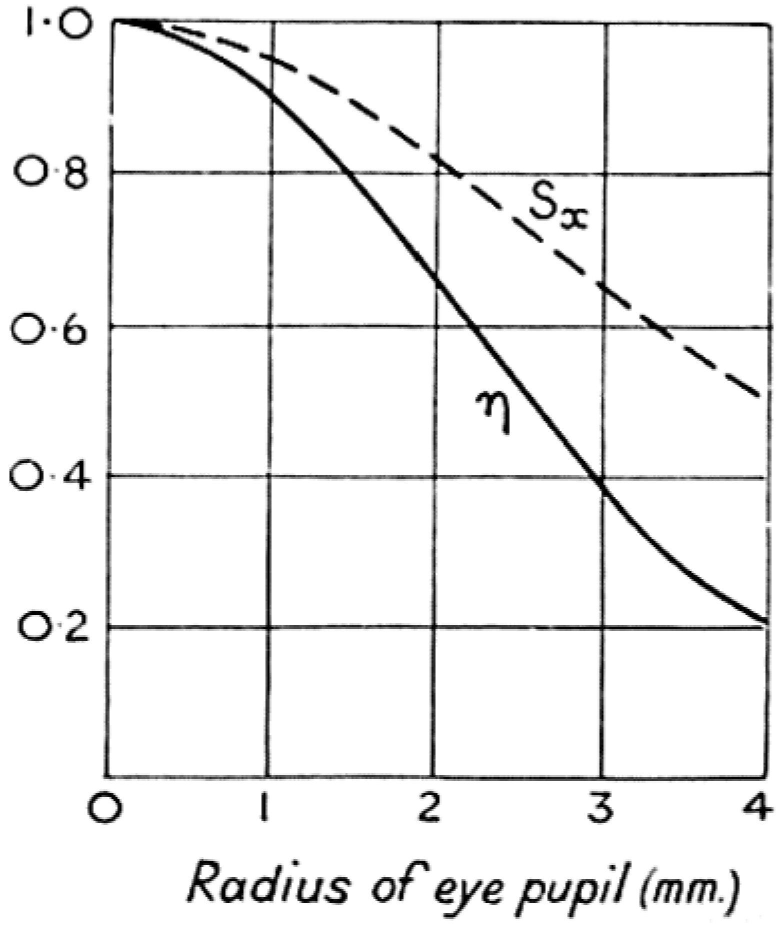

FIG. 196. STILES-CRAWFORD PHENOMENON

As will be seen below, cases arise when the image projected by a system is viewed so that the pupil of the eye is not wholly illuminated, as, for example, when using a microscope. The apparent luminance of the image will be further altered in the proportion $\left(\frac{p}{p_o}\right)^2 \frac{S_p}{S_{po}}$, where p and p_o are the restricted and full radii of the pupil respectively. This assumes, however, that the restricted pupil is uniformly illuminated. Owing to the reflection and absorption losses in instruments, losses which are usually larger for the more oblique pencils, the assumption is rarely strictly true; but it may be used to obtain a first approximation to the answer of several problems, as will be seen.

Transmission Factor of Photographic Lenses and other Instruments. The theorem of (6.09) indicates a ready means of measuring the transmission factor of an optical instrument such as a photographic lens or a telescope. There are instruments, known as brightness photometers which allow of the direct measurement of the luminance of a luminous surface. We have, therefore, only to measure (1) the luminance of the surface observed directly; (2) the luminance of the image of this surface as projected by the photographic lens or other system. Then the ratio of image to object luminance gives the transmission factor. It is to be noted that the relative positions and distances of the luminous surfaces are immaterial, provided that the photometer, whether "physical" or "visual" is used under suitable conditions, since the brightness of a surface is, as shown above, independent of the distance or direction of view provided that the angular subtense does not decrease below a certain value.

A suitable luminous surface is obtained by illuminating a piece of opal glass from behind. A photographic lens may be supported near the surface, and the luminance of the image of the surface is observed, holding a visual photometer aperture close to the lens. The latter is then removed and the luminance of the surface determined directly. For measurements on telescopes, the surface is held in the exit pupil of the instrument; its image is then observed in the entrance pupil, and the luminance can be determined with the photometer. These methods are much quicker and more accurate than the older methods using the optical bench, but in recent years most visual methods have been superseded by photo-electric measuring devices.

Transmission of Light by Optical Instruments. Physical Measurements. Vacuum photo-electric cells, gas-filled cells, photronic or "barrier-layer" cells, and electron-multiplier cells, when used with suitable circuits and galvanometers, all afford means (though with differing sensitivities) of measuring relative amounts of light.* In most cases, the use of special filters (or other means) of weighting the spectral components of the light in proportion to its luminous effect to an average human eye is required. Care has also to be taken that the mode of incidence of the light into the cell is standardized if strictly comparable readings are to be obtained; for example, a spherical integrator may be employed (Fig. 197). This can be a hollow metal sphere, say about 9 in. diameter, of which the interior is painted matt white. The sphere is pierced by circular holes, perhaps 1 in. diameter, symmetrically opposite to each other, and the photo-cell is applied to one aperture so as to receive light from the interior of the sphere. A parallel beam of light, entering by the opposite aperture, would fall directly on the photo-cell were it not for the interposition of a white matt opaque disc near the centre of the sphere which intercepts the light and reflects it to the surrounding walls. A great deal of multiple reflection takes place so that in the end the amount of light falling on the photo-cell is reasonably great, and its response will be closely proportional to the *total* amount of light entering the sphere and falling on the inner opaque disc; the diameter of the beam and its exact mode of incidence make no appreciable difference.

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* Vol. I, p. 236.

In order to measure the transmission of a telescope, a collimating lens is used to produce a parallel beam of light from a brightly illuminated pinhole. The diameter of the beam should be appreciably less than the entrance pupil of the telescope, and also small enough to enter the spherical integrator without obstruction.

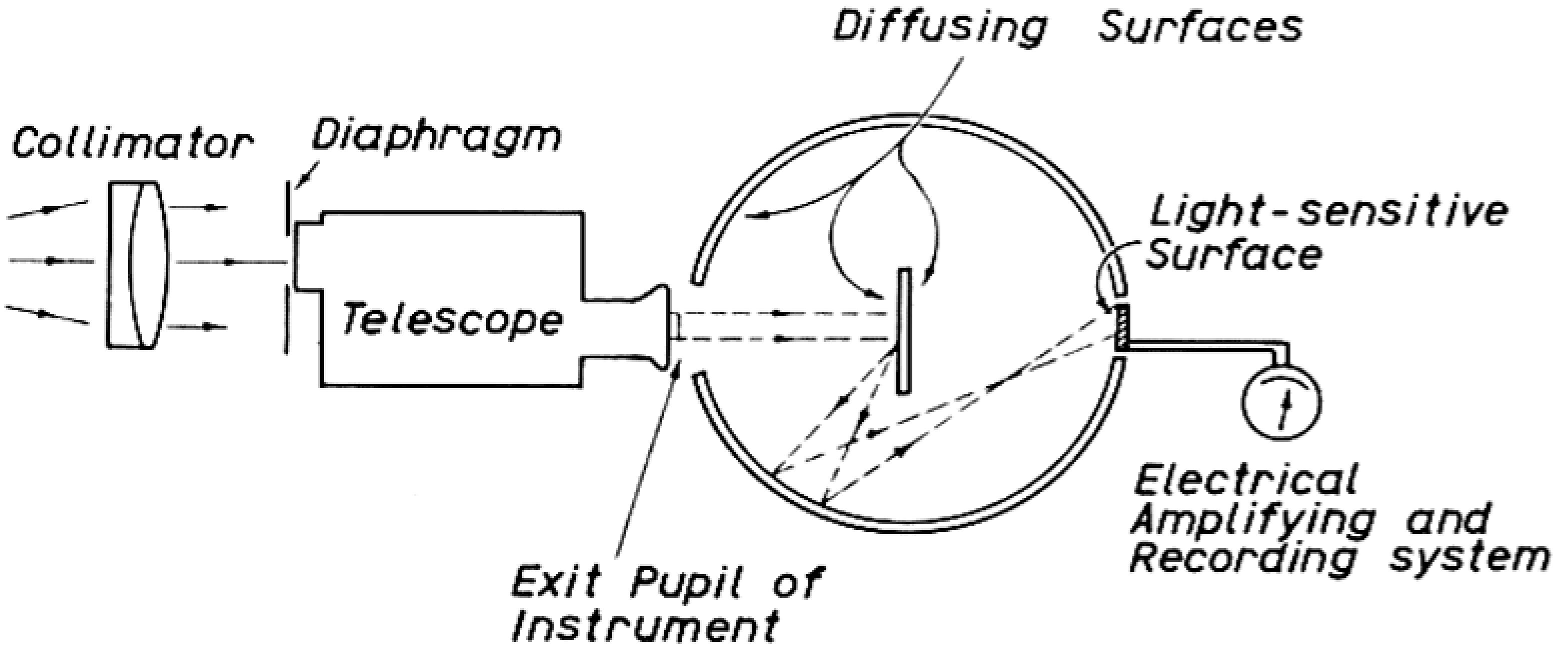

FIG. 197. USE OF SPHERICAL INTEGRATOR FOR MEASUREMENT OF THE TRANSMISSION OF A TELESCOPE (SCHEMATIC ONLY)

The beam is received in the integrator first *without* and then *with* the intervention of the telescope. In the second case the diameter of the parallel beam emergent from the eyepiece is smaller, but the *total* amount of light is the significant quantity. Subject to suitable experimental precautions, the ratio of the two photometer readings represents the "physical" transmission factor of the telescope. It has been found possible to use photronic cells for this measurement, the cell being simply connected in series with a suitable galvanometer.

In order to measure the transmission of a photographic lens we may first set up a diffusing surface S (matt, white, illuminated from the rear by a suitable lantern). Let its luminance be B. In front of this, at a distance of, say, 12 in. is placed an opaque screen with a small hole (*a*); this hole now acts, over a certain solid angle, as a small diffusing surface of the same luminance B. A suitable auxiliary lens is placed within this solid angle so as to form a real image of (*a*) on a photo-cell or better on a diffusing window in a box containing such a cell. This box is screened from all light except that reaching it from the auxiliary lens. The galvanometer reading is a measure of the illumination received from the aperature (*a*) with its luminance B.

Now place the lens to be tested behind the aperture (*a*) so that light from the diffusing surface must pass through it before reaching (*a*). The aperture is now illuminated by the image of S projected by the lens, and its luminance will be kB where k is the transmission factor of the lens. It is necessary to make sure that the interposition of the test lens does not restrict the solid angle of illumination from *a* so much that the auxiliary lens is no longer fully illuminated. The mode of incidence of the light into the photometer should be exactly the same in both cases.

The galvanometer reading is now somewhat smaller; the ratio of the new reading to the old (with suitable experimental precautions) is the transmission factor of the lens.

Needless to say the full technique of precise absorption transmission measurements is hardly appreciable from the foregoing brief descriptions, but if the fundamental principles of the present chapter have been grasped the general features of other physical measurements should present little difficulty, e.g. the use of a diffusing surface near the entrance pupil of a system to simulate the effect of a whole "sky" of uniform brightness and then the use of a box containing a photo-cell behind a diffusing window to explore the relative illumination of the focal plane of the lens.

Brightness of the Image in a Telescope. The above theorem regarding the "brightness" of the image projected by an optical system dealt with the luminance as a physical quantity. Given two surfaces of the same physical luminance, and radiating in accordance with the cosine law, they will only *appear* of the same brightness if the eye observes them under the same conditions; for example, with the same diameter of the eye pupil for each observation.

Consider first the case of the telescope; suppose it is used to view a distant luminous surface such as a region of the moon over which the luminance B is constant. Let us now regard B as the external luminance, and suppose that when we take account of the appropriate Stiles efficiency factor corresponding to the radius of the eye pupil, the effective luminance is BS_o; let this pupil area in unaided vision be ds_2.

Now suppose that the telescope of magnifying power M is held before the eye. Let the area of the object glass be A, and let the effective light transmission factor of the instrument be K. The luminance (in the limited physical sense) of the image projected by the telescope is now KB, and the area of the exit pupil of the instrument is $\frac{A}{M^2}$. If this area is larger than that of the eye pupil the impression of the luminance of the telescope image will be KBS_o since the eye pupil is wholly illuminated; but if the exit pupil is smaller than the eye pupil, the impression of luminance will now be proportional to the product of the available aperture of the eye pupil and the relevant efficiency factor. Thus

$$\frac{\text{Apparent luminance with telescope}}{\text{Apparent luminance without}} = \frac{K}{ds_2}\left(\frac{A}{M^2}\right)\left(\frac{S_1}{S_o}\right), \quad (6.11)$$

where S_1 is the Stiles efficiency factor for the radius of the exit pupil.

It is easy to show that when there is obstruction of the full pencils owing to the exit pupil being larger than the eye pupil, then the effective entrance pupil of the telescope is of area $M^2 ds_2$, so that the ratio of the retinal illumination in the two cases is simply determined by the transmission factor of the telescope, provided the eye pupil is of constant size.

Case of Star Images. The above considerations do not apply unless the distant luminous surface is of appreciable angular magnitude. In the case of such an object as a star, the area of the retinal image, in the case both of direct and aided observation, will usually be so small that its magnitude will be determined mainly from the optical imperfections of the eye system and the physical spreading of the image.

At any rate, there will not be a great difference in the retinal area illuminated in aided and unaided observation; but whereas in unaided observation we merely have the small area of the pupil to receive light, we can with a telescope capture the light received by the much greater area of a large object glass, and concentrate most of it into a retinal image of much the same extent as that found in unaided observation of a star.

Let J be the candle-power of a star, and r_o the maximum distance at which it can be perceived by the unaided eye. If p_o is the radius of the eye pupil the flux reaching it will be $J\pi p_o^2/r_o^2$ and the effective flux F_o reacting in the retina will be expressible as

$$F_o = k_o S_o J\pi p_o^2/r_o^2,$$

where k_o represents the appropriate physical transmission factor of the eye and S_o is the Stiles efficiency factor. If the star is just visible, F_o represents some threshold amount.

On the other hand, when a telescope with an exit pupil smaller than the eye pupil is employed, the effective light, F_1, reaching the retina from a star of the same candle-power at a maximum distance r_1 will be

$$F_1 = Kk_1S_1J\pi y^2/r_1^2,$$

where K and k_1 are the physical transmission coefficients of the telescope and eye system respectively, S_1 is the new Stiles efficiency factor, and y is the radius of the entrance pupil of the telescope. On the above assumptions the threshold values of the flux must be the same in both cases; thus, equating the above amounts F_o and F_1 we get

$$\frac{r_1}{r_o} = \frac{y}{p_o}\left\{\frac{Kk_1S_1}{k_oS_o}\right\}^{\frac{1}{2}};$$

i.e. the range of visibility for stars of the same candle-power is proportional to the radius of the entrance pupil of the observing system subject to the correcting factor in the bracket. Probably k_1 and k_o are not appreciably different in practical cases; any such difference in the eyes of the observers who originally measured the Stiles-Crawford effect would have been absorbed into the values of the efficiency factor; we will assume them equal. Taking a numerical case where $p_o = 4{\cdot}0$ mm., and $y = 500$ mm., we will suppose that sufficient magnification is employed to reduce the exit pupil radius to 1·0 mm. Then $S_1 = 0{\cdot}95$ and $S_o = 0{\cdot}484$, and

$$\begin{aligned} r_1 &= 125r_oK\ (1{\cdot}4) \\ &= 175r_oK \end{aligned}$$

The efficiency factor will clearly do much to offset the loss of light by absorption and reflection in the instrument. Again, if there is any general luminosity in the "background" of the sky, this will be diminished in the telescope image as calculated above; this effect tends to increase the contrast between the brightness of the star and the background, but the discussion of the threshold in such a case is a complex matter.

Night Glasses. It was mentioned above that the eye has the power of integrating the light in feeble stimuli spread over a retinal area corresponding to an angular diameter of about one degree in the field of vision. The "threshold" of perceptible luminance* is, therefore, inversely proportional to the square of the angular subtense of the stimulus. The smaller the "threshold," the easier the vision in faint light! When this angle rises above one degree, the relation tends to change, and after the angle reaches 2·5° the threshold was found by Piper[4] to be approximately inversely proportional to the angular subtense itself, and not to the square; this is true up to a subtense of about five degrees, after which the threshold tends to become independent of the angular size of the stimulus. We may put these results into symbols; let t be the threshold luminance, and α the angular subtense, then (approximately)

---

* Luminance is defined as the candle-power per unit area, in the usual photometric sense.

$$t \infty \frac{1}{\alpha^2} \quad . \quad . \text{ up to one degree}$$

$$t \infty \frac{1}{\alpha} \quad . \quad . \text{ from two to five degrees}$$

When using a telescope in faint light we have two cases to consider. In the first the exit pupil is larger than the pupil of the eye; the luminance of the image is $k\mathrm{B}$ and independent of the magnification. *While this is true*, we shall obtain continuous advantage by increasing the magnification since the angular size of the image will be proportional to m; this will hold good till the image we wish to observe subtends more than four to five degrees.

In the second case, the magnification has been so far increased that the exit pupil is now smaller than the pupil of the eye. The effective luminance of the image is now (neglecting the Stiles-Crawford effect) inversely proportional to the square of the magnification, so that to maintain a given apparent brightness of the field the external object luminance B must be proportional to m^2; but if the stimulus is a small one, so that its magnified image subtends less than one degree in the visual field of the eye, the allowable "threshold" varies as $\frac{1}{m^2\alpha^2}$ by reason of the physiological effect (where α is the angular subtense of the stimulus to the unaided eye). Hence

$$t \infty \frac{m^2}{m^2\alpha^2}$$

i.e.

$$t \infty \frac{1}{\alpha^2}$$

so that the threshold for small stimuli is almost independent of the magnification of the telescope. For larger stimuli with an apparent angle, in the instrument, of over one degree, the increase of magnification will obviously cause a disadvantage.

Experiments by the writer[5] seemed to show, however, that with small stimuli the visual threshold for the unaided eye was inversely proportional to a slightly higher power of the angular subtense than the second power up to about one degree of subtense. Confirmatory experiments with telescopes seemed to indicate that the expected slight advantage with increasing magnification was very noticeable. This was not fully understood at the time, but is now ascribed partly to the Stiles-Crawford effect. It is advantageous to use the smaller pupil. See also an article by Koehler.[16]

The majority of night glasses are binoculars for hand use, and the magnifying power should then not be greater than about six times, while the greatest efforts are made to secure the largest possible exit pupils and the least possible internal losses of light. This is only one aspect of the wider question of the efficiency of visual systems, which is dealt with in Chapter VII.

Brightness of the Image in the Microscope. It is seldom that the object in the microscope is self-luminous; in the majority of cases the condenser projects the image of some luminous surface into the object plane. Let K be the transmission factor of the microscope (supposed uniform for various ray paths), and b the transmission factor of an object element; then, if the eye pupil were wholly filled with light when observing the image, the effective luminance of the image element would be B Kb. But in the majority of cases the eye pupil is only partially illuminated. It was shown above that the radius p' of the exit pupil of the microscope is

$$p' = \frac{(NA)\beta}{m}$$

where NA is the numerical aperture, β is the distance of distinct vision, and m is the visual magnification. Hence, if p_o is the radius of the eye pupil when observing the source directly, we have, to a first approximation,

$$\begin{aligned}\frac{\text{Luminance of microscope image}}{\text{Luminance of source}} &= \mathrm{K}b\left(\frac{p'}{p_o}\right)^2\left(\frac{\mathrm{S}_1}{\mathrm{S}_o}\right)\\ &= \mathrm{K}b\beta^2\left(\frac{NA}{mp_o}\right)^2\left(\frac{\mathrm{S}_1}{\mathrm{S}_o}\right)\quad(6.12)\end{aligned}$$

where S_1 and S_o are the Stiles efficiency factors for the exit pupil and normal pupil size respectively.

Note that if the aperture of the objective is not filled by the condenser, then the illuminated area of the exit pupil will be smaller; the "numerical aperture" to be used in the above equation is, therefore, that numerical aperture of the objective which is effectively filled by the condenser.

Exposure in Photomicrography. In photomicrographic work we shall be concerned with the illumination of the image projected on to the screen. The condenser, we will suppose, projects an image of the source or effective source into the object plane, which may, as a first approximation, be supposed to radiate as a perfectly diffusing source into that aperture of the objective which is filled by the condenser. Let the luminance be B′, then the light received by the objective from a small area *ds* will be

$$\pi B' ds \sin^2 \alpha = \frac{\pi B' ds (NA)^2}{n'^2}$$

where α is the angular divergence of the extreme rays from the condenser, NA is the numerical aperture, and n' is the refractive index of the medium. But if B is the original luminance of the source (in air, say), and K_c is the transmission of the condenser, then the light entering the objective will be

$$\pi b K_c B ds \ (NA)^2$$

where b is the transmission of the object element as before, since

$$B' = K_c b B n'^2$$

The transmission of the microscope is K_m say, and the size of the corresponding image patch will be $M^2 ds$ where M is the linear magnification. Hence the illumination of the image will be

$$\text{Illumination} = \frac{\pi b K_c K_m B ds \ (NA)^2}{M^2 ds}$$

$$= \pi b K_c K_m B \left(\frac{NA}{M}\right)^2 \quad . \quad . \quad (6.13)$$

In practice the useful part of this result will be that the illumination is proportional directly to the square of the NA, and inversely to the square of the magnification; a relation which is of service in making photomicrographs of the same object (say) with different objectives. Assuming the reciprocity relation, the *exposure* will be proportional directly to M^2, and inversely to $(NA)^2$.

The Projection Lantern. The most widely used types of projection apparatus are those for the projection of transparencies, "lantern slides," and kinematograph film images. The "magic lantern" is said to have been invented by Roger Bacon, but detailed information on the lantern was first given by *della Porta* (1538–1615). One of the chief early difficulties was the lack of suitable light sources, as the early oil lamps without chimneys were not superseded by the Argand burner till the end of the eighteenth century.

The provision of a suitable source, of high intrinsic brilliancy, is still the *sine qua non* for successful projection. A small screen can be successfully illuminated by a coiled filament lamp, in which the coils are formed "in one plane" and occupy a restricted area. Flame arc lamps or Xenon arcs (enclosed in a fused silica envelope) are used for full-size kinematograph projection.

First Arrangement. In a very usual optical arrangement, where the source of light is small in comparison with the transparency, it is arranged that the condenser shall project an image of the source into the entrance pupil of the projection lens; the transparency is placed immediately after the condenser. The arrangement is shown diagrammatically in Fig. 198.

Assuming the angular divergence of the beams to be small, we may calculate an approximate expression for the illumination of the screen. Let ds_1 be the area of the source and B its average luminance, and let ds_1' be the size of the corresponding image in the entrance pupil of the condenser projection lens L. Let α' be the angular convergence (relatively to the axis) of the extreme rays from the condenser C to the lens L. If K is the transmission factor for the condenser, then the average luminance of the image is KB, and the total amount of light F passing into the projection lens is therefore given by

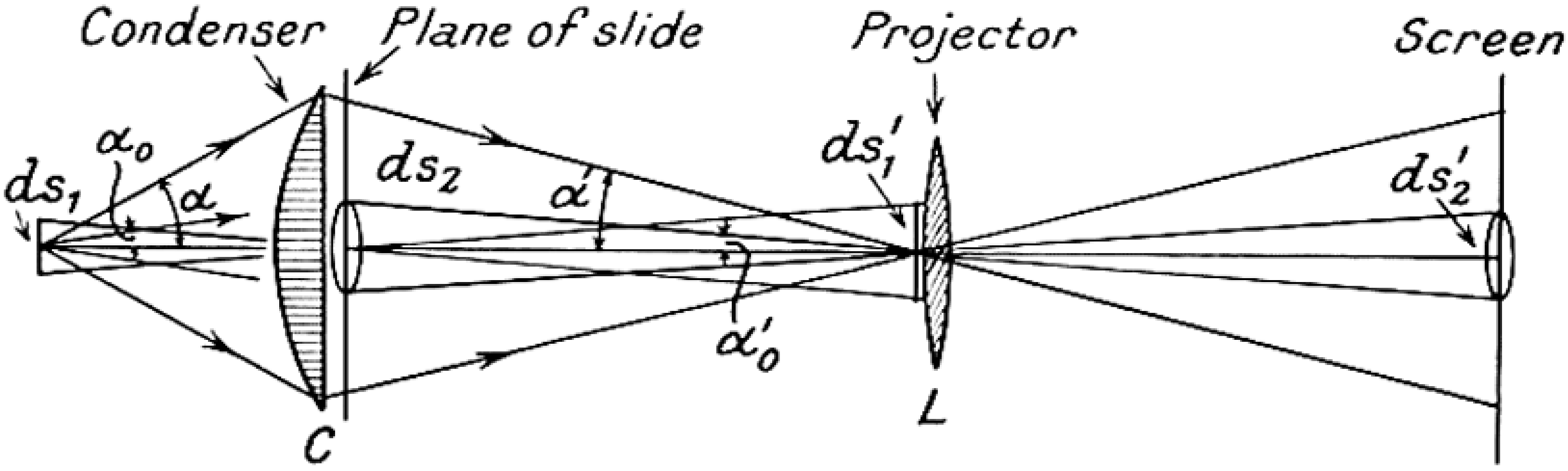

FIG. 198. FIRST ARRANGEMENT FOR PROJECTION
(Diagrammatic only)
(The image of the source is formed in the entrance pupil of the projector lens)

$$F = \pi K B ds_1' \sin^2 \alpha'$$

The entrance pupil of the projection lens will not lie far from its nodal point; assuming the angular divergence of the rays after passing through the projection lens to be α', the area of the screen illuminated at a distance l' will be (very nearly) $\pi (l' \sin \alpha')^2$; hence the ***illumination*** of the screen is

$$\frac{\pi K K_1 B\, ds_1' \sin^2 \alpha'}{\pi l'^2 \sin^2 \alpha'} = \frac{K K_1 B\, ds_1'}{l'^2} \quad . \quad . \quad (6.14)$$

where K_1 is the transmission factor of the projection lens.

This illumination is evidently proportional to the luminance and area of the source, provided that the projection lens is sufficiently large to transmit the whole of the beam from the condenser.

Further, our equations imply that the illumination of the transparency is equivalent to illumination by an extended source having the same average brightness as the actual one and placed just behind the slide, but allowing, of course, for the losses in the condenser lens.

Consider a small circular area ds_2 in the plane of the transparency. If B_1 is the effective luminance of this plane, the light sent from this area to the area ds_1' (the image of the source) in the entrance pupil of the projector is

$$\frac{B_1\, ds_2\, ds_1'}{l^2}$$

where l is the distance from transparency to entrance pupil.

Assuming that ds_2 is a small circular area of radius a we obtain

$$\frac{B_1\, ds_2\, ds_1'}{l^2} = \pi B_1\, ds_1' \frac{a^2}{l^2} \approx \pi B_1\, ds_1' \sin^2 \alpha'_o$$

where the *angle of convergence of the rays* between the axis and the most oblique ray from ds_2 passing through the axial point of the image ds_1' is α'_o.

But we know that the effective luminance of the source image ds_1' must be KB, where K is the transmission of the condenser and B the luminance of the source. Hence the light passing through this image and derived from ds_2 is

$$\pi KB\, ds_1' \sin^2 \alpha'_o$$

Equating the two values for this light from ds_2 we get

$$\pi B_1\, ds_1' \sin^2 \alpha'_o = \pi KB\, ds_1' \sin^2 \alpha'_o$$

so that

$$B_1 = KB \quad . \quad . \quad . \quad . \quad (6.15)$$

The effective luminance of the plane of the "object" is the luminance of the source multiplied by the transmission factor K of the condenser system.

Provided that all the light goes unhindered through the projection lens, there is no need for the strict condition that the image of the source shall be formed in the entrance pupil, but we can easily see that if the condenser is free from spherical aberration, so that a sharp image of the source is formed, then the above arrangement will allow us to use the minimum aperture of the projection lens while obtaining the maximum possible light with the given source and condenser, and at the same time illuminating as large a transparency as is possible. If under these conditions the entrance pupil of the projection lens is not completely filled with light, advantage may be obtained by using a larger source, or by pushing the source nearer to the condenser (see below).

Again, there is no need to place the transparency immediately behind the condenser. Provided it is wholly illuminated it may be placed anywhere between the condenser and the projected image of the source, provided that the projector lens can still receive all the light, and that the required projection can be effected.

Second Arrangement (Kinematograph Projector). We noticed above that the illumination of the screen is dependent (granted a sufficient aperture of the projection lens) on the area of the projected image of the source of light. If this source is practically uniform its image may be projected into the plane of the transparency, provided that the picture can be completely covered. In practice, the arrangement is mainly of interest in kinematograph projection, since the small area of the film picture can be covered by the projected image of the arc crater. The "condenser" is usually represented in practice by a mirror, but the theory can be illustrated simply by Fig. 199, in which the condenser is shown diagrammatically by a lens. The symbols *ds*, *ds'*, and *ds"* represent the areas of the source, intermediate image, and final image respectively. It will be clear that the illumination of the final image will be dependent on the angular aperture of the beam which is transmitted.

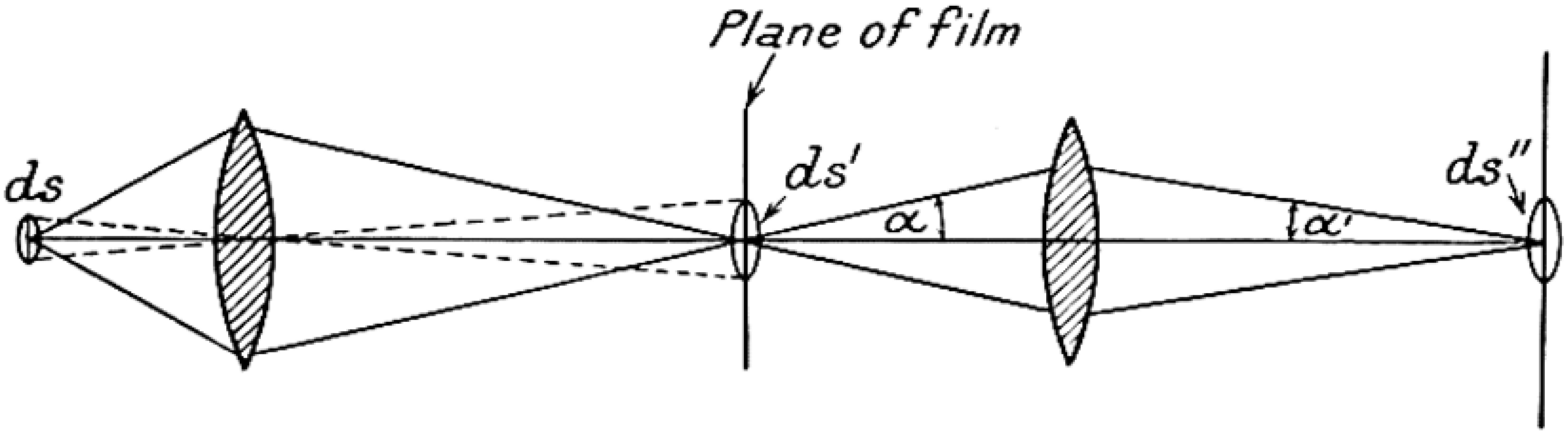

FIG. 199. SECOND ARRANGEMENT FOR PROJECTION
Aplanatic condenser projects image of uniform source into plane of transparency to be projected

If the angles α and α' represent the angles with the axis made by the extreme rays diverging to the projector lens and converging to the final image, we can write, using the symbols in the same sense as above,

$$\text{Total light received by the projector lens} = \pi K B ds' \sin^2 \alpha$$

$$\text{Hence the illumination of the screen} = \frac{\pi K K_1 B ds' \sin^2 \alpha}{ds''}$$

But the optical sine relation gives

$$\frac{\sin^2 \alpha}{ds''} = \frac{sin^2 \alpha'}{ds'}$$

so that the illumination E of the screen

$$E = \pi K K_1 B \sin^2 \alpha' \quad . \quad . \quad . \quad (6.16)$$

$$\approx \frac{KK_1B \text{ (utilized area of the stop of the projection lens)}}{\text{Square of distance from lens to screen}}$$

With such an arrangement, a projection lens of large aperture (low stop number) can be usefully employed if the condenser gives a beam of equally large angular aperture. The actual illumination on a screen, therefore, depends in each case, first on the illuminated aperture of the projection lens which would be seen to be illuminated when looking through a pinhole in the screen; secondly, on the intrinsic luminance of the source; thirdly, on the transmission of the optical system. With a condenser of large aperture we shall be able to utilize a given area of the stop with a projector of shorter focal length than with a condenser of smaller aperture, and hence to secure a proportionately larger picture of the same brightness. The advantage of the second arrangement is, however, that a mirror can be used as condenser as shown in Fig. 200. The plane of the image of the crater of a carbon arc is uniformly illuminated, but other sections of the beam will show more or less shadow due to the carbons. When high intensity arcs are employed, the flame of the arc contributes a proportion of the light much greater than in the case of the ordinary arc lamp, in which practically all the light is derived from the crater. It is advisable to use some special device to ensure that a negative carbon crater maintains its position very

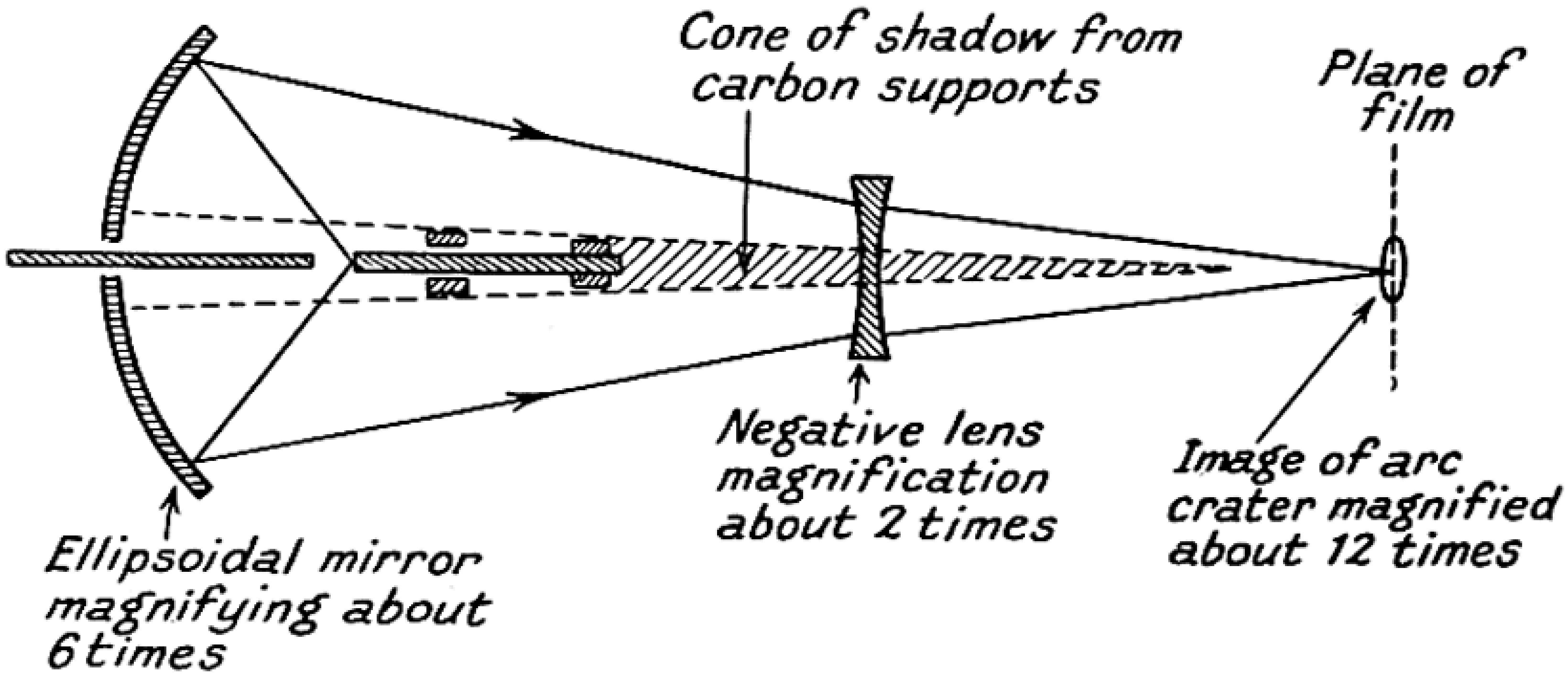
Cone of shadow from carbon supports
Plane of film
Negative lens magnification about 2 times
Image of arc crater magnified about 12 times
Ellipsoidal mirror magnifying about 6 times

FIG. 200. USE OF MIRROR CONDENSER

exactly, and that the length of the arc also remains constant. The electrodes of the enclosed Xenon arcs do not burn up like those of the carbon arc, so that the problem of continuous adjustment does not arise.

Flame or Xenon arcs constitute sources with appreciable axial lengths and finite volumes; hence discussions which relate to a small plane source are only partially relevant and the required uniform illumination of the transparency is secured by a compromise adjustment; it should be remembered that the various zones of a deep ellipsoidal condenser work at very different magnification.

Selection of Projection Lens. The focal length of the projection lens required under given conditions can be found from the approximate relation—

$$\text{Focal length in inches} = \frac{(\text{Throw from lens to screen in feet}) \times (\text{diameter of transparency in inches}}{(\text{Corresponding diameter of image in feet})}$$

For example, take a standard size slide with a picture 3 in. sq., say, and suppose it is desired to project a picture 8 ft sq. at a distance of 40 ft. Then

$$\text{Focal length} = \frac{40 \times 3}{8} = 15 \text{ in.}$$

In the case of the kinema projector, the width of the "gate" aperture exposing the film may be about 0·91 in.; to find the focal length of a lens to give a 13 ft picture at a 50 ft throw we have

$$\text{Focal length} = \frac{50 \times 0{\cdot}91}{13} = 3\tfrac{1}{2} \text{ in.}$$

Lantern Slide Projection Systems. It commonly occurs that the projection systems in ordinary use employ condensers having large residuals of spherical aberration. A common type of condenser has two "bulls-eye" lenses, each plano-convex, the convexities being turned together. The powers of the components are arranged so that roughly parallel light passes between the two members. The plano-convex lens is cheap to manufacture, and represents an approximation to the crossed lens giving minimum spherical aberration.

The size of the best concentration of light in the beam focused by the condenser is dependent partly on the spherical aberration of the condenser, partly on the chromatic aberration but only to a very small extent, and partly on the size of the source; the latter more especially when the source is of considerable magnitude, as in the case of a gasfilled lamp. The dimensions of the image of the source (neglecting spherical aberration for the moment) will be the greater the shorter the distance from the source to the condenser; but the total amount of light taken up by the condenser will be the greater if the source is pushed closer to it. It is, therefore, advisable to use a projection lens having a fairly large diameter of the back lens in relation to the focal length; then the source may be pushed in towards the condenser till the cones of light begin to be cut off by the mounts of the projection lens. The best condition will usually be such that the image of the source formed by the paraxial regions of the condenser lies beyond the projector, and since the spreading of the beam will be less with smaller sources, they may be brought closer to the condenser than when using larger ones without loss of light. The condition is illustrated in Fig. 201.

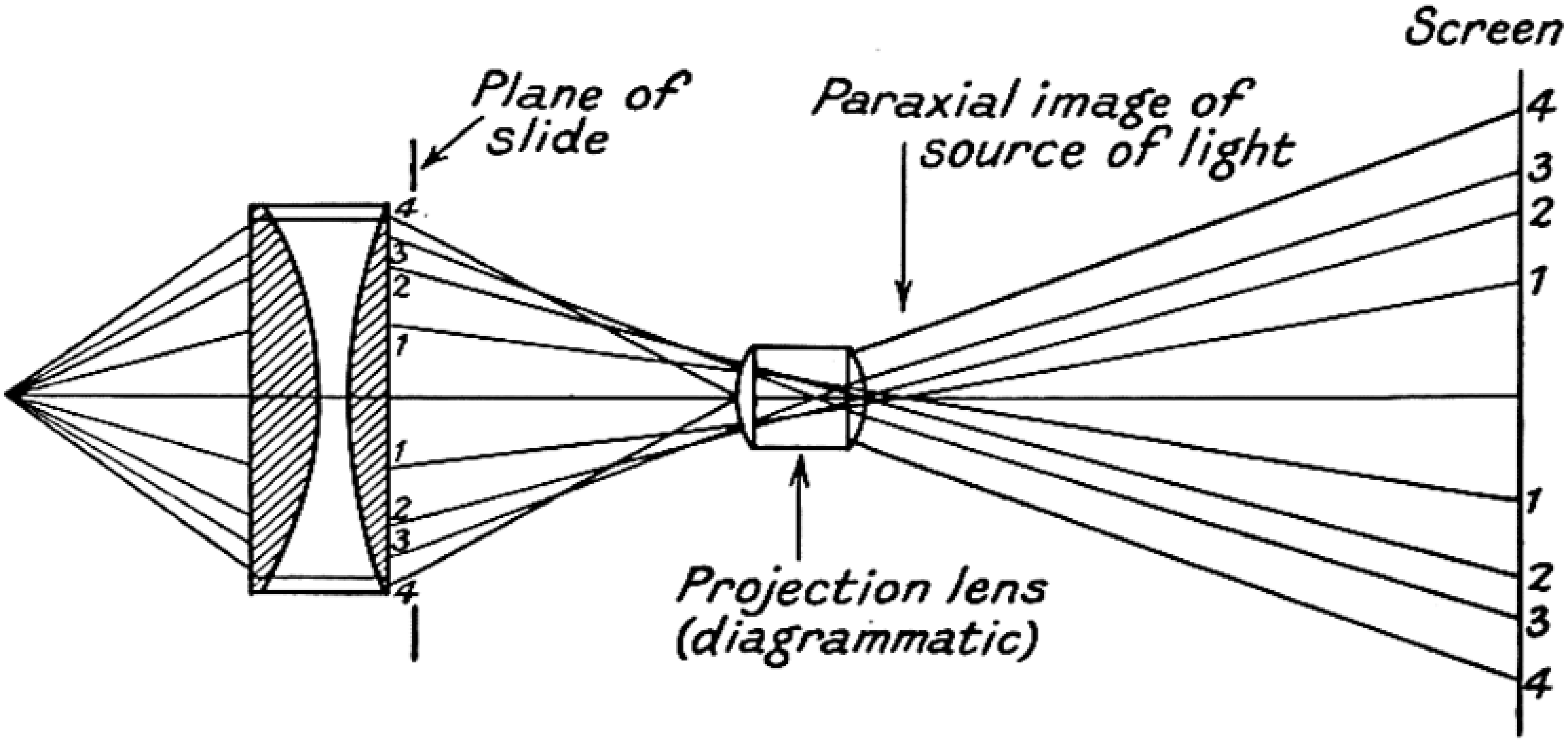
Screen
Plane of slide
Paraxial image of source of light
4
3
2
1
1
2
3
4
Projection lens (diagrammatic)
4
3
2
1
1
2
3
4

FIG. 201. PROJECTION WITH CONDENSER LENS SUFFERING FROM SPHERICAL ABERRATION (DIAGRAMMATIC)

If the projection lens is moved too far to the right the outer part of the field (ray 4) will be cut off; if moved too far to the left an annular zone of light (rays 2 and 3) will be cut off

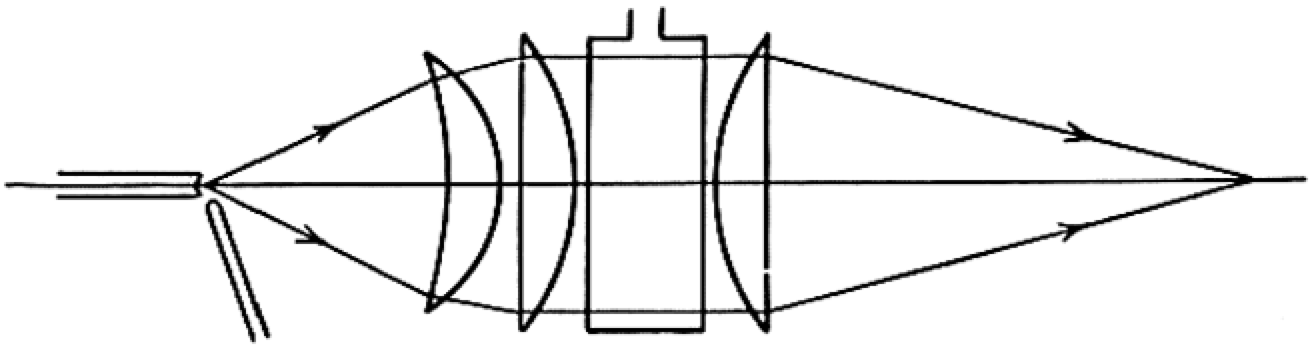

FIG. 202. TRIPLE LENS CONDENSER (WITH DIMINISHED SPHERICAL ABERRATION) INCORPORATING WATER-CELL FOR HEAT ABSORPTION

If specially figured or corrected lenses are employed in the condenser system (see p. 339), then the beam will be free from any spreading due to spherical aberration, and this again will allow of the maximum amount of light being taken up by the system consistent with given dimensions of the source. Fig. 202 shows a triple lens condenser which gives considerably diminished aberration. It has a trough containing water for cooling purposes.

The lens type chosen for projection is usually on the lines of the Petzval portrait objective, and may work at a high aperture, *f*/2·5 to *f*/4 or thereabouts. The Petzval type has then the advantage of the large diameter of its lenses for a given focal length, and consequent economy of light. These lenses are usually well corrected for spherical aberration and coma, so that the centre of the field is well defined. Owing to the long throw which is usually required, the angular field to be covered is not usually very large, and there is then no need for a more complex and expensive anastigmatic system, although cases do arise in which a high magnification is required for a comparatively short throw, and another type of lens giving a wider field may then be advisable; an anastigmat may then be chosen. The condenser must naturally be capable of giving the angular field required.

Loss of Light in Projection Systems. The average efficiency of ordinary projector systems is very low, only about 5 per cent of the total light from the source reaching the screen in many cases. In kinematography a further loss occurs through the cutting off of light by the revolving shutter.

Even though a mirror may be used behind the source of light, it is difficult to get much more than 10 per cent of the light through the slide carrier with ordinary systems, and reflections at the surfaces of the slide and projection lens cause further serious weakening. Numerical details of the kinematograph projector may be of some interest. It may be reckoned that at least 4 per cent of light is lost by reflection at a glass-air surface, and an absorption of 5 per cent per centimetre thickness of glass is not uncommon in the inferior glass often employed in condenser lenses. A two-lens condenser may therefore transmit only 70 per cent of the incident light, and the loss in a triple condenser would be considerably greater. Reflection losses can be considerably diminished by using "coated" lenses.

The gate at which the film appears is situated either at the focus of the condenser system, or at the "waist" of the converging cone if the system is subject to spherical aberration, and the loss here varies greatly with the source and the optical system employed. It is evident that the area of uniform illumination must overlap the gate widely in order to allow for slight variation in the position of the source of light due to wandering of the arc, or maladjustment of the carbons in the focus. Even in favourable circumstances a loss of about half the light seems difficult to avoid if a safe overlap is to be given.

The film itself, even in the most transparent part, removes about 20 to 25 per cent of the direct light, and an uncoated objective, usually with six "air-glass" surfaces, may remove a further 40 per cent, even if the absorption in the glass is negligible. We then have to consider the loss due to the kinematograph shutter which masks the image while the film is in motion from one picture to the next. This inevitably cuts off 50 per cent. Hence the transmissions of the various parts are likely to yield the following approximate table—

APPROXIMATE PERCENTAGE TRANSMISSIONS

Condenser	Gate	Film	Projector	Shutter	Combined
70	50	75	60	50	7·8

The arc lamp employed in kinema work may have a candle-power of 12,000, according to the current taken. Assuming that the condenser intercepts a cone of radiation of unit solid angle, which is an approximation to the truth, the total light entering the system will be approximately 12,000 lumens, of which 7·8 per cent reach the screen, i.e. 936. If the screen area is, say, 250 sq. ft, this means an illumination of 3·7 lumens per sq. ft, which is fairly satisfactory in practice. The use of coated surfaces will improve the result considerably.

The condenser losses are considerably reduced with the mirror and negative lens combination. The glass surface, instead of reflecting the light back, in this case sends it onward in the required direction, and the only serious losses are those due to any imperfection of the mirror reflection and reflection at the negative lens, which should not amount to more than 15 or 16 per cent when the surfaces are in good condition.

Episcopes. Episcopes are instruments designed for the projection of images of opaque objects. Such objects are illuminated very intensely with the aid of powerful lamps and reflectors, and their images are projected by anastigmat lenses of very large aperture.

Let us assume that an element of perfectly reflecting and diffusing surface, of area *ds*, is illuminated in the first place by a powerful electric lamp of 2000 candle-power, say, at 6 in. from the surface, as in Fig. 203. The lamp may be furnished with a reflector to enhance the illumination, and although the incidence of the light must be oblique, it may be possible to attain an illumination of 8000 ft candles on the surface, i.e. 8000 lumens per sq. ft must be re-radiated; but the total radiation = πB lumens per sq. ft. Hence, B = $8000/\pi$. The candle-power of the elementary area ds_1 is therefore 8000 ds_1/π, and the lumens radiated into a cone of semi-apical angle θ will be $\pi \sin^2 \theta ds_1 (8000/\pi)$. If the lens is working at $f/2$, say (the ratio of diameter of entrance pupil to focusing distance from object to pupil is $\frac{1}{2}$, so that $\tan^{-1}\theta = 1/4$, and we find $\theta = 14°$), $\sin^2\theta = 1/16$ approximately, so that the amount of light is 500 ds_1 lumens. This light, or a proportion of it depending on the transmission of the lens, is to be distributed over the magnified image of ds_1. If the magnification would be sufficient to enlarge a 4 in. picture up to 6 ft, the linear magnification will be 18 times, and the area magnification $= 18^2 = 324$. Hence, if the transmission of the lens is 80 per cent, we shall have 400 ds_1 lumens distributed over an area 324 ds, so that the lumens per unit area $= 400/324 = 1{\cdot}23$. This is the illumination of the screen in foot-candles, but allowance must further be made for the reflection factor of any actual object surface. It is difficult to get a bright image.

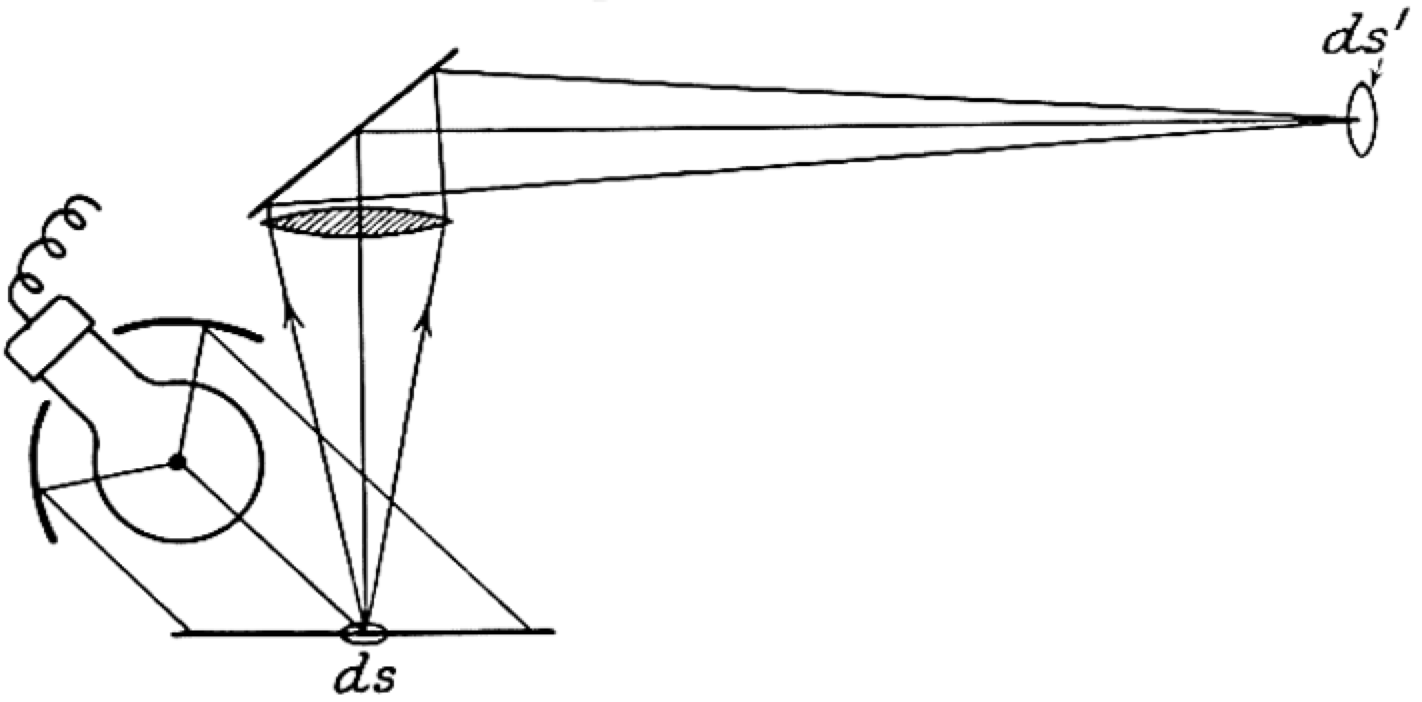

FIG. 203

The principle of the Ulbricht integrating sphere has been applied by Bechstein to the problem of Episcopic projection. Fig. 204 illustrates the principle. The sphere S has two main openings, one of which is filled with the projection lens L, the other is closed by the picture or surface, the image of which is to be projected. The interior of the spherical surface is coated with a matt white paint. Suitable paints can be secured with very high total reflection coefficients, i.e. up to 98 per cent.

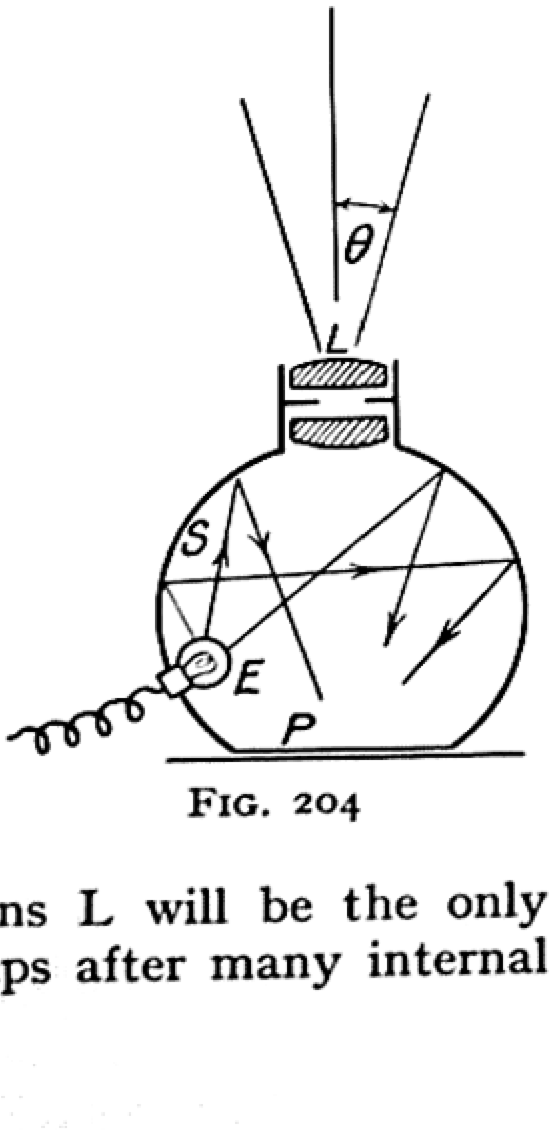

FIG. 204

Suppose for a moment that the paint was totally reflecting and absorbed no energy, also that the surface of the diagram P absorbed a negligible amount of energy, then if we introduce a source of light E into the sphere, the aperture of the lens L will be the only path by which the light can escape (perhaps after many internal reflections), and thus clearly *all* the light radiated by E would have to pass through L. It can be shown that the proportion of the light derived from the circular area P to the total light will be represented by $\sin^2 \theta : 1$, where θ is the semi-angular subtense of P (provided that we neglect the light directly escaping from E through L without internal reflection, or assume that a small diffusely reflecting screen is placed to stop such direct escape).

The above performance would be an extremely efficient example of projection, even as compared with the case of transparent objects; but in practice no such efficiency is attainable since the object for projection will absorb much light, the lens L will have a very appreciable aperture and may allow some direct light to escape, and a considerable absorption of light energy will take place at the walls of the sphere. In spite of these drawbacks, a good performance can be obtained.

Epidiascopes. A number of makers have introduced systems combining an episcope with an ordinary projection lantern; a mirror and condenser can be brought into action when required in order to change from one system to the other; such instruments are known as "epidiascopes." Lantern slides for projection are often called "diapositives" in Continental literature.

The Projection of Light. *Apparent Brightness of Image-forming System* (*Maxwellian view*). In a previous section we have calculated the effective luminance of the image projected by an optical system, assumed to be free from spherical aberration and to fulfil the optical sine condition. Referring back to Fig. 195 and the accompanying discussion, the *illumination* of the area covered by the image in the image plane (light per unit area) is

$$\pi \mathrm{BK} \sin^2 \alpha'$$

if α' is small, $\sin \alpha' = y/l'$ where y is the radius of the exit pupil of the image-forming system, so that

$$\text{Illumination} = \frac{\mathrm{BK}\pi y^2}{l'^2}$$

$$= \frac{\mathrm{BK}\ (\text{area of exit pupil})}{l'^2} \quad . \quad . \quad (6.17)$$

We therefore find that the whole exit pupil of the image-forming system is now radiating as a source of luminance BK, i.e. the luminance of the source multiplied by the transmission factor of the optical system. The "candle-power" of the radiant area is thus BK *times* (area of exit pupil).

It is not very easy to give an entirely satisfactory general proof that the same thing holds good whenever the eye views an optical system projecting the image of a uniform source in such a way that any ray from the eye traced back through the system intersects the source; the following treatment, however, may indicate that the principle is wider than might be inferred from the special case above.

Let A be a small area of an object of luminance B, and let RT (Fig. 205) be the trace of a refracting surface having its normal in the plane of the diagram; let the small area be rectangular with one diameter in the plane of refraction and one diameter perpendicular thereto. Let i and i' be the angles of incidence and refraction, then the angular width of the fan of rays (in the plane of the diagram) from the element to a point R of the surface is di, and the corresponding width when refracted is di'. The breadths perpendicular thereto are proportional to $\sin i$ and $\sin i'$ respectively. In order to see this, imagine perpendiculars dropped from each fan to the normal and that the diagram is rotated through a small angle $d\theta$ about the normal. The *locus* of the surface will be constant in the neighbourhood of the point R, and the arcs described by corresponding object space and image space elements at distances l and l' from R will be $l \sin i d\theta$ and $l' \sin i' d\theta$. The solid angle ω_1 subtended by A at R can be written

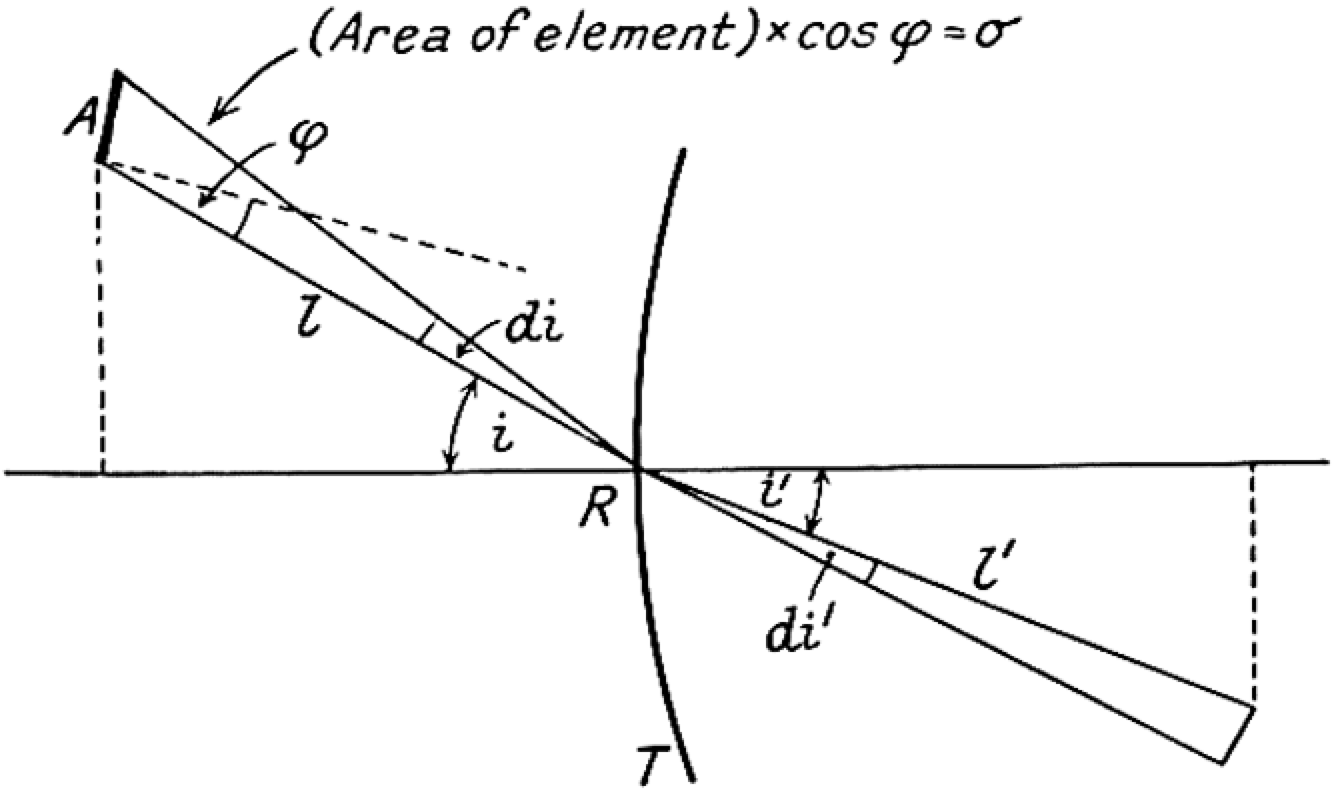

FIG. 205

$$\omega_1 = (l di)\,(l \sin i d\theta)/l^2$$

A similar expression is obtained for ω_2, the solid angle of the refracted bundle of rays. Then,

$$\frac{\omega_1}{\omega_2} = \frac{\sin i\, di}{\sin i'\, di'}$$

But
$$n \sin i = n' \sin i'$$

and
$$n \cos i\, di = n' \cos i' di'$$

Hence
$$\frac{\sin i\, di}{\sin i'\, di'} = \frac{n'^2 \cos i'}{n^2 \cos i}$$

so that

$$\frac{\omega_1}{\omega_2} = \frac{n'^2 \cos i'}{n^2 \cos i}$$

Let the projected area of the element A in the direction of R be σ, then the light sent from the radiating element to a small area ds on the surface is $(B\sigma\, ds \cos i)/l^2 = B \cos i\, ds\, \omega_1$

and the amount of light radiated by ds is

$$B_1 \cos i'\, ds\, \omega_2$$

where B_1 is the effective "normal luminance" of the element R, which will agree with its effective luminance to an observing eye of which the pupil is filled with the light. In order that we may have the amount radiated equal to k times the amount received (k being the transmission factor), we must have

$$B_1 \cos i'\, ds\, \omega_2 = kB \cos i\, ds\, \omega_1$$

so that

$$B_1 = \frac{kB\omega_1 \cos i}{\omega_2 \cos i'}$$

$$= \frac{n'^2}{n^2} kB$$

Hence the effective luminance of the optical surface would be equivalent to that of the source if there were no reflection losses; and if the initial and final media were the same, since the equation could be applied to any number of refractions. Thus for four refractions

$$B_4 = k_1 k_2 k_3 k_4 \left(\frac{n_4'}{n_4} \cdot \frac{n_3'}{n_3} \cdot \frac{n_2'}{n_2} \cdot \frac{n_1'}{n_1}\right)^2 B$$

$$= K\left(\frac{n_4'}{n_1}\right)^2 B \quad . \quad . \quad . \quad . \quad . \quad (6.18)$$

Reflection can be looked upon as a particular case of refraction.

This theorem is a very important one; it indicates, *let us repeat*, that when the eye pupil or other receptor is filled with the beam from a projector of any kind, and when all rays which could be traced from the eye back through the projector intersect the source, then the whole of the projector system has the effective luminance of the source, except for the losses due to reflection and absorption.

The appliances to be dealt with under this heading comprise searchlights, motor-car headlights, light-house projection systems, signalling lamps, and the like. It will not be possible to do more than to give the briefest outline of the theory and practice, since a very large technical literature exists in regard to all of these.

We showed above that the effective luminance of the radiating aperture forming an image of a surface of luminance B is KB, where K is the transmission of the system, and this agrees with the effective luminance of the image when *it* is observed by the eye. If the eye moves to various distances, the apparently illuminated part of the radiating aperture of the system will always have the same effective luminance KB, provided that the pupil of the eye is filled with light, or lies within the cone of radiation from the optical system; if, however, the pupil is not completely filled with light, the radiating aperture will appear less bright; or if there are possible ray paths between the eye and the optical system which, on being traced backwards, fail to intersect the source of light, then the corresponding parts of the optical system will appear dark. Some complication of the question of *apparent* luminance is caused by the Stiles and Crawford effect, but this (and questions of adaptation, etc.) will necessarily be neglected in elementary discussions.

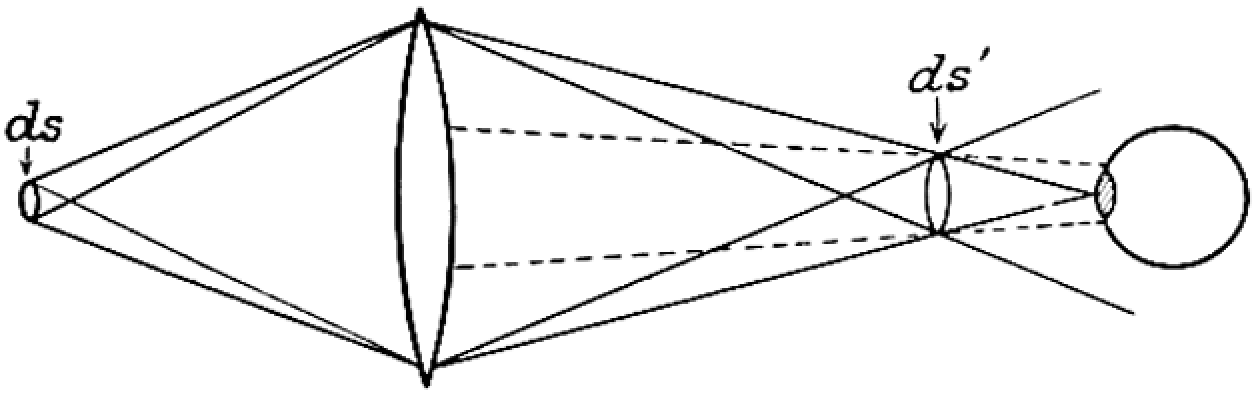

FIG. 206A

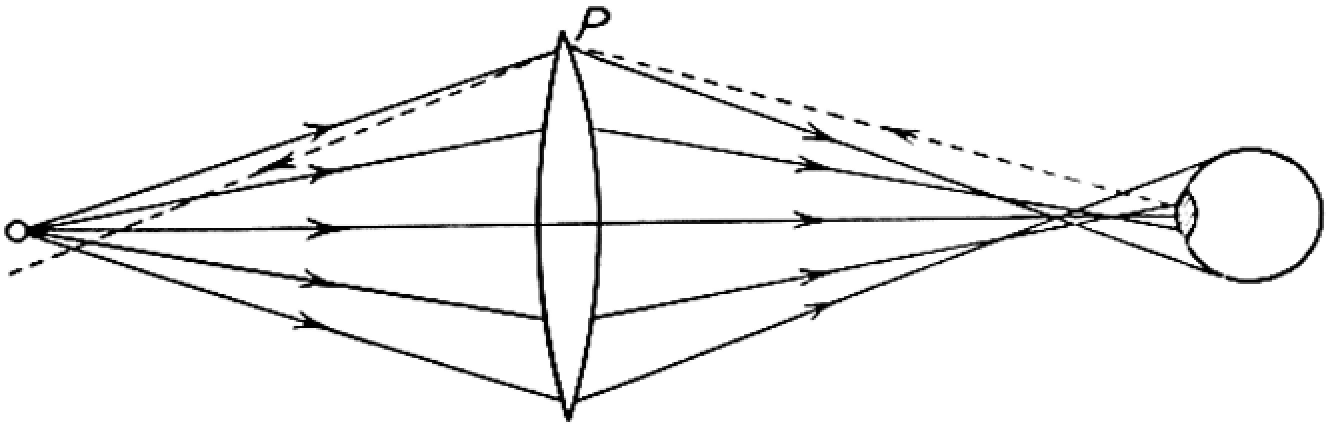

Fig. 206B

To make these principles more definite we will refer to Fig. 206A. The eye is withdrawn behind the image *ds'* of a radiating element *ds*; when the eye pupil was coincident with *ds'* the whole back surface of the lens appeared to be of luminance KB, since *ds'* was, we will say, slightly larger than the eye pupil. When the eye is withdrawn behind the image *ds'*, the area of the latter acts like a circular stop, and only a limited area in the centre of the lens can send light to the whole of the eye pupil; this part, therefore, appears of the full luminance KB, and is surrounded by a penumbral shadow. Again, consider the case of Fig. 206B, in which the image of a very small source is projected near the eye by a lens exhibiting strong spherical aberration; the centre of the lens appears filled with light, but the marginal zones are in shadow, since the marginal rays pass outside the pupil. We could infer the same thing if we trace a ray (dotted) from the pupil centre backwards through the margin of the lens system, and find that it passes outside the source; if, again, we take a ray through the marginal point P from the bottom point of the pupil and find that it intersects the source, while a ray from the top of the eye pupil fails to intersect it, we shall infer a partial illumination of the surface of the system near P.

If the aperture appears wholly illuminated to the observing eye, technical parlance speaks of a "complete flash"; otherwise we may have a "partial flash" if the apparent illumination is incomplete.

Effect of Non-fulfilment of the Sine Condition. We have seen that in a simple case the effective luminance of the flash is KB, and this was independent of any fulfilment of the optical sine *condition*, i.e. the constancy of magnification for the different zones of the system, although the optical sine relation expressing the magnification for a zone was used in the simple discussion. If the system does not fulfil the sine condition, the images of the source formed by different zones will have different sizes; hence the projected image will have a diffuse boundary. If the outer zones give the larger image, then an eye observing the lens from a point in the outer region of the image will see the margin of the lens bright while the centre parts are dark, and *vice versa*. Evidently, in order to obtain a sharply-bounded image, the fulfilment of the sine condition will be of importance.

Searchlights and Headlights. Searchlights almost invariably employ mirror reflectors rather than condensing lenses, since mirrors give freedom from chromatic aberration, and, moreover, involve less loss of light. The geometrical form of the paraboloid of rotation renders rays from the focus strictly parallel on reflection, although the system does not fulfil the optical sine condition. Hence, rays diverging from a point (in the focal plane) away from the axis are not strictly parallel.

The divergence from the sine condition for a zone of diameter D of a parabolic mirror of focal length f is shown by the relation (see (2.11))

$$\frac{y}{\sin\alpha'} = f\left\{1 + \frac{1}{16}\left(\frac{D}{f}\right)^2\right\}$$

If the sine condition were fulfilled (p. 42), we should have $y/\sin\alpha'$ as a constant. The defects are quite easy to realize geometrically (see Fig. 207); $y/\sin\alpha'$ is the distance PF for any zone; the above equation can easily be calculated from the equation to the parabola (see p. 82). The rays FP and FQ are rendered parallel on reflection, but we can see that the angle $\widehat{FPB}$ is greater than the angle $\widehat{FQB}$ if BP and BQ are rays from the extra-axial point B; the paths of these rays after reflection are PC and QD respectively, and hence the angle $\widehat{RPC}$ is greater than $\widehat{SQD}$, so that PC and QD are clearly convergent. Therefore the searchlight beam departs from parallelism, with a finite size of the source, on account of the geometrical properties of the image formation.

In practice, a single reflecting surface is too liable to tarnishing and damage. Hence, reflectors of silvered glass are generally employed, the silver backing being suitably protected by coppering and painting.

One plan is to grind a glass reflector of which both surfaces are congruent paraboloids of rotation[7]; thus the reflected component from the front glass surface should be "parallelized" in addition to that from the back. But, in practice, the finite thickness of the glass introduces a certain amount of spherical aberration into the beam reflected from the back, so that it is advisable to modify[8] the shape of the back surface in order to avoid this defect, and it is no longer a true paraboloid.

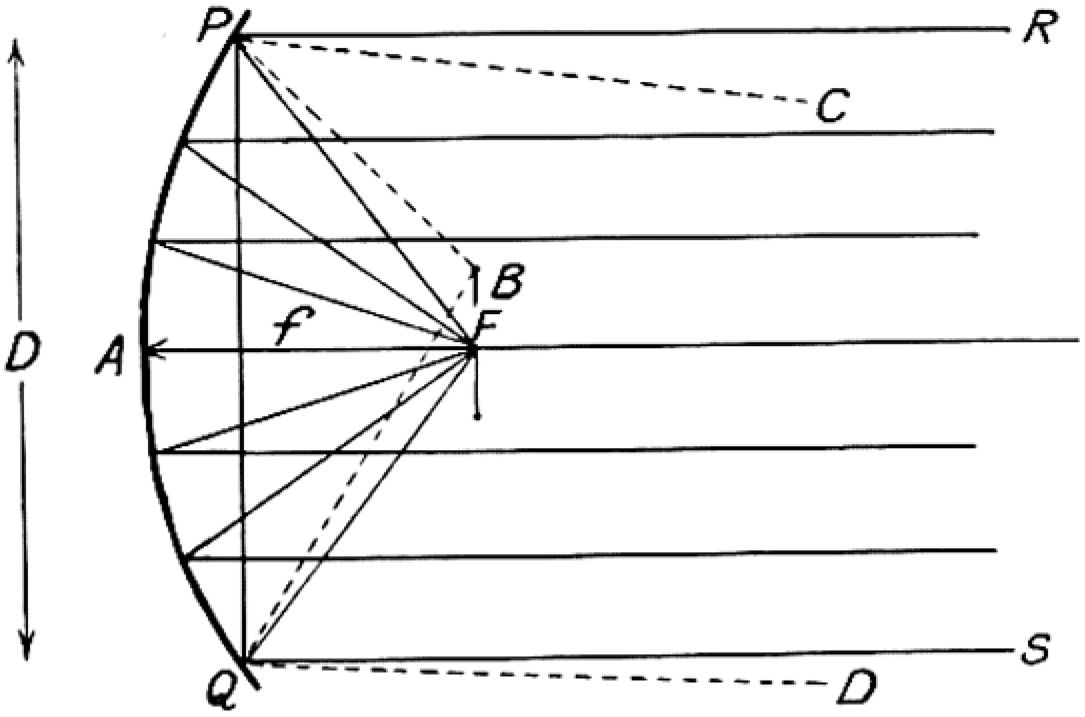

FIG. 207

In both the above cases, then, the reflected components from both front and back surfaces are more or less parallelized, and this is of importance in long distance projection where the utmost economy of light is required.

In the case of motor headlights, however, the lateral spreading of a certain amount of the light is highly desirable, and it is then possible to use reflectors (as in the Zeiss systems) in which the back reflecting surface is truly spherical, and the front surface is figured to a suitable non-spherical curve, refraction at which corrects the spherical aberration arising from reflection at the back surface.

The Mangin mirror (Fig. 208) (described by A. Mangin in 1876) consists of a glass reflector, silvered on the back, each surface of which is a true sphere. The curvatures may be so chosen that the system is freed from spherical aberration, and also from coma; thus it fulfils the optical sine condition. In consequence of this the projected image of a small source placed at the focus is fairly sharply defined, and the lateral spreading of the light is greatly restricted as compared with the effect of a parabolic reflector. The mirror is, consequently, very useful for signalling lamps where only a very limited region near a receiving station may receive the light.

The form of the glass reflector itself is that of a diverging meniscus lens, and the centre of curvature of the hollow side may nearly coincide with the focus. Since the thickness of the glass rapidly increases towards the margin, this sets a limit to the aperture ratio which can be effectively used; but the mirror can be made to subtend 130° to 140° at the focus.

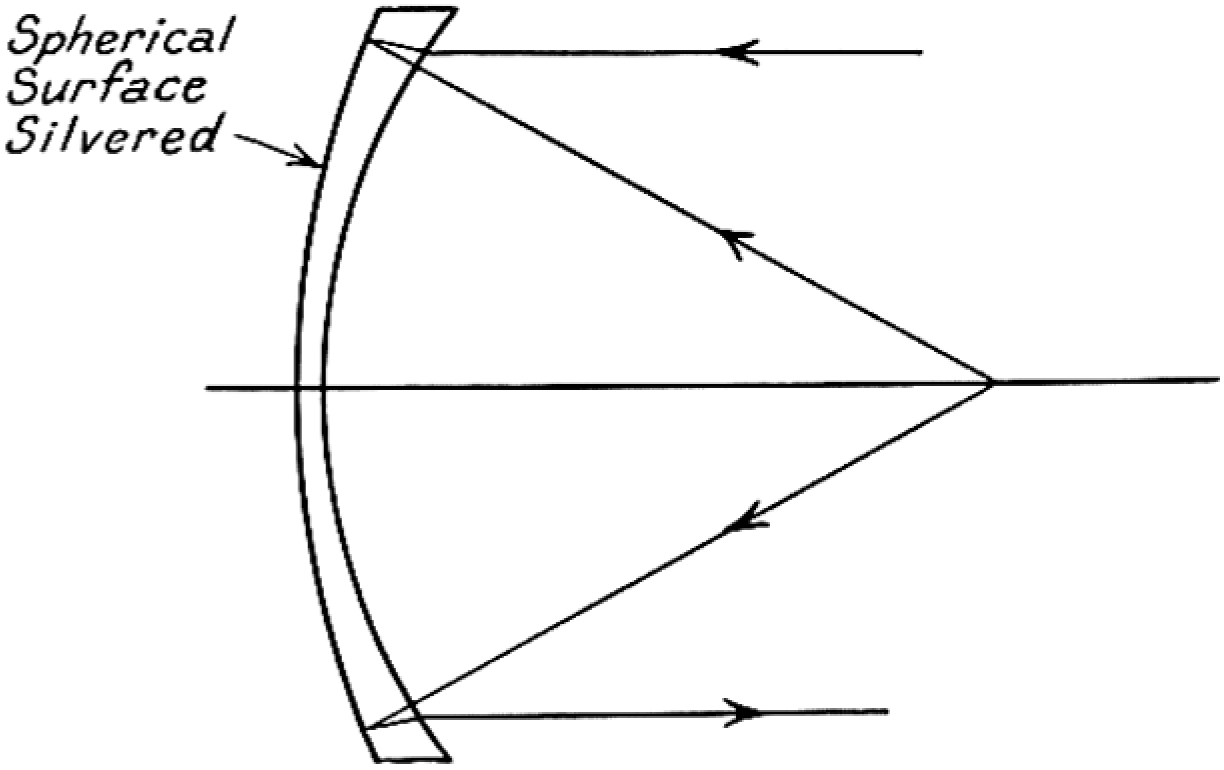

FIG. 208. THE MANGIN MIRROR

Though the Mangin mirror is effectively employed for the smaller searchlights up to about 20 in. in diameter, the parabolic reflectors (which avoid too great a marginal thickness of glass) are employed for the larger sizes.

Illumination due to Searchlight. Consider a small area in the field of illumination taken near the axis in a plane perpendicular to the axis. It is to be at such a distance from the projector that any ray taken from this area to the exit pupil of the system and traced onwards, will intersect the source of light. Under these conditions an eye placed at the position of our area would observe a "complete flash" in the projector.

According to the theory the illumination will then be

$$\frac{\text{BK (area of projector)}}{l'^2}$$

so that the effective "candle-power" of the projector in this area is, as before, BK × (area of exit pupil). The coefficient K must, however, include the possible effects of atmospheric absorption.

Large searchlights are made with mirrors up to 5 ft in diameter, but the difficulty of producing an accurate figure of the surfaces, as compared with the smaller 3 ft mirrors, causes the results obtained from the larger mirror to fall short of expectations. Assuming a 3 ft mirror and a carbon arc crater of an intrinsic brightness of 10^5 candles per square inch, a complete flash would give a gross candle-power of

$$\pi \cdot 18^2 \cdot 10^5 = 10^8 \text{ candles (approx.)}$$

But this figure will, in practice, be reduced very considerably by the obstruction of the negative carbon of the arc, the reflection losses in the mirror and in the front "window" which may be fitted to protect the arc from wind; so that the net result is not likely to exceed 60 per cent of the above figure in a clear atmosphere, even with a fairly perfect reflector. The majority of reflectors fail, however, to give a really complete flash owing to optical imperfections.

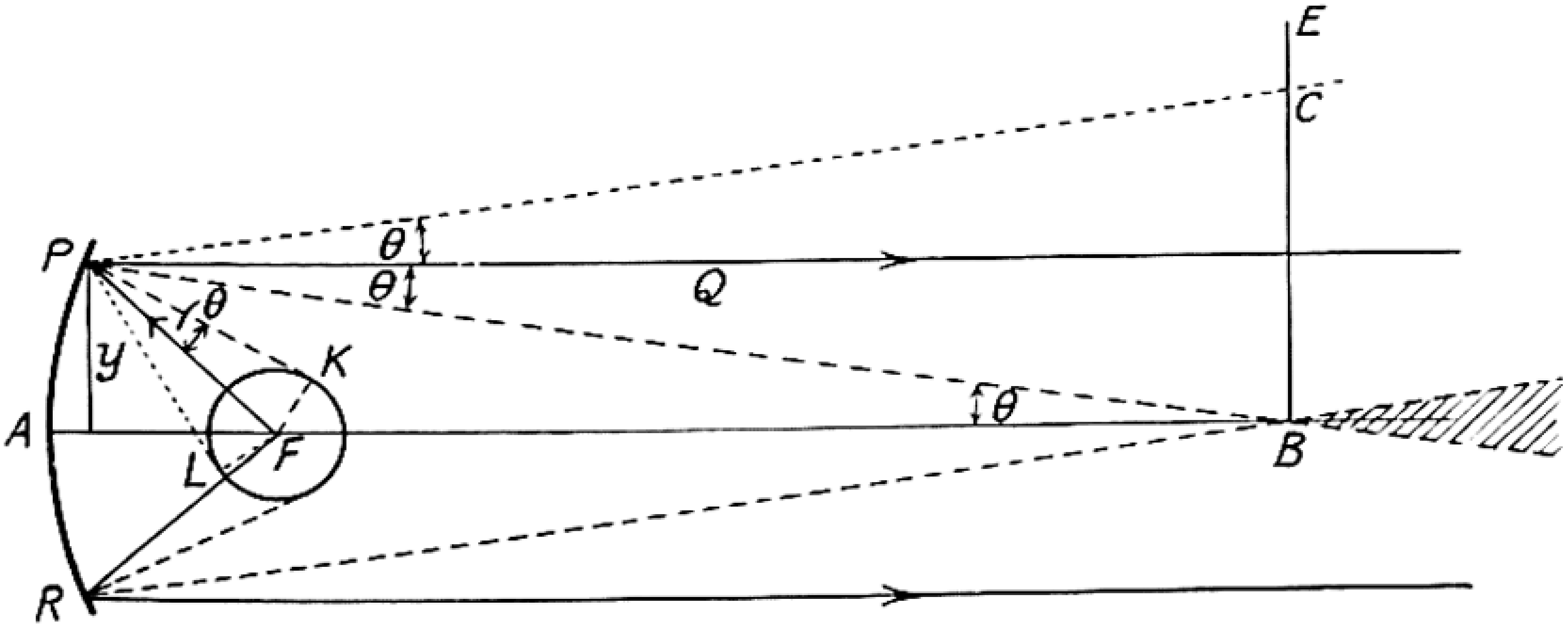

FIG. 209. OPTICAL PERFORMANCE OF PARABOLOIDAL REFLECTOR

High intensity mercury arcs are replacing the carbon arc in many searchlight systems.

Effect of Various Sources. In the simple theory above, the source was assumed to be an elementary disc perpendicular to the axis. The nearest approximation to this in practice is the crater of the arc in a searchlight, but smaller projectors may employ other sources, such as filament lamps or acetylene flames. We can usefully consider the illuminated area and the "flashing" of the radiant area under other conditions.

A typical example, Fig. 209, concerns a spherical source of radius $FK = r$, situated so that its centre falls into the focus. A ray following the path FP would be reflected parallel to the axis, but one derived from the extremity of the source as seen from P (i.e.

one following the tangential direction KP) will be reflected through B, where $\widehat{FPK} = \widehat{QPB} = \theta$. Since PQ and AB are parallel, the angle $\widehat{ABP}$ is also θ, and we have

$$\sin\theta = \frac{r}{PF} = \frac{y}{PB} = \frac{y}{AB} \text{ (approx.)}$$

if AB is very large in comparison to y, the radius of the zone marked by P. We have, by the geometry of the parabola,

$$FP^2 = y^2 + \left(f - \frac{y^2}{4f}\right)^2 = \left(f + \frac{y^2}{4f}\right)^2$$

$$\therefore \qquad \frac{r}{\left(f + \frac{y^2}{4f}\right)} = \frac{y}{AB} \text{ very nearly, if } r \text{ is not large}$$

and

$$AB = \frac{y}{r}\left(f + \frac{y^2}{4f}\right) \text{ very nearly.}$$

It is clear that a "complete flash" will be obtained at any axial point beyond this distance, and that the illumination at such points will vary (neglecting absorption effects) according to the "inverse square" law. A greater distance is required as the aperture of the mirror is increased.

A ray from the apparent extremity of the source L will be reflected so as to make an angle $\widehat{CPQ}$ with the line PQ. If ECB marks a plane through B perpendicular to the axis, the point P of the mirror will be apparently illuminated to an eye placed anywhere between C and B; on the other hand, a point R on the other extremity of the mirror diameter will be dark, and a complete flash will not be obtained. The flashing of the whole mirror will therefore only be obtained from points within the cone, represented by the shaded area in the figure, the generating line of which is the prolongation of the line PB or RB. At a distance D from the mirror the diameter of the fully illuminated area taken perpendicular to the axis is, writing d for AB,

$$\text{Diameter of fully illuminated area} = 2\,(D - d)\tan\theta$$

$$= \frac{2\,(D - d)\,r}{\left(f + \frac{y^2}{4f}\right)} \text{ very nearly.}$$

The "inverse square" law can only be supposed to hold along those parts of straight lines from the point A which lie within the shaded area.

Lantern and Lighthouse Projection Systems. Lenticular or "dioptric" condensers for the projection of light are of considerable importance in connection with lanterns and lighthouses. In a ship's lantern, concentration is only required in the sense that light should not be wasted in going much above or below the horizontal, but must spread freely in azimuth. The lighthouse beam must often be restricted in both directions. In the lighthouse the "lens" can be made of much larger aperture than would be practicable for any single parabolic reflector of such size, and several projectors can be grouped around a single source. Fig. 210 shows the section of a typical lighthouse "lens"; the whole would be realized by rotating this section about the horizontal axis through the source; a "ship's lantern" system would be obtained by rotating the central elements of the section about a *vertical* axis through the source. These stepped lenses were invented by Fresnel in 1822.

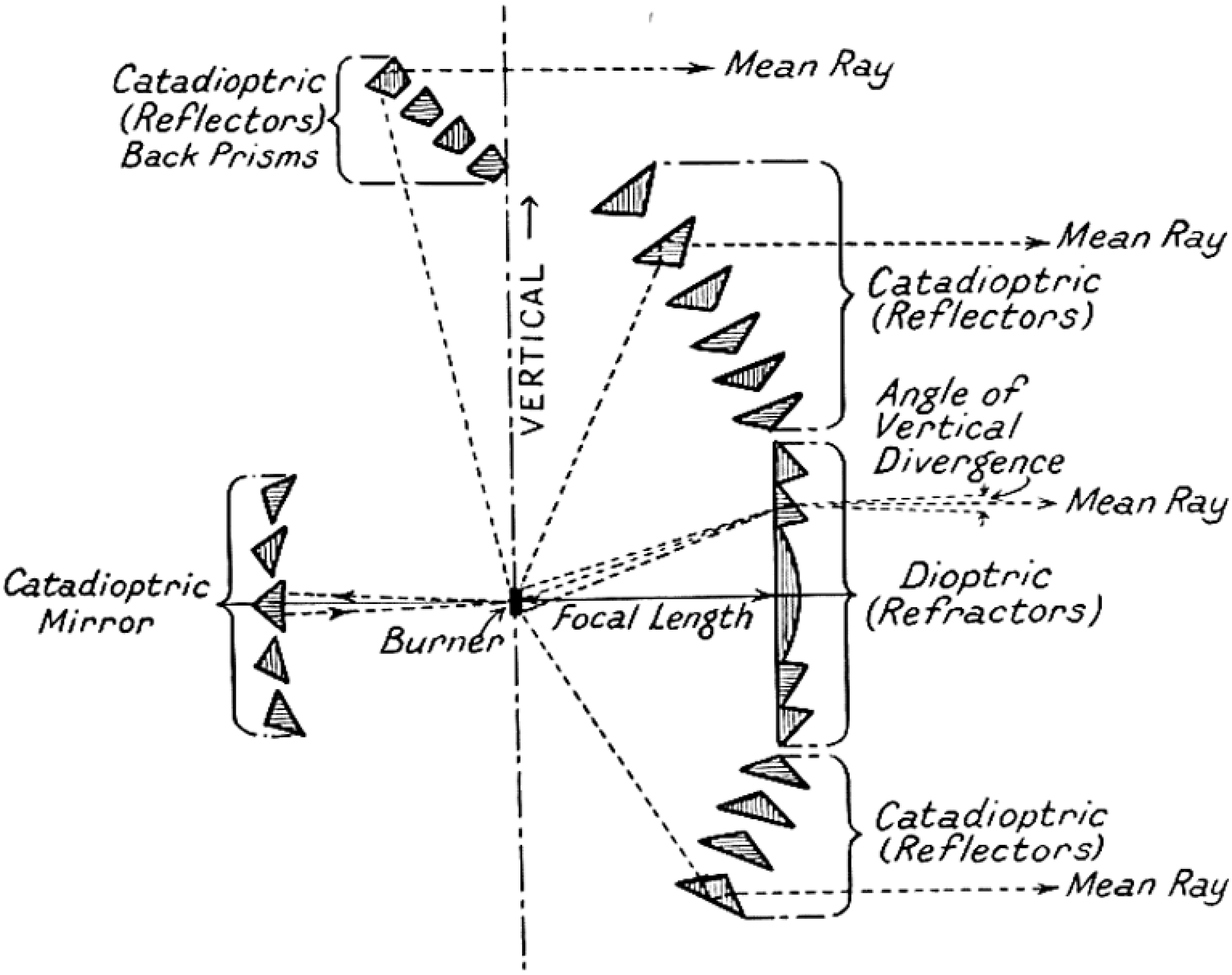
Catadioptric (Reflectors) Back Prisms
Mean Ray
VERTICAL
Mean Ray
Catadioptric (Reflectors)
Angle of Vertical Divergence
Mean Ray
Catadioptric Mirror
Burner
Focal Length
Dioptric (Refractors)
Catadioptric (Reflectors)
Mean Ray

FIG. 210. LIGHTHOUSE PROJECTION SYSTEM

The construction of the inner elements is a method of overcoming the great thickness of the lens which would be necessary if the front face were continuous; the discontinuous elements allow a great saving of weight and of absorption of light in the glass. For the outer zones, the elements are reflecting prisms, but one of the refracting surfaces of these, like the surface of the inner prism elements, is not of perfectly straight section; they are given a curvature which should make a ray from a point source at the focus emerge in a horizontal direction. The refraction at the inner prism elements is subject to chromatic aberration, but extreme spherical aberration of the outer refracting zones, which would be characteristic of a single lens with spherical surface, can be avoided by giving the correct form to the outer ring elements.

A theoretical account of the distribution of light in the beams has been given by W. M. Hampton.[9]

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CHAPTER VII

THE TESTING OF OPTICAL INSTRUMENTS AND THE STUDY OF THEIR PERFORMANCE

THE fundamental test of any instrument or appliance is that of satisfactory performance under ordinary conditions of use; but tests based on qualitative estimates of performance under working conditions demand much experience, and are best conducted by comparing any unknown instrument with one already proved to be satisfactory. The best laboratory methods of testing are those which tend to be as free as possible from errors due to defective eyesight or other disability of the observer. Particulars of qualitative tests are given in special treatises.

Quantitative tests of performance are usually to be preferred to qualitative or subjective estimates. For example, we may measure the magnification or variation of magnification (the distortion) numerically; also the "definition" of the image in terms of the numerical resolution limit; and the "contrast" of the image as a function of the contrast of the object field for details of given magnitude; such particulars may have to be given for various parts of the field of view. It is necessary, however, to guard against the too easy assumption that the whole of the possibly useful information about a system can always be expressed by one or two numerical data of such kinds.

Occasionally when a new system is being developed by a manufacturer, or in the investigation of optical systems for exact measurements (e.g. astronomical observations) the physical measurements of the defects of a system (aberrations) must be undertaken. It is not possible, in general, to infer the aberrations or necessary corrections of a system merely from measurements of resolution; and though it is possible to determine the whole performance of a system from a complete measurement of its aberrations, the process is so tedious as to be impracticable. The functions of such tests and measurements are therefore generally distinct and should not be confused.

Star Tests. The so-called "star" tests, though involving subjective reports, can be made to yield roughly quantitative estimates of aberration defects. The star, real or artificial, is a source of light, subtending an angle so small that its geometrical image is small in comparison with the resolving limit of the system. An artificial star for telescope testing may consist of a pinhole in tinfoil illuminated by the projected image of a small electric lamp, carbon or zirconium arc, etc. The pinhole need not be very small if it can be mounted at a distance great enough to make its angular size sufficiently small.

Extremely small round pinholes in tinfoil can be made by the use of a finely-honed needle point. The tinfoil is mounted on glass, and the needle is lowered by a thread so that it pierces the hole by its own weight; this technique is required in the most careful aspects of the Foucault test (see below).

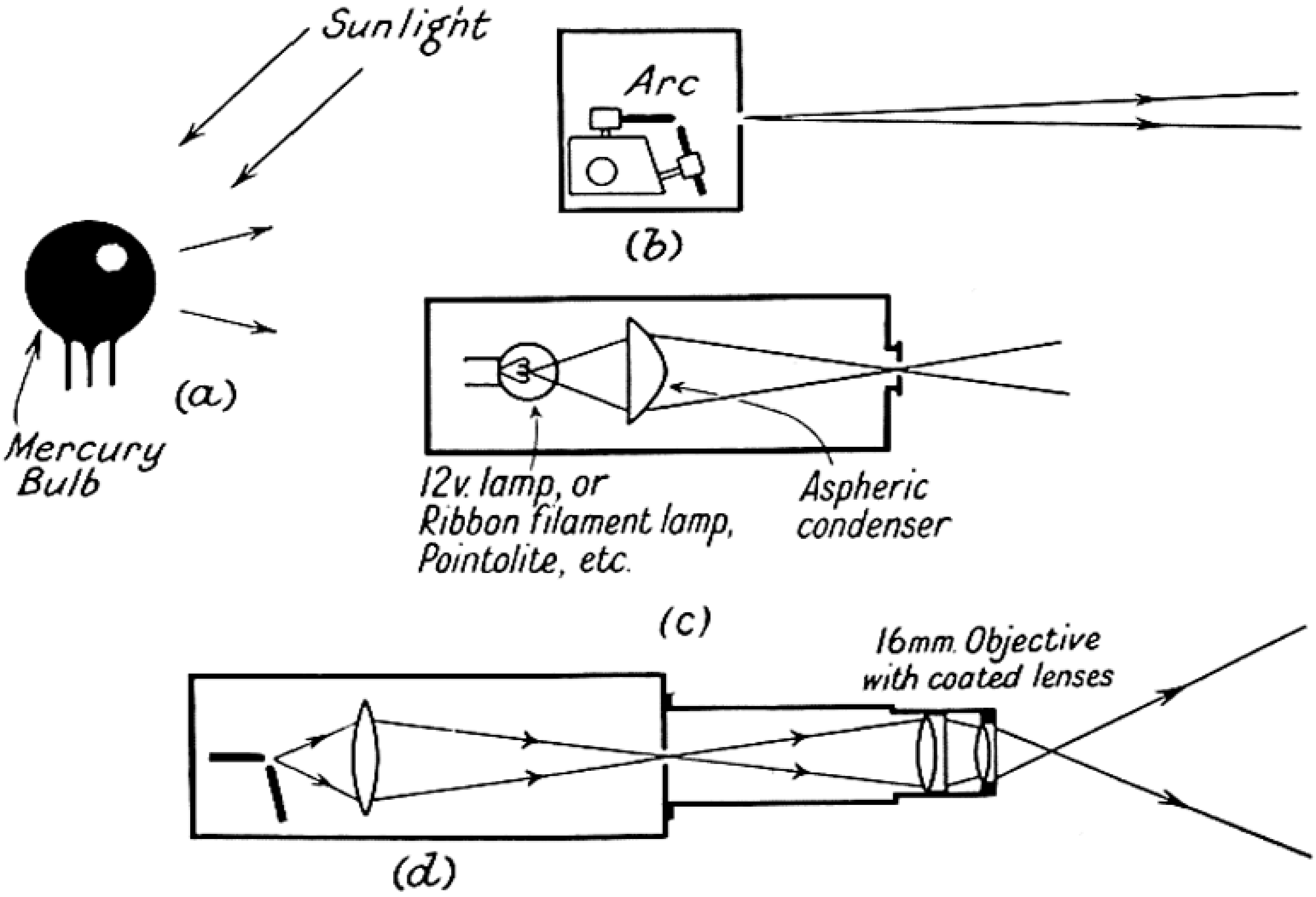

Sunlight
Arc
(b)
(a)
Mercury Bulb
12v. lamp, or Ribbon filament lamp, Pointolite, etc.
Aspheric condenser
(c)
16mm. Objective with coated lenses
(d)

FIG. 211. VARIATIONS OF AN ARTIFICIAL STAR FOR TELESCOPE TESTING

a = Mercury bulb
b = Source behind pinhole (may give narrow beam)
c = Arrangement for wider angle beam
d = Arrangement for very small star for close work

Artificial "stars" for the microscope may be obtained by silvering the undersides of a number of cover-glasses of varying thickness, without taking too great pains to remove dust and specks from the surface. It will usually be found that a number of small holes result in the film, which are well below the resolution limit of ordinary objectives. The glasses are cemented down on 3 in. × 1 in. slides, silvered faces down, and the "stars" are then illuminated by the microscope condenser, which forms the image of an arc or Pointolite lamp, etc., in the plane of the film.

Another device useful with low powers is to produce, by holding a slide above boiling mercury, a deposit of extremely minute mercury globules which form correspondingly small images of a source of light suitably disposed. The light may be thrown on the globules by the help of a small mirror, or even by a vertical illuminator system in the microscope.

Having secured a suitable "star" source, and made sure that it can send unrestricted light into the whole aperture of the objective, the appearance at the focus is then examined with the eyepiece or eyepieces supplied with the instrument. In many cases, nowadays, the objective and eyepiece systems are designed together and should, therefore, be tested together; but if odd objectives are to be tested, a good idea of their individual performances can still be obtained with the aid of a good set of Huygenian eyepieces of focal lengths ranging from, say, 25 mm. to 7 mm., if objectives of known performance are available for comparison. When using the highest power the Airy disc will be clearly seen in appreciable extension at the best focus with its surrounding rings, that is, with telescope and microscope objectives of usual types.

A complete discussion of the various points in the "star" tests would take too long; space can only be found for brief notes. Fuller details will be found in a booklet by Mr. H. D. Taylor[1] (*The Adjustment and Testing of Telescope Objectives*).

Axial Images; Centring. The first test is for *centring* of the lenses. The star image is brought into the centre of the field and the appearances are observed, both at the best focus and also when the eyepiece has been moved slightly within and without the best position. Complete symmetry of the distribution of light and colour around the axis indicates the correct centring and mounting of the lenses, but if one component lens has its centre displaced from the general axis of the system, unsymmetrical colour effects and coma will appear. The eyepiece is assumed to be sufficiently well centred by the tube of the instrument.

Squaring-on. Slight asymmetry of the appearance of the Airy disc rings near the best focus may indicate imperfect "squaring-on" of the objective; the optic axis of the objective, or one of the lenses, does not pass through the centre of the eyepiece. If this is suspected, tests should be made with a special "squaring-on" eyepiece.[2]

Colour Correction. An uncorrected objective, such as may be encountered in old telescopes or microscopes, shows brilliant colour effects with the artificial "star," especially on each side of the best focus. Inside the focus, the disc is fringed with bright red, passing through yellow to blue in the centre; outside, the disc is fringed with blue-violet passing through green and yellow to red in the centre.

An *achromatic objective*, if corrected for visual observation (Vol. I, p. 245), has its minimum focus for the apple-green region of the spectrum. Practically no colour is seen at the best focus, but inside the focus a yellowish disc (greenish at the centre) is fringed with an orange-red border, but the colour is very much less marked than with a non-corrected lens; outside the focus there appears an outer fringe of apple-green surrounding a yellowish disc with a faint reddish-violet centre.

The beginner should be warned that some of the colour effects, especially the red fringe inside the focus, must be partly attributed to the eyepiece and the eye, which are not chromatically corrected, and that the amount of colour may vary with different eyepieces, being less marked with the higher powers. It is advisable for him to make a start by observing with lenses known to be well corrected, and to work with different eyepieces so that the eyepiece effect may be allowed for; so, until much experience has been obtained, any unknown lenses should be tested against similar ones of proved performance. Strictly speaking, objectives should only be tested in conjunction with the eyepieces for which they are designed. For example, an objective designed for use with a four-lens terrestrial eyepiece will be considerably over-corrected (chromatically) as compared with an astronomical objective computed for a Huygenian eyepiece.

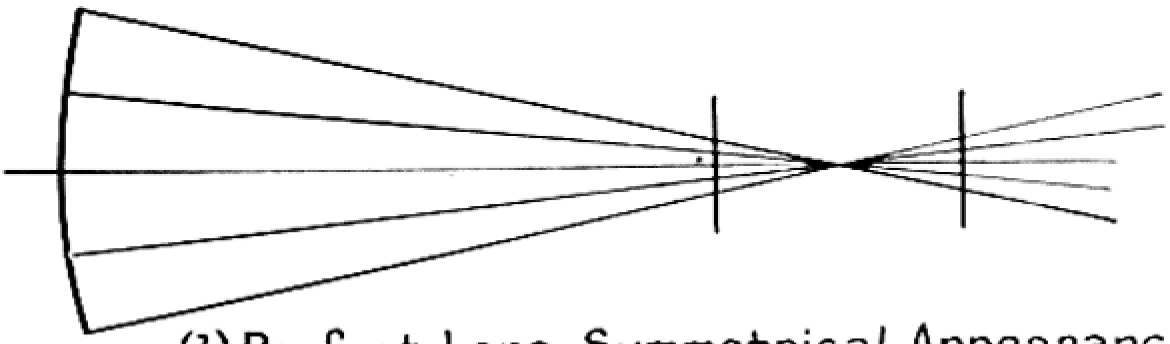
Apochromatic objectives should exhibit practically negligible axial colour (as also will, of course, reflector objectives): any considerable colour effects must be ascribed to the residual errors of eyepiece and eye, unless they are marked enough to be ascribed to faulty construction. Experience is necessary.

"Photographic" Colour Correction. It will be remembered that lenses designed for photography are usually achromatized by uniting the foci for the G′ and D lines of the spectrum, thus bringing the minimum focusing distance for the lens into the F line region (blue-green). The result of this will naturally be to enhance the red fringe inside the focus, and to make a blue-green instead of an apple-green appearance in the outer fringe outside the focus. Here, again, the performance of any lens should be compared with one of known colour correction—or the actual chromatic variation of the focus can be measured. (See below.)

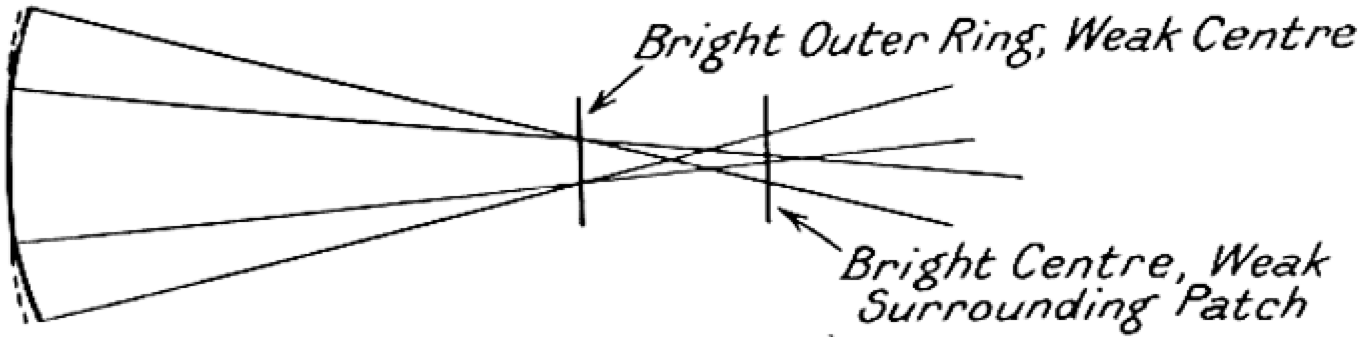
Spherical Aberration. This defect may affect axial images from telescope and microscope objectives, and may be due to various causes such as incorrect figure, wrong working distance, etc. The simple theory was discussed in Vol. I, Chapter IV, and the distribution of light near the focus in the presence of various aberrations was explained. In "under-correction," the type of aberration due to a double convex lens, the marginal zones have the shortest focusing distance, and the focal point for any zone is farther from the lens the smaller the radius of the zone. The reverse is the case in "over-correction." In "zonal" spherical aberration, however, the rays from an intermediate zone of the lens may have too small or too great a focusing distance in the cases of zonal under-correction and over-correction respectively. Some typical results may be left to the accompanying diagrams and pictures, which will explain them better than a great deal of verbal description. (See Figs. 212, 213, and 214.)

It will need a very high-power eyepiece (X20 or X15) clearly to distinguish the Airy disc with an objective (in telescope or microscope) of average type, but the out-of-focus effects can readily be observed with lower powers.

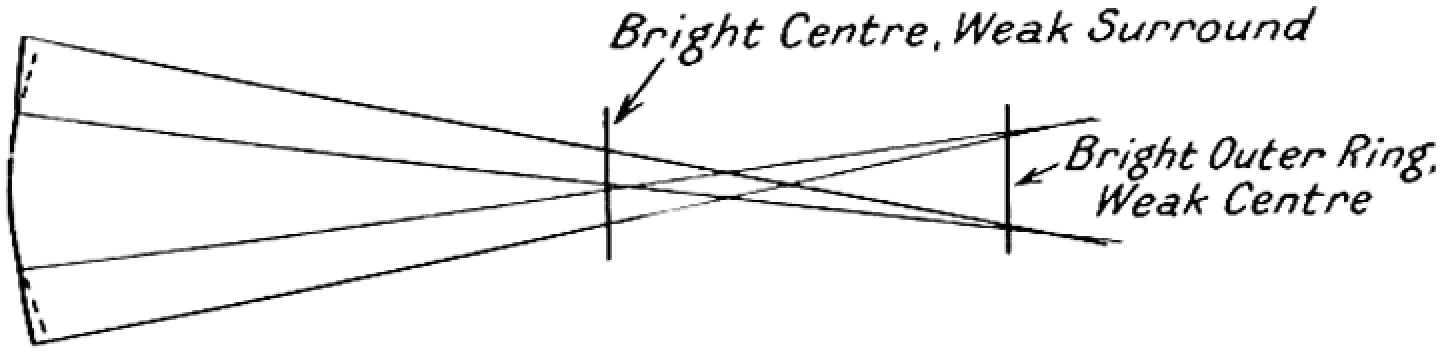
Defects in the Lens, Striae, Strain, etc. Striae in the glass of the lens may produce a marked "fuzziness" at the best focus and irregularities in the extra-focal appearances. Strain in the glass may be due to bad annealing, or undue pressure by the mount. The resulting distortion of the surfaces is also manifested clearly in the distortion of the extra-focal rings. Striae and strain are to be tested for independently by the



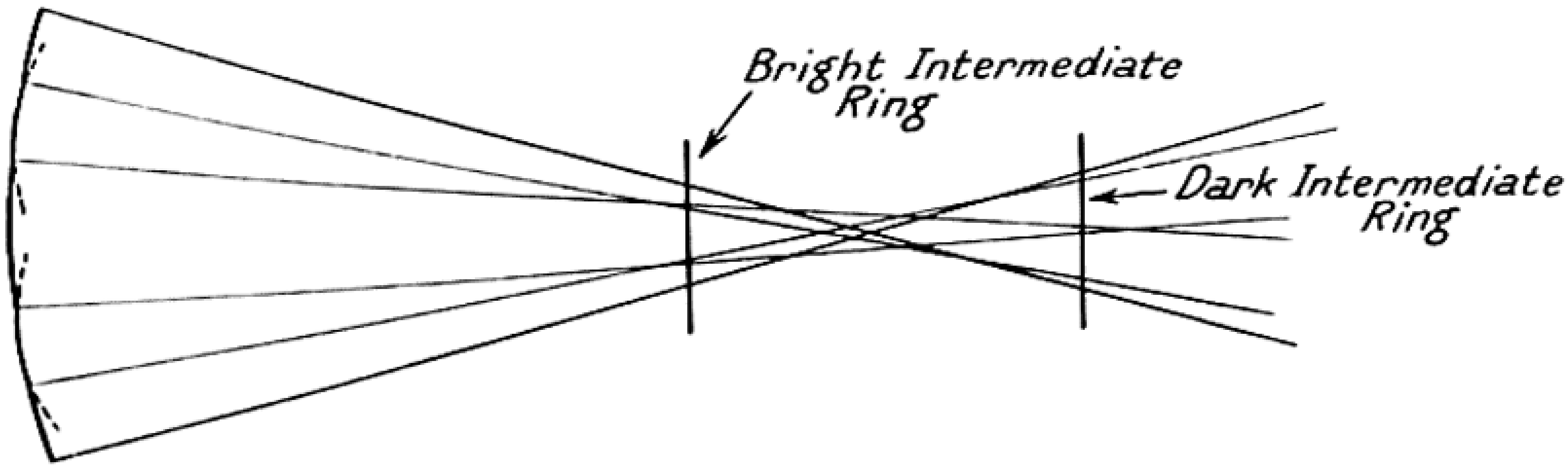
(1) Perfect Lens. Symmetrical Appearances on Each Side of the Focus


Bright Outer Ring, Weak Centre
Bright Centre, Weak Surrounding Patch

(2) Spherical Under-correction



(3) Spherical Over-correction



(4) Zonal Under-correction

Fig. 212. Ray Diagrams for Star Test

Foucault test and the polariscope respectively. (See Vol. I, pp. 283, 259, and below.)

Unless there is a very marked physical defect in the glasses or surfaces, there will usually be one stage of the focus in which a concentration very closely approximating to the Airy disc appears, but when aberration is present light leaves the centre disc and appears in the surrounding rings.

Extra-axial Aberrations. If chromatic or spherical aberration are present on the axis they will persist over the whole field, but if they are absent on the axis, the primary aberrations which may still afflict the image definition in the outer part of the field are *chromatic difference of magnification*, *coma*, and *astigmatism*. The general nature of these defects have already been explained in Vol. I, Chapter IV. Again, it is important to test an objective with the proper eyepiece, for we cannot easily dissociate the effects of the two. Chromatic difference of magnification causes an extra-axial

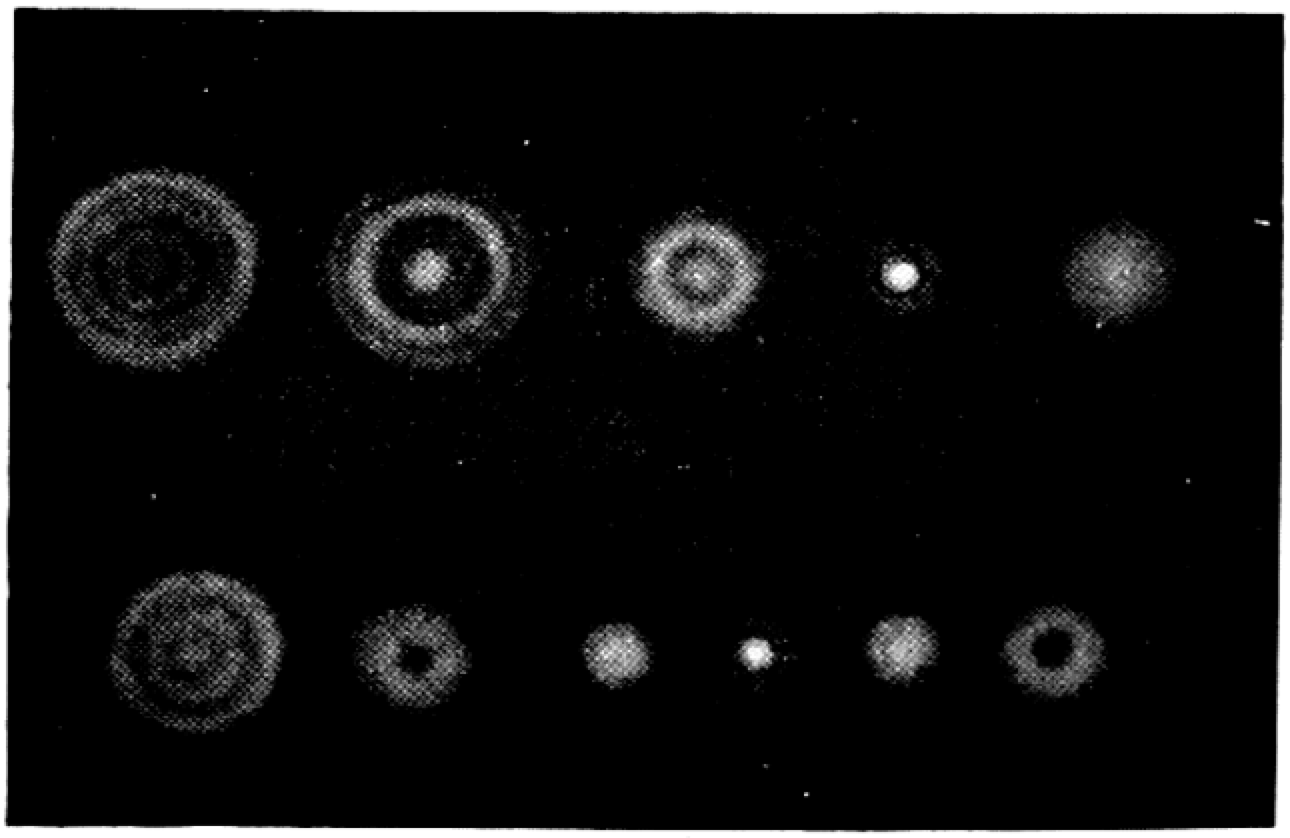


FIG. 213. STAR DISCS NEAR THE FOCUS

Top: spherical aberration present
Below: no aberration

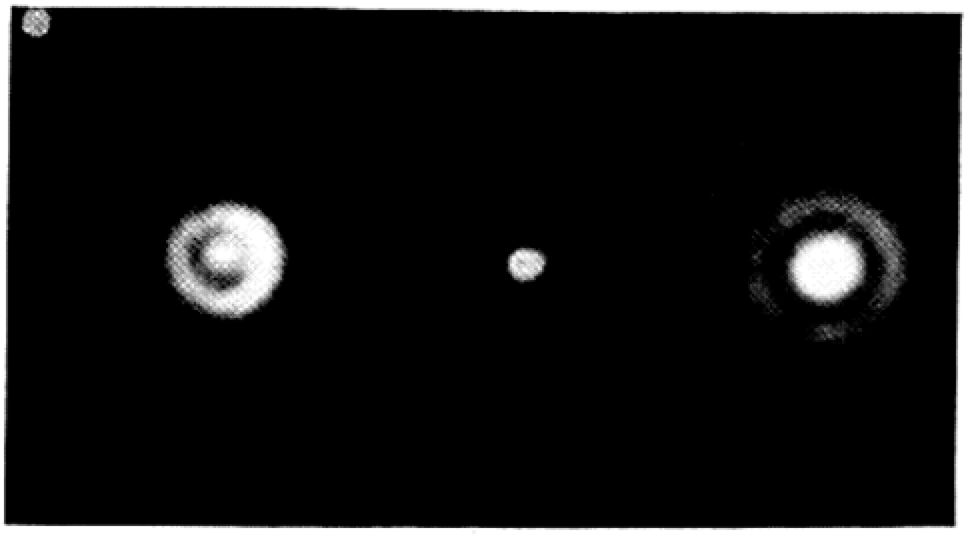


FIG. 214. STAR DISCS NEAR THE FOCUS IN THE PRESENCE OF ZONAL SPHERICAL ABERRATION

"star" to appear as a short radial spectrum (with the colours crowded together and mostly overlapping). Coma gives an unsymmetrical side distribution of light (see Fig. 215); while astigmatism gives a radial or tangential line. All ordinary telescope objectives show astigmatism of the oblique pencils, and if more or less coma happens to be present also, the appearances are very complex. Astigmatism does not appear on the axis unless one or more of the surfaces of the lens have a cylindrical error or a lens is badly tilted. If the lens suffers slightly from coma, then a small error of squaring-on will cause the central image to show a coma effect.

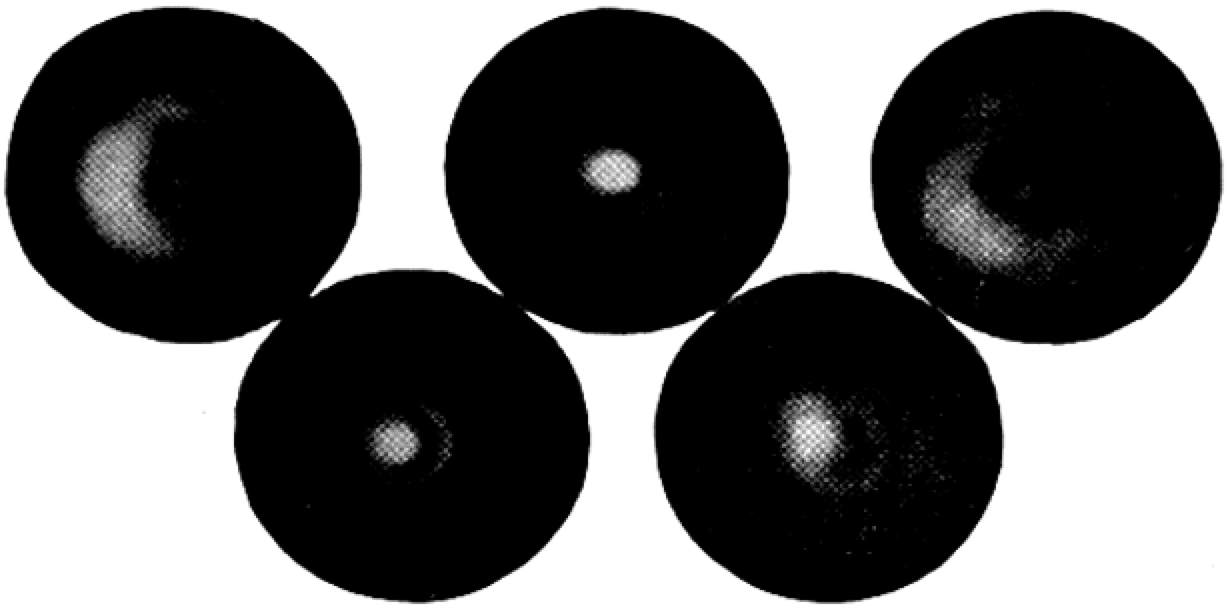


FIG. 215. EFFECTS NEAR THE FOCUS IN THE PRESENCE OF PRIMARY COMA

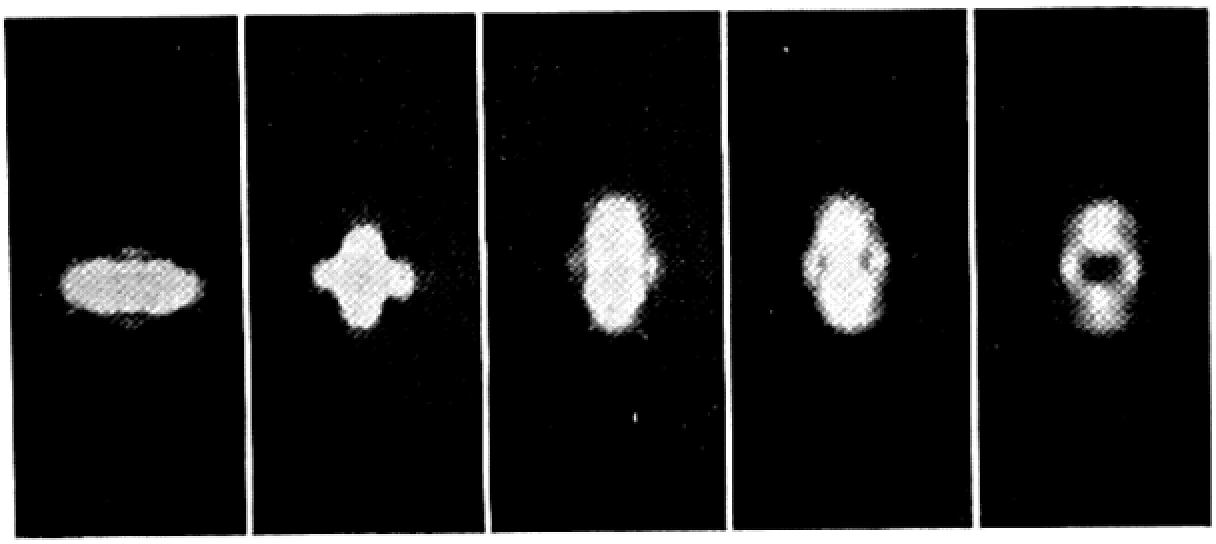


FIG. 216. STAR DISCS NEAR THE FOCUS IN THE PRESENCE OF PRIMARY ASTIGMATISM

Other Test Objects. Numerous test objects are employed by practical workers in addition to, or even to the exclusion of, the "star" test, especially in microscopy; reference should be made to the various text-books. The "star" test yields very satisfactory results for experienced workers, but should be supplemented by a test of another type. The fact that at one focus the "star" image suffering from spherical aberration approximates very closely in size and general appearance to the Airy disc means that any image, even in the presence of this aberration, will show more or less sharp detail. Hence, mere sharpness of detail is not a sufficient indication of a well-corrected system. The supplementary test is one for the contrast of the image, and the object should be one with sharp demarcation between black and white. For the microscope this is represented by Abbe's test plate, consisting of transparent *rulings* in an opaque silver film on the underside of a cover glass.

It is easily possible to interpret the extra-focal appearances on the image so as to infer the defects of the objective, but less simply than with the "star" test.

For the telescope, contrast objects are numerous; the twigs of trees; towers against the sky, and so on, are easily found. For other test objects see Johnson's *Practical Optics*,[3] p. 45.

Performance of Optical Systems. The experimental determination of resolution is likely to vary with the type of test-object used.

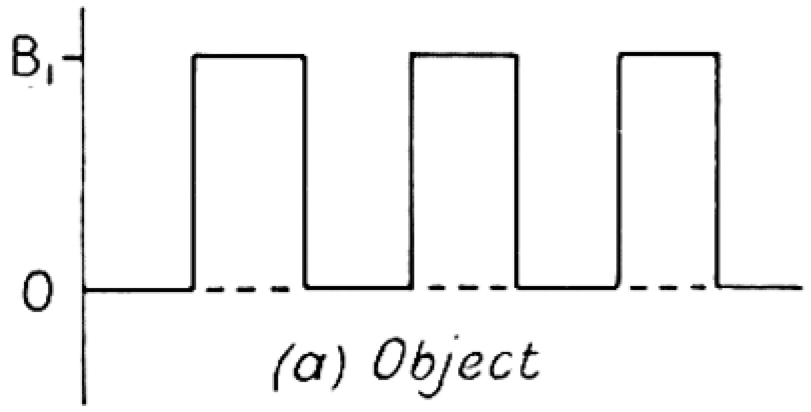
When a test-object similar to a double star is employed, especially in visual tests, and the separation of the "stars" is decreased till there is no fall of intensity between the images, the impression of an elongated object may still persist; and this will lend an undue measure of uncertainty to the estimate, given by various observers, of the point at which resolution ceases; for this reason a "grating" of equal bars and spaces (Foucault grating) is often employed as an object. On increasing the distance, or in other ways, the angular subtense of the grating element can be decreased until "resolution" can no longer be detected in the image, by whatever means it is examined. A more definite conception is the measure of contrast in the image of the periodic (Foucault) object of given frequency.

If B_1 is the brightness (or luminance) of the spaces, and B_2 that of the "bars" in the object, a *measure of contrast*, γ, as defined by Arnulf[4] is

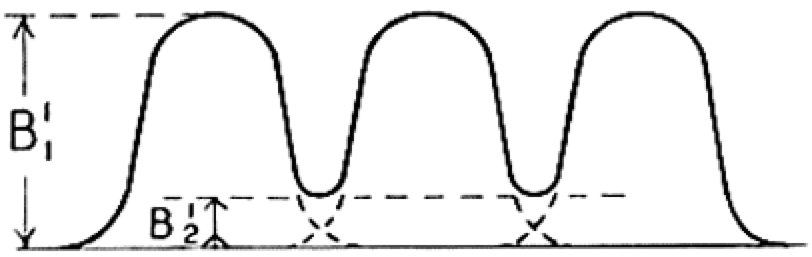
$$\gamma = \frac{B_1 - B_2}{B_1} \quad . \quad . \quad . \quad . \quad (7.01)$$

when B_2 is zero, we have the maximum contrast, unity. If the image of such a grating is formed by an optical system, diffraction, aberration and stray light will cause spreading of the light from the geometrical images of the spaces into the dark intervals; if the angular width of the grating elements decreases, this spreading will begin to fill up all the dark space and the maximum contrast of the image will thus decrease; Fig. 217. The calculation of this effect for a perfect lens, supposing the spreading to be entirely due to diffraction, has been carried out by Arnulf.[4] The image contrast is plotted, Fig. 218, against the width of the geometrical image of the grating element expressed as a multiple p of the radius of the first dark ring of the Airy disc in the image field. When p is about 0·8 the image contrast sinks to zero. Results, partly based on Arnulf's, for objects of varying contrast are shown. The contrast of images produced by imperfect systems will be discussed below.

Visual Limitations. Even if the image is physically resolved by an instrument, the performance in practice may be dependent on the resolving limit of the observer's eye under the relevant conditions of image brightness, diameter of pupil, and state of adaptation. Supposing an optically perfect eye with pupil diameter s, the angular resolving limit ω in radians would be (p. 43)



(a) Object



(b) Image

FIG. 217. INTENSITY VALUES FOR OBJECT AND IMAGE
(Grating)

$$\omega = 1{\cdot}22\lambda/s$$

and this value will be nearly true for the angular subtense of a grating element of contrast unity. Arnulf defines the "specific limit of separation" σ by the product

$$\sigma = s\omega$$

where s is given in millimetres and ω in minutes of arc. If now the value of σ is plotted against s, the result for a perfect eye would be "σ = constant," i.e. a straight line parallel to the s axis. Arnulf carried out a lengthy series of visual observations from which values of σ were found for various conditions of field brightness and contrast, using differing diameters of the effective eye-pupil, obtained by vision through small holes of varying diameters. The object was a Foucault grating of variable angle, luminance, and contrast. In his curves the field-luminance unit is the "stilb" (candles per square centimetre). An idea of this is obtained if it is remembered that a white surface illuminated by sky-light in summer (not direct sunlight) would have a luminance of about 3 stilbs. Fig. 219 shows typical results, for a field of luminance 0·148 stilb.

Various tendencies prevent any constancy of σ. For the larger pupil diameters the spherical and chromatic aberrations of the eye oppose the shrinkage of the diffraction disc, and the discrete structure of the retina helps to keep the value of ω nearly constant. Hence the curves of σ against s become nearly linear. However, when the pupil diameter is very small the loss of illumination makes itself felt and ω rises steeply on this account; there is also the deficient homogeneity of the eye media to be remembered.

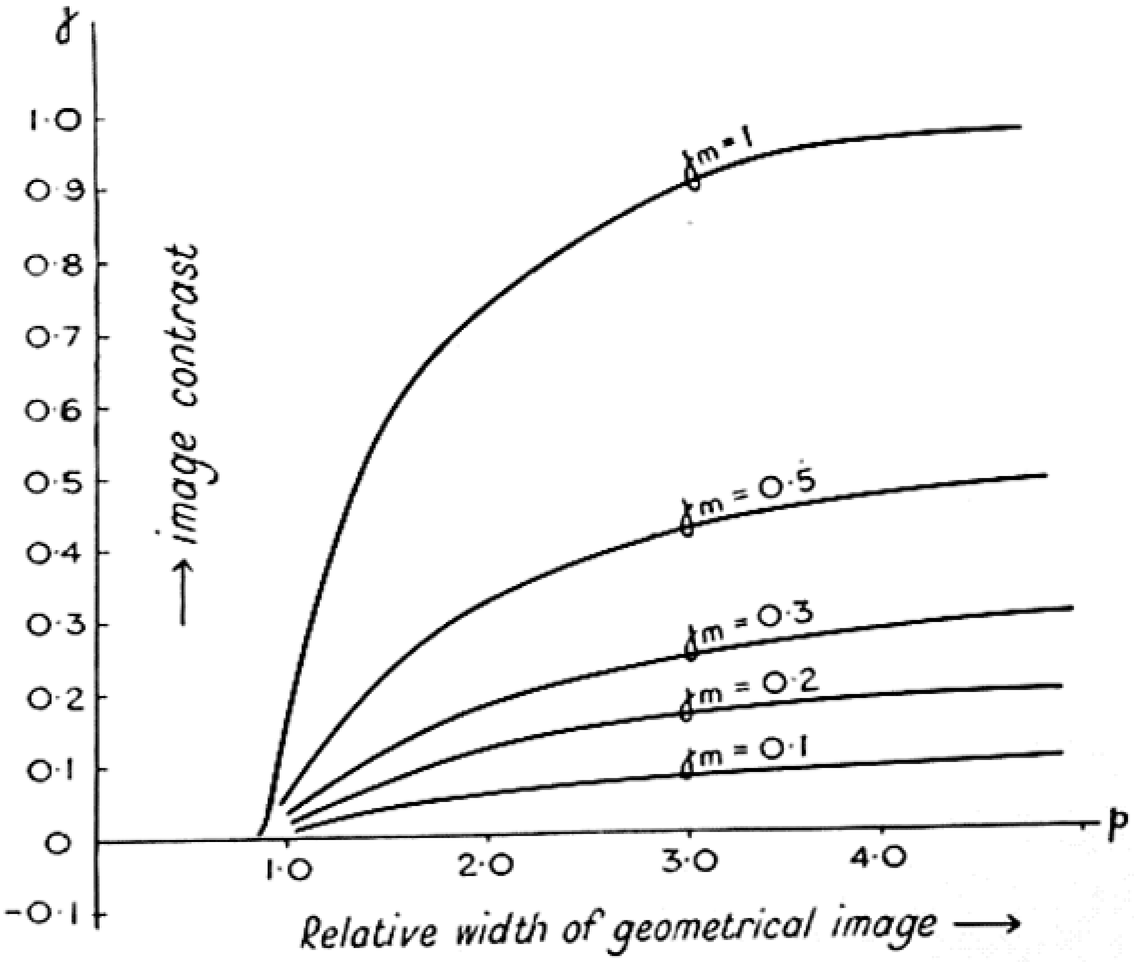


FIG. 218. IMAGE CONTRAST, SHOWING THE EFFECT OF DIFFRACTION

For fields of lower brightness the minimum of the σ curve falls at a much larger pupil diameter. The original paper should be consulted for the complete set of curves; they will throw much light on the probable performance of any instrument of known characteristics when used on an object of known structure and contrast.

It will be found that, provided the field illumination is adequate, and the optimum resolution is required, the best eyepiece for a telescope will be one which will reduce the exit-pupil to about 0·6 mm., but under low field illuminations the pupil size must be larger; and up to 8·0 mm. may be desirable for night glasses. The image contrast also has an important bearing on the optimum magnification and pupil size.

Such information as the above can also be used to discuss the optimum visual conditions in microscopy, though the relation between the object contrast and the image contrast is then, for reasons explained in the chapter on the microscope, a complex matter.

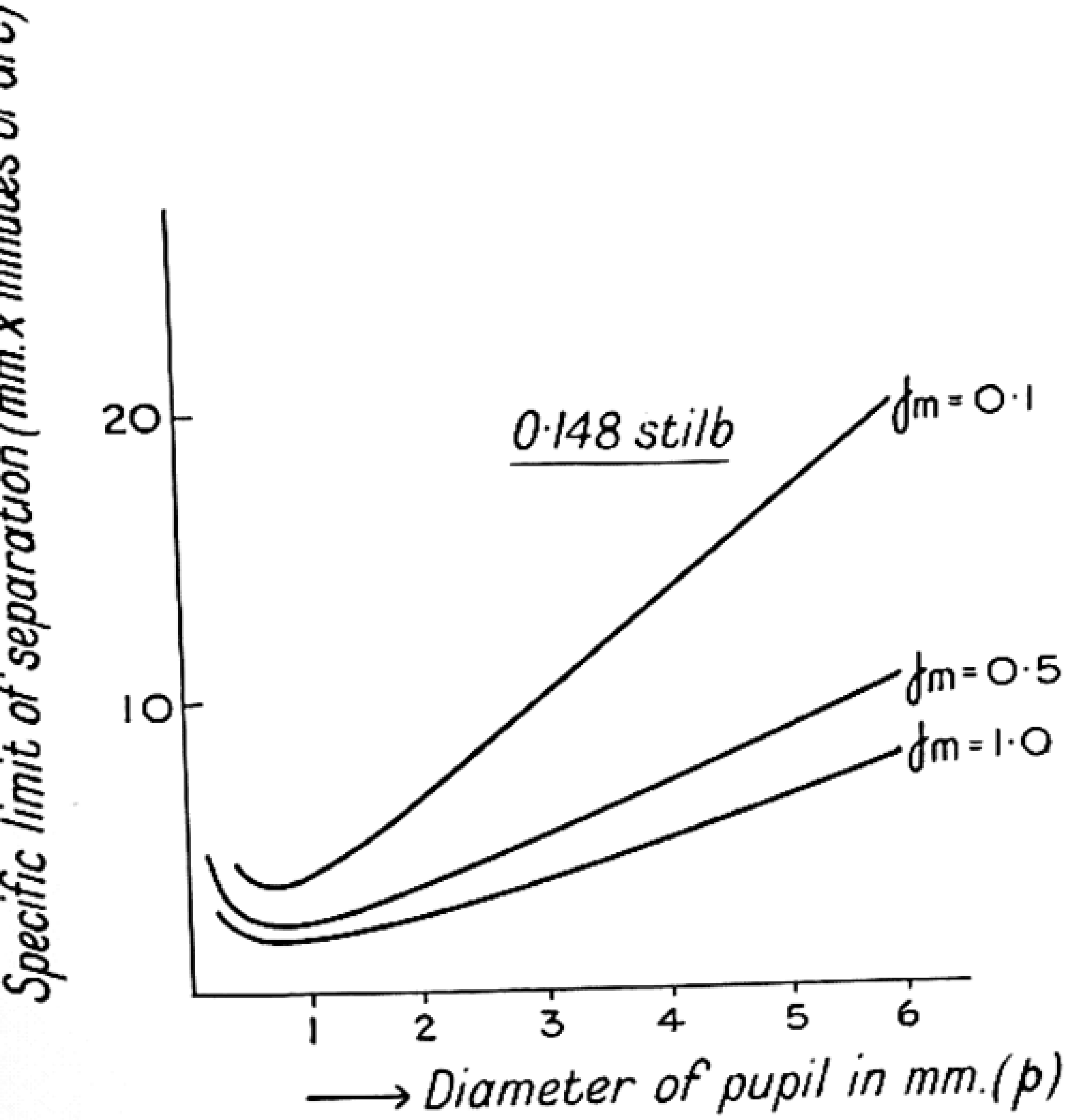


FIG. 219. SPECIFIC RESOLUTION AS A FUNCTION OF PUPIL DIAMETER AND CONTRAST

Instruments for Night Vision. Special consideration has to be given to instruments intended for use at night. Lord Rayleigh's observation, that he was short-sighted in "darkness" has been confirmed by later observers who have found amounts of myopia up to — 2D in their eyes when working at very low levels of illumination. This is attributed partly to spherical under-correction of outer zones of the optical system of the eye exposed by the enlargement of the pupil, and partly to the Purkinje effect which gives more "weight" to the shorter wave-lengths of the visible spectrum in scotopic vision. The performance of "night" instruments can sometimes be improved if suitable dioptric correction is given by special focusing.

Contrast Reduction Factors. E. W. H. Selwyn and J. L. Tearle[5] have suggested that the performance of an instrument (apart from the eye, etc.) could be described in terms of a *contrast reduction factor.* Selwyn's measure of contrast,* α, (differing from the γ of Arnulf in (7.01)) is

$$\alpha = \frac{B_1 - B_2}{B_1 + B_2}, \qquad (7.02)$$

which might also be written,

$$\alpha = \{B_1 - \tfrac{1}{2}(B_1 + B_2)\}/\tfrac{1}{2}(B_1 + B_2),$$

that is "*the difference of the maximum luminance from the mean luminance* divided by *the mean luminance.*" If the grating is such that the luminance variation is sinusoidal, being expressible by

$$B_1 = A + B_o(1 + \sin x),$$

where x is a displacement perpendicular to the lines, and A and B_o are constants, the measure of contrast is by the above definition

$$\begin{aligned} \alpha &= \{(A + 2B_o) - (A + B_o)\}/(A + B_o), \\ &= B_o/(A + B_o) \end{aligned}$$

It can be shown that for most practical purposes the image of such a grating formed by an instrument will have also sinusoidal luminance variation, though its contrast will *always* be reduced. If we start with a grating of contrast unity, where

$$B_1 = B_o\,(1 + \sin x),$$

and the luminance of the corresponding image is, say,

$$B_1' = A_o' + B_o'(1 + \sin x),$$

we know that the average luminance of the image must bear a definite ratio to that of the object; let the ratio be k, so that

$$A_o' + B_o' = kB_o$$

We can thus write the contrast reduction factor P as

$$\begin{aligned} P &= B_o'/\{A_o' + B_o'\} \\ &= B_o'/kB_o \qquad (7.03) \end{aligned}$$

Suppose, however, that we start with a grating of contrast less than unity where,

$$B_2 = A + B_o\,(1 + \sin x);$$

and the contrast is

$$B_o/(A + B_o).$$

We now think of the object as consisting of a constant field luminance

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* This measure of contrast is used in classical studies of the visibility of interference fringes (Michelson).

A *plus* the grating of unit contrast, i.e. $B_o(1 + \sin x)$. The lens reproduces the constant object background A with a constant image background kA, and we have to add to this the amount $A_o' + B_o'(1 + \sin x)$, as before. The total image luminance is then

$$B_2' = kA + A_o' + B_o'(1 + \sin x)$$

and the contrast is therefore

$$B_o'/(kA + A_o' + B_o') = B_o'/k(A + B_o)$$

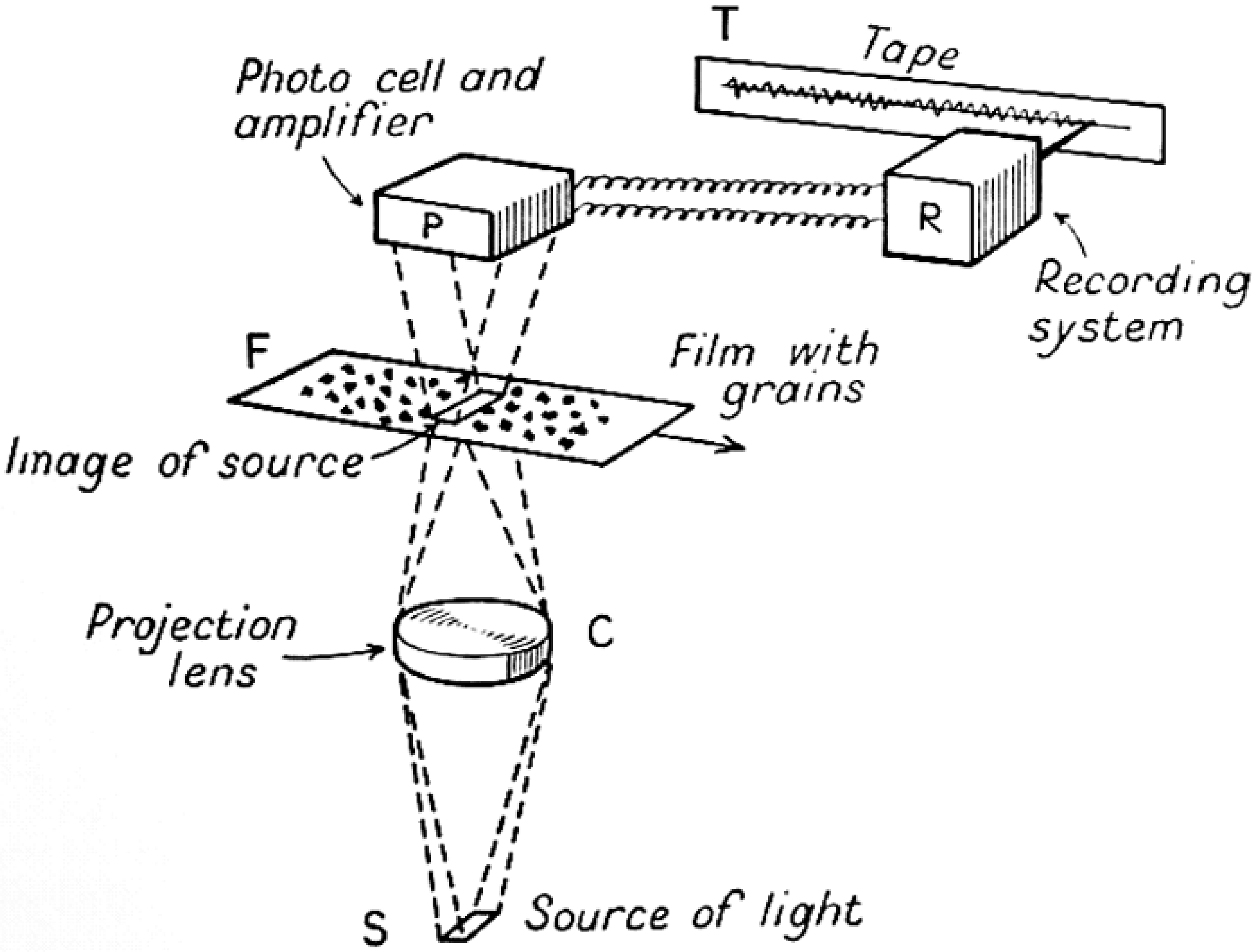

FIG. 220. MEASUREMENT OF GRANULARITY
(Symbolic)

The contrast reduction factor is therefore the same as before, i.e.

$$P = \frac{B_o'/k(A + B_o)}{B_o/(A + B_o)} = \frac{B_o'}{kB_o}$$

This is obviously independent of the background constant A, and (given the above type of object) the contrast reduction factor for a given optical system is therefore independent of the original contrast of the object.

Performance of Photographic Systems. In testing photographic lenses the image will be recorded by the plate or film. The lens itself will first produce some degradation of contrast in the image before recording; but, further, the emulsion is a non-homogeneous dispersion of silver bromide and silver iodide crystals in the gelatine. When the light of a sharply defined image falls thereon, some of it is scattered in the film outside the boundary of the image; the action is assisted by reflection at the back surface of the film or plate (halation); the image contrast therefore suffers degradation from these causes also.

Another factor has to be considered in the recognition or detection of the periodicity in the developed image (supposing always the object to be a grating as above); it is the granularity of the negative material, which may be defined as the "average fluctuation of density." In order to relate this to the relative size and distribution of the grains of metallic silver in the negative, consider the record made by a recording microphotometer of the density of a uniformly exposed and developed film. The symbolic diagram, Fig. 220, suggests how the condenser C projects an image of the rectangular illuminated slit S into the plane of the film F. The transmitted light enters a photo-cell P whence the current passes to an amplifier and recorder R; these record the apparent density fluctuations on a tape T which moves with F. If the aperture-image in the film is large in comparison with the grains, the recorded average density for the aperture will show little change, but as the size of T is reduced till its area is comparable with the grain size, the fluctuations will increase. The area between the recorded curve and the middle line taken over a reasonable length of the record can be taken as a measure of the fluctuations and thus (when expressed in suitable units) of the granularity. It can be shown that the variation of density is roughly proportional to the average grain size (in linear measure), and inversely proportional to the square root of the area of the aperture image.

The granularity tends to hide the presence of any pattern of varying average density in the developed image; the greater the density variation of the image and the greater the image-size, the less will it be obscured; but small patterns of low contrast will tend to become unrecognizable. The exact theoretical relation is somewhat debatable, but arguments have been adduced to show that the minimum visible density difference is proportional to GR, where G is the granularity and R is the number of lines per mm. in the grating pattern.

Let the contrast of the sinusoidal object grating be α then let P_l be the contrast reduction factor of the lens; we can further consider that the action of the film in reducing the contrast of the image before it is recorded is represented by a further contrast reduction factor P_e. Further, let γ be the usual photographic contrast factor.*

The expression for the maximum luminance in the image is now

$$I_o(1 + \alpha P_l P_e)$$

where I_o is the mean value; the minimum is, of course,

$$I_o(1 - \alpha P_l P_e)$$

Then, provided the exposures corresponding to this luminance range lie

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* Not to be confused with Arnulf's measure of contrast mentioned above. If the object has (large) patches of luminance B_1 and B_2 respectively and the corresponding densities of the image are D_1 and D_2 respectively, the γ value here used is (within certain limitations of exposure)

$$(D_2 - D_1)/\log_{10}(B_2/B_1)$$

within the limits of the straight part of the characteristic curve of the film (see footnote),

$$D_2 - D_1 = \gamma \log_{10} (B_2/B_1)$$
$$= \gamma \log_{10} \frac{I_o (1 + \alpha P_l P_e)}{I_o (1 - \alpha P_l P_e)},$$

where D_2 and D_1 are the maximum and minimum densities in the negative. If $\alpha P_l P_e$ is reasonably small (as may be expected for cases where the resolving limit is being approached) the term on the right will approximate to $\gamma \log_{10} (1 + 2\alpha P_l P_e)$, and since by the logarithmic series expression

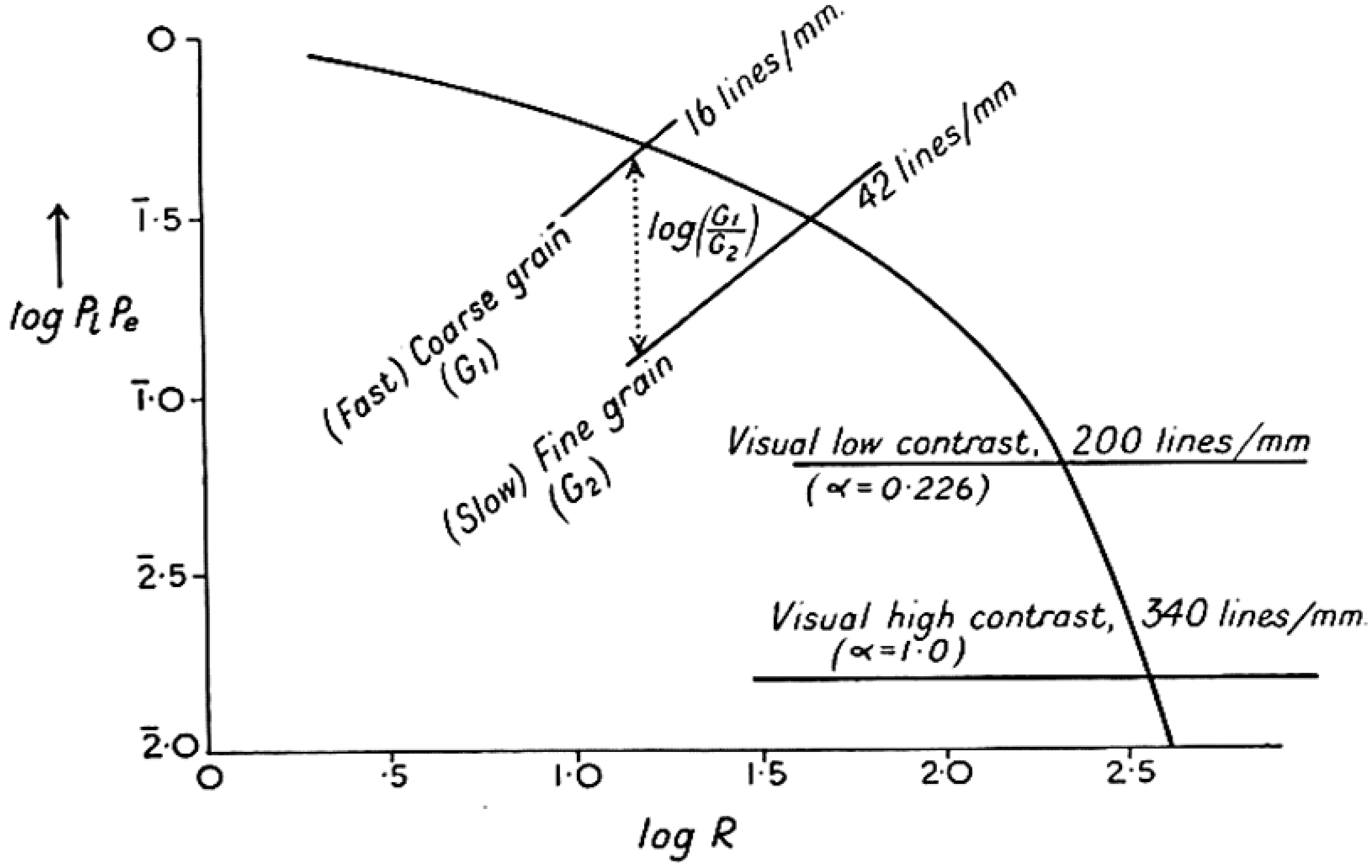


FIG. 221. CONTRAST FACTORS AND RESOLUTION

$$\log_e(1 + y) = y - \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 + \text{etc.},$$

(the series approximating to a sum y when y is very small) we shall have $\log_{10}(1 + 2y)$ equal approximately to $0{\cdot}87y$. Hence

$$D_2 - D_1 \approx 0{\cdot}9\gamma\alpha P_l P_e$$

And thus (from above) $0{\cdot}9\gamma\alpha P_l P_e = k\text{GR}$,
where k is a constant.

In order to solve this equation, Selwyn plots log $P_l P_e$ and log $(k\text{GR}/0{\cdot}9\gamma\alpha)$, both against log R on the same diagram. It is found from general experience that the first curve can be expected to approximate to the curve represented by

$$\log_e P_l P_e = -(R/R_o)^n,$$

and by studying the performance of a typical lens it was found that the above equation could be represented in the graph, Fig. 221, by

$$\log_{10} P_l P_e = -(0{\cdot}434)(R/32)^{0{\cdot}6}$$

And this is the equation of the curved line. Observation suggested that the degradation of the image by scatter in the film before the recording is negligible in comparison with the degradation produced by the aberrations of the lens, so that P_e was assumed equal to 1 for the first approximation. Now observation showed that with the given lens, the limit of resolution of a coarse-grain fast film was about 16 lines per mm. and with a fine-grain film about 42 lines per mm. Hence if points for these values ($\log R = 1{\cdot}204$ and $1{\cdot}623$) are marked on the curve, they should represent the intersection of the curve with the lines representing the plot of

$$\log\left(\frac{kGR}{0{\cdot}9\gamma\alpha}\right) \text{ against } \log R$$

These lines are obviously straight lines at 45° to the axes, and if γ is constant, any two lines should differ in their vertical height by the amount of the difference of log G for the two cases. It is found in fact that the ratio of the granularity experimentally found for such films is in good agreement with the value indicated by this height difference. It can therefore be considered with some claim to justification that the theory is fairly well founded. The essential feature of the account is that the low resolving power observed with photographic materials is not due to the spread of the light in the emulsion layer (there seems little reason to think that this would differ much between fast and slow films) but to their granularity.

It has been judged advisable to give a short account of this theory because the photographic testing of optical systems may lead to great difficulties unless some of the fundamental features of the use of the materials are well understood. The original papers should be consulted.

Visual tests on the resolution of a photographic lens usually show a very much higher resolution than can be obtained with any film if the magnification of the image is sufficient to bring the resolvable detail up to an angular subtense exceeding the limiting resolution of the eye; under such conditions the limit of perception will be set by the degree of contrast which is perceptible, δ (say). Experiments also described by Selwyn show that this is reasonably constant over a wide range of subtense values. Hence the condition for resolution is then

$$\alpha P_l = \delta$$

We may conveniently plot $\log P_l$ against log R, and $\log(\delta/\alpha)$ against log R; since P_e was taken as unity, the *curve* is the same as that in Fig. 221. The limiting visual value of δ is about 0·015; if $\alpha = 1$, (high contrast), $\log(\delta/\alpha) = \bar{2}{\cdot}18$, and the indicated value of R is about 340 lines per mm. If α is 0·226 (low contrast), $\log(\delta/\alpha) = \bar{2}{\cdot}83$, and R drops to 205 lines per mm. These values again correspond reasonably well with observations for an actual lens.

It might be thought that if only the *contrast reduction factor* for a lens could be determined (visually, say) its photographic performance with a given film could be predicted. While this is true to some extent, the visual prediction unduly "weights" those radiations which are brightest in the spectrum, whereas a panchromatic film will respond to a wide range of wave-lengths; moreover, any useful figure should be applicable to the field of the lens as a whole, and the integration of such factors for the whole field will lead to great difficulty. Nevertheless, this general conception ought to find many applications in specifying the performance of visual optical systems.

Further notes on the object-function and contrast transmission factors for optical systems will be found in the Appendix, p. 370.

Tests on Optical Systems: Scanning Methods. It is possible in principle to find the "energy distribution function" for the image, of a "point" source, projected by the system under test, by "scanning" the image with a small aperture backed by a suitable photo-electric cell (photo-multiplier). The image projected by the system may receive one or two stages of optical magnification before it is scanned; the main difficulty is to retain sufficient energy to actuate the cell when both source aperture and scanning aperture are small enough. The apertures should have width representing not more than about one-quarter of the appropriate "limit of resolution" at the point in question.

However, it has been shown by Hopkins[6] that, provided that the measurements are carried out in various orientations, all the image-forming properties of a system can be investigated by using *slits* instead of point apertures, thus obtaining much greater photometric efficiency. It is possible, furthermore, to make corrections[7] for the errors resulting from finite slit widths.

The "distribution functions" themselves, when found and plotted for a series of lenses for comparison purposes, afford some basis for the assessment (by an experienced worker) of the likely properties of the system; however, the matter can easily be taken a stage further, in terms of the transmission factor of the instrument for a given spatial frequency in the object function as discussed in Appendix III.

In the definition of the transmission factor D(s) of an optical system, we write (in effect), see p. 374, for the distribution of energy in the focal plane in the direction u'

$$\mathrm{B}'(u') = \sqrt{\frac{2}{\pi}} \int_0^\infty \mathrm{D}(s)\beta(s) \cos\{u's + \phi(s)\}\, ds$$

where $\beta(s)$ is the relative intensity of the intensity contribution of period s in the object function. If the object is suitably shifted we can make $\phi(s)$ zero for any value of s. Further, if the object is a single line of light, we can look on it as containing all frequencies equally in the Fourier sense so that $\beta(s)$ becomes unity. If this is difficult to grasp, remember that a cosine series such as

$$\frac{1}{\mathrm{N}}\,(\cos a + \cos 2a + \cos 3a + \; . \quad . \quad ., \text{ etc., to N terms})$$

has a value close to unity for $a = 0, 2\pi$, etc., when N is large, and the sum is very small at all other values of a; as against this the corresponding cosine *integral* has an appreciable value only for $a = 0$. Written in this way we have

$$\mathrm{B}'(u') = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{+\infty} \mathrm{D}(s) \cos u's\, ds$$

which is a Fourier transform of the transmission coefficient. The converse transform

$$\mathrm{D}(s) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{+\infty} \mathrm{B}'(u') \cos u's\, du'$$

shows that the transmission coefficient D(s) can be obtained by a straightforward integration from the distribution function B′(u'); this can be effected numerically or by suitable electro-mechanical devices.

As an alternative to such calculation, it is possible in principle to use actual cosine gratings of unit contrast, prepared by special techniques, in either the object or the image plane; thus it is possible to scan the image of such a grating by means of a fine slit, or to scan the image of a line source by a grating in the plane of the image; in the latter case the aperture of the photometer must be large enough effectively to integrate all the light derived from an area reasonably wide in comparison with the slit image. In either case the grating can be moved for the scanning. However, in such a case it is only possible to obtain the transmission factor for such frequencies as are represented by the gratings available. In a case such as the microscope objective, where the conjugate distances differ, it will be necessary to use the grating at the longer conjugate, as fine gratings are relatively difficult to make. For example, if the numerical aperture of a lens is about 0·25; the gratings for use in the image when the magnification is 100 would have about 10 lines/mm. for a frequency approaching the limit of resolution. A paper by Lindberg[8] describes the use of a grating of continuously variable interval; a disc (Fig. 222A) is made with equal radial sectors of very small angle, alternatively transparent and opaque; thus the spatial frequency depends on the relative position of the short and narrow scanning aperture, i.e. the frequency is greater the nearer to the centre. If the disc rotates at constant speed the aperture transmits a periodic light pulse with an amplitude depending on the image-contrast. For an aperture short in comparison with the radius the grating approximates closely to a "square-topped" effect; however, the amplifier attached to the photo-cell can be made to respond only to the fundamental time-frequency; hence the equivalent of using

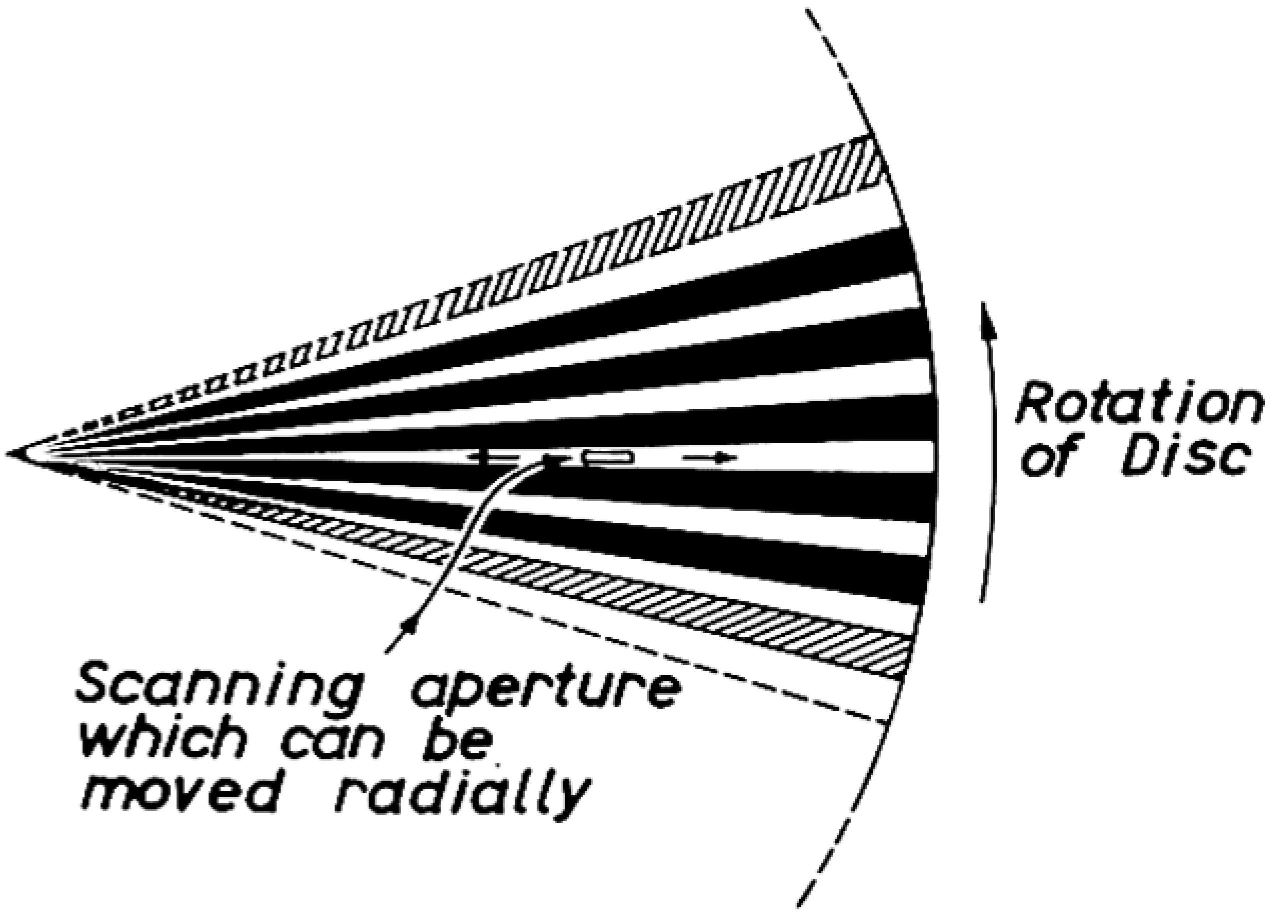


FIG. 222A. SUGGESTS (VERY ROUGHLY) THE SECTOR DISC FOR SCANNING

a true cosine grating is obtained. The disc or aperture has only to be displaced by half a period to pass from the maximum to the minimum response. A uniform movement of the aperture during the steady rotation of the disc will produce a response of the photometer similar to that shown on Fig. 222 (B), from which the relative response of the imaging system for different spatial frequencies is readily obtained.

The use of such scanning gratings is much more efficient as regards the conservation of the light, and the limit of sensitiveness is often set by the allowable width of the source.

Tests of Resolution. Photographic tests are, for the reasons cited above, still of great importance for the users of instruments, and gratings suggest themselves as suitable test objects for resolution. However, owing to the diffraction rings in the star image (often exaggerated by aberration), there is some danger that if the angular interval of an object grating is diminished there may be a total loss of resolution followed by a spurious image with reversed contrast. In order to avoid such anomalous effects, test objects[9] of the types shown in Fig. 223 (*a*) and (*b*), are employed built up into a test chart with elements of progressively decreasing size (Tearle[5]). The contrast in such charts can be given any value desired in order to test the lens under conditions resembling those of practice. (The object in aerial reconnaissance photography is almost always of low contrast.)

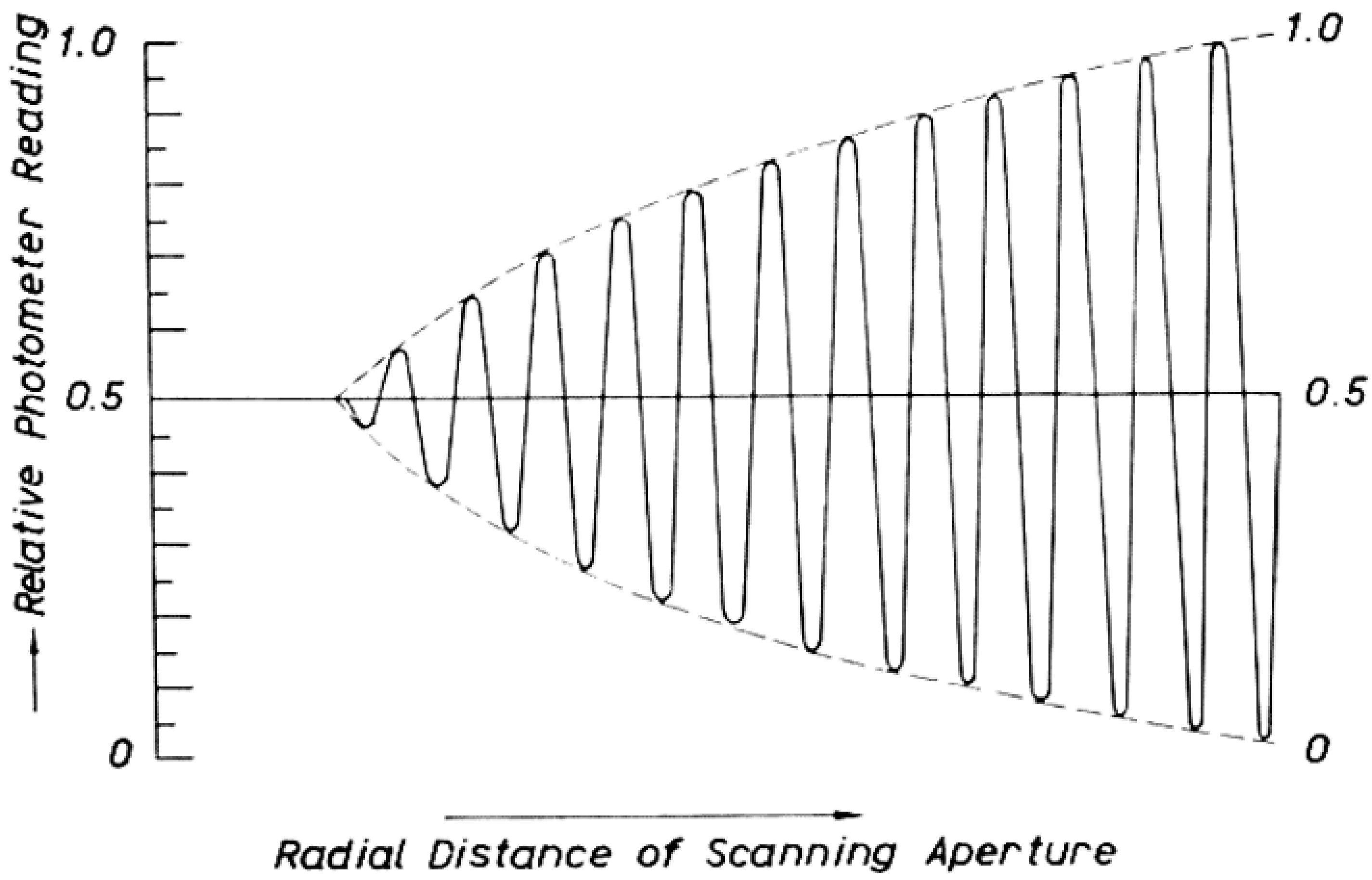
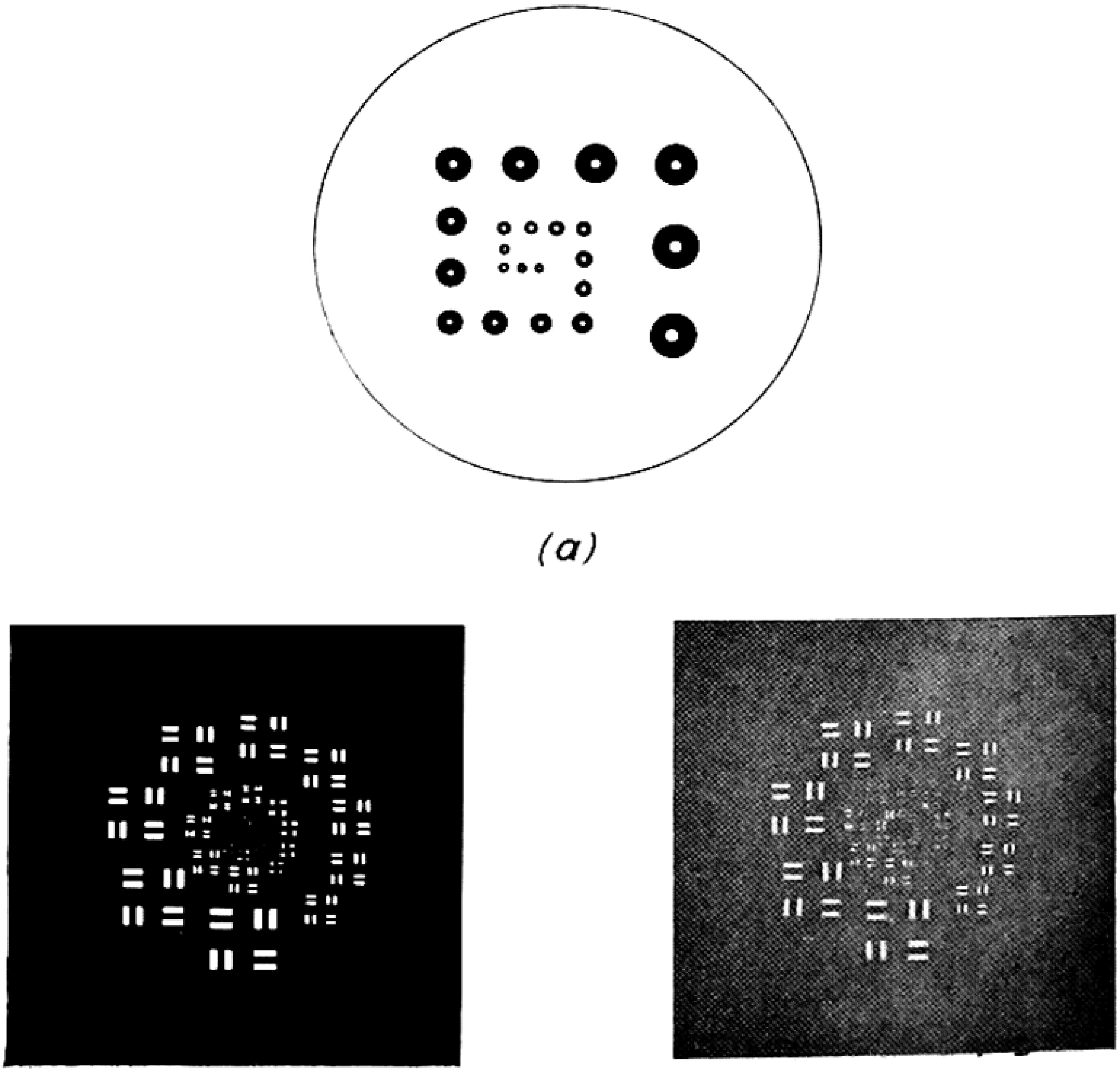


FIG. 222B. SHOWS THE KIND OF RESPONSE OBTAINED WHEN THE SCANNING APERTURE IS GIVEN A STEADY RADIAL MOVEMENT WHILE THE SECTOR ROTATES

Such a group of objects can be mounted as a transparency in the focal plane of a collimator and photographed with the lens set at various angles with the collimator (see Fig. 224).

Let Λ be the width of the "grating element" in a group just resolved (distance from the centre of one dark line to the next), then the angular resolving limit θ is Λ/f_c' where f_c' is the focal length of the collimator lens. Suppose now that the collimator is set at an angle ϕ with the axis of the lens of focal length f', and that a grating element just resolved on the plate measures λ mm. If the element separation λ_s is in the sagittal direction perpendicular to the plane containing the axis the angular resolution is $\lambda_s/f' \sec \phi$; thus



(a)

(b) (c)

FIG. 223. TEST CHARTS

(a) Ring chart (Howlett)
(b) and (c) High and low contrast Cobb charts (Tearle)

$$\theta_s = \frac{\Lambda_s}{f_c'} = \frac{\lambda_s}{f' \sec \phi}$$

Where the minimum resolvable separation λ_t of the elements is in the axial plane (the "lines" of the grating having a tangential direction in the field) the angular resolution is $\lambda_t \cos\phi / f' \sec\phi$; thus

$$\theta_t = \frac{\Lambda_t}{f_c'} = \frac{\lambda_t}{f' \sec^2 \phi}$$

Some workers define the "resolving power" R of the lens as a product, viz.

R = focal length in mm. *times* resolution, in lines per mm., on film.

$$= f'/\lambda$$

Hence $$R_s = \frac{f_c' \cos\phi}{\Lambda_s}, \text{ and } R_t = \frac{f_c' \cos^2\phi}{\Lambda_t}$$

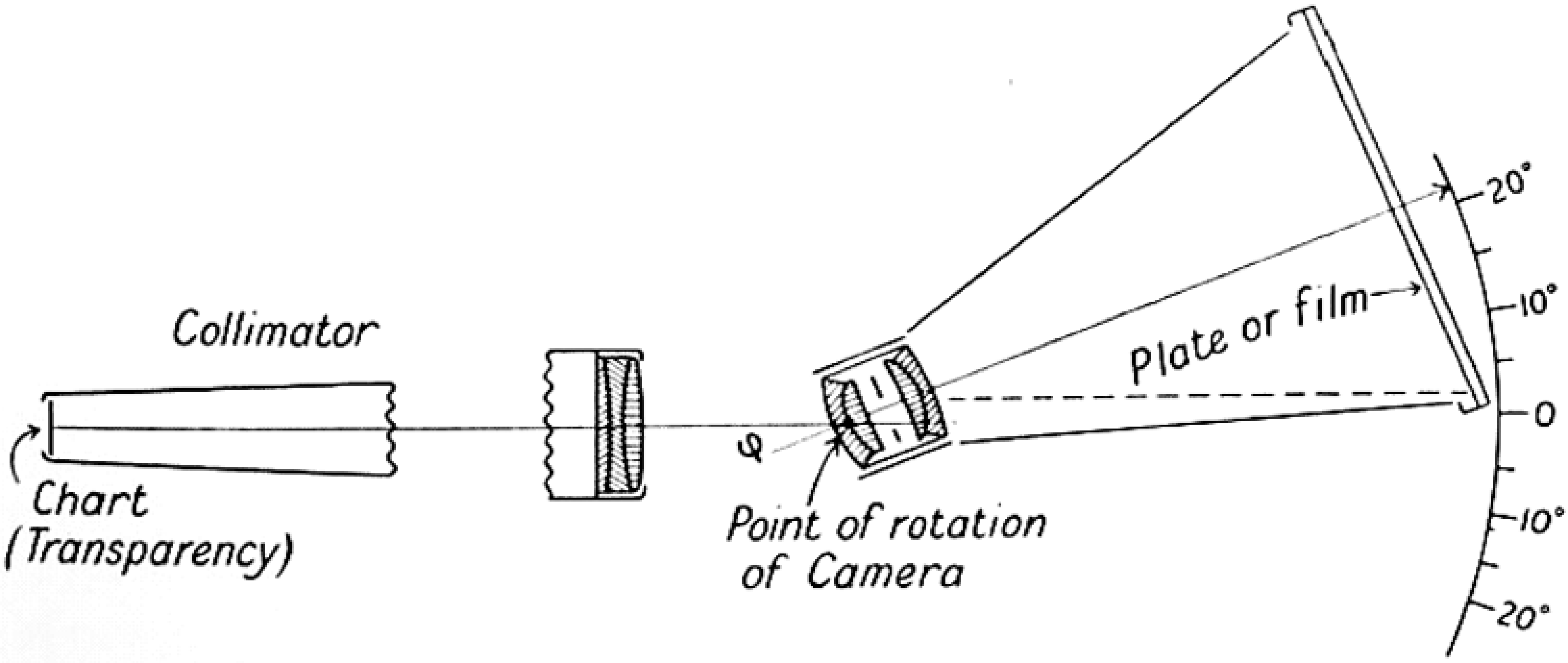


Fig. 224. Bench for Resolution Tests

It may be noted that R may be used to calculate the size of the least resolvable detail in the object-field supposed at infinity, since

$$R = \frac{\text{Object distance}}{\text{Size of object detail}}$$

The merit of a topographical photograph will depend on the maintenance of a good performance in resolution all over the field, and a mean overall resolution can be calculated. In a method employed by some laboratories the negative is divided into 20 annular zones of equal area, and it is assumed that the resolution at the boundary between the $2n$ and the $(2n-1)$th areas gives the mean resolution over the two areas. Then the mean resolution over the whole film is given by the means of the resolutions at the 10 boundaries, which are read off from the curves of R against ϕ.

Some variations of practice occur; as for example in whether the rectangular shape of the film or plate is allowed for, etc. In comparing such figures given independently for various lenses, it is important to know what size of field has been assumed.

A complete test of a lens will involve such a set of measurements over a range of trial positions of the focal plane. It is sometimes the case that the best over-all mean resolution, taken as the mean of R_s and R_t, does not coincide with the plane of best axial focus; if the camera were of a fixed focus type in which the focus is set by reference to the best axial focus, this might be a serious disadvantage.

Veiling Glare. Some amount of stray light is always present in an image formed by an optical instrument. Diffraction at the edges of the lenses, and scattering of light by defects in surfaces and media, may produce a more or less uniform illumination of the image plane. If a small bright source of light such as the sun is present in the field, any pair of transmitting surfaces perpendicular to the axis can (after double reflection) produce an image of such a source which will not in general coincide with the main image and will therefore illuminate a certain *area* of the actual image plane. If there are n surfaces there will in general be $\frac{1}{2}n(n-1)$ such images in space. If the images are near the focal plane they may constitute "glare spots," and a certain number of promising designs for photographic lenses have had to be rejected on account of the presence of such images—since they are apt to give trouble, in particular if the lens is used "facing the sun." On the other hand, even if there is no special bright source, each element of the object forms its corresponding glare area, and the integration of the effect for the whole object plane produces a certain amount of diffused illumination which tends to veil the contrast of the main image and is thus (with the rest of the unwanted light) called "veiling glare."

The coating or blooming of lens surfaces* was at first expected to remove the greater part of the light due to "reflections," but in fact the improvement is usually less than might be expected owing to the greater amounts of light scattered from the sub-microscopic structure of the films coating the surfaces. Hence while the local glare spots may be made insignificant the overall glare may not be lessened unless the coating is done with technical perfection.

Glare Index. Let a telescope be directed towards a uniform luminous surface subtending an angle large enough to fill the whole field. The apparent field luminance in the telescope is now B, say. Imagine that an opaque perfectly black disc is now held up so as to screen a part of the axial region of the field, the distance of the disc being sufficient to ensure that the image of the disc is focused in the field of the eyepiece. Owing to the veiling glare the image of the disc is not perfectly dark but is illuminated to some degree; let the apparent luminance be b; then the glare index is equal to b/B.

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* See Vol. I, pages 190 and 266.

$$\text{Glare index} = \frac{\text{Luminance of image field due to unwanted and stray light}}{\text{Total luminance of image field}}$$

In laboratory tests[10], the uniform luminous surface is represented by a sheet of white diffusing medium uniformly illuminated at the rear by suitable lamps and reflectors; a black disc screens its centre and the image of this is projected to infinity by a suitable single lens. If the total field is say 8 degrees, the opaque spot might subtend 1 degree. The image of the field is projected by the eyepiece into a suitable photoelectric photometer, when it is not difficult to measure the relative luminance of the region of the disc image in the presence and absence respectively of the black disc. Allowance has to be made for the effect of the auxiliary projecting lens, which may itself produce some veiling glare.

Similar methods can be used in measuring the glare indices of photographic lenses, though in this case it is necessary to imitate experimentally the presence of an illuminated field, on the object side, of much wider extent. For this reason it is sometimes advisable when investigating large lenses to find the amount of stray light for a series of elementary areas of the object field, and then to integrate numerically for the total stray light.

Measurements of these kinds are now accounted of first-class importance in the testing of lens systems. Even apart from measurements it is very instructive to hang, over a window, a circular disc of black card large enough to fill the real field of a small telescope directed towards it from the side of the room; the exit pupil of the instrument is now examined with a hand magnifier, when the presence (and origin) of stray light will often be easily seen.

The Measurement of Aberrations. Aberrations may be expressed in terms of the geometrical ray paths, or in terms of phase relations of vibratory disturbances in the image; it is usually possible to calculate the one expression from the other. Likewise, the methods of measurement can be divided into two general classes; one which aims at tracking ray paths by various means, the other which gives a direct indication of the optical phase aberrations in an interferometer pattern. Broadly speaking, the first class involves little apparatus, but much time and trouble; the second involves expensive apparatus, but comparatively little time.

Visual Ray-path Method. The "ray" is a mathematical conception and cannot be physically realized. If we place a diaphragm containing a small aperture of finite size in the path of a convergent wave-front, the maximum concentration of energy will be for practical purposes in the centre of curvature of the element thus exposed. If the aperture has symmetry with respect to some point in its plane, the distribution of light in the convergent beam will show a corresponding symmetry with respect to the line joining this point to the centre of curvature; the centre of symmetry of the diffraction pattern in the beam may thus be conceived as a ray track.

The most direct method of procedure is to arrange a diaphragm behind the lens which forms the image of an artificial "star." The distribution of light in any plane can then be examined with the aid of an eyepiece. (Fig. 225 (*a*) and (*b*).)

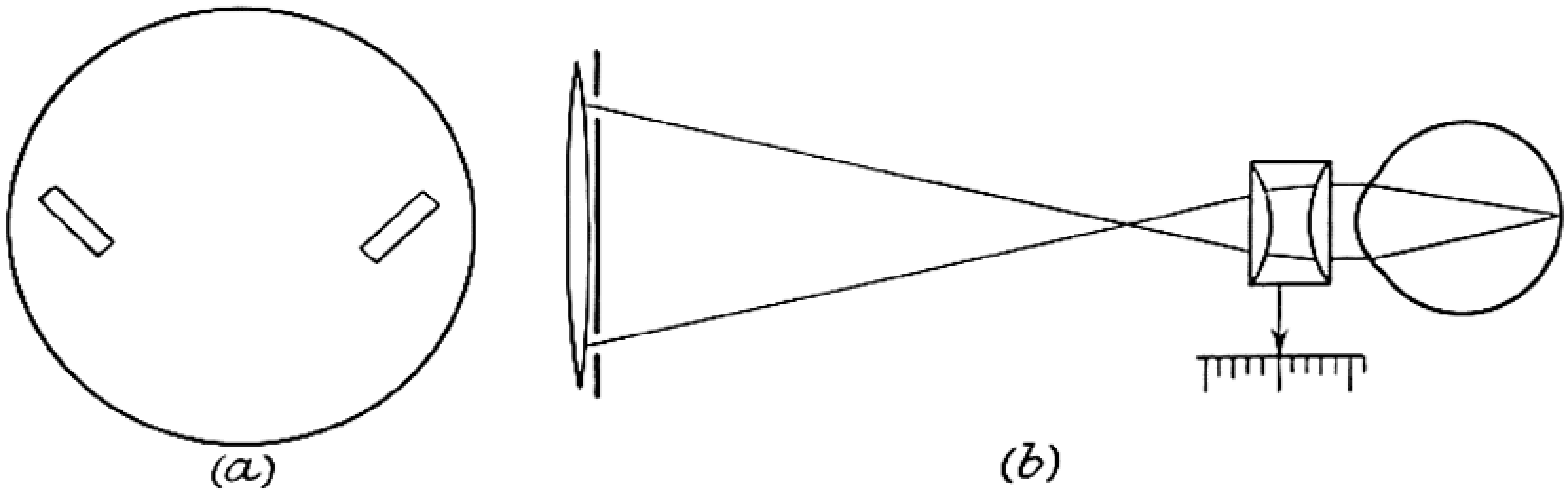

FIG. 225. DIRECT OBSERVATION OF ZONAL FOCUS

A useful method of finding the foci of various zones is suggested by C. Beck, who uses a pair of apertures disposed in the diaphragm, as shown in Fig. 225 (*a*). The position of the focus of the corresponding zone is recognizable by the cruciform symmetry of the double diffraction pattern. With the aid of a series of diaphragms, it is thus possible to find the focusing points of a series of zones, or for a series of different wave-lengths of light, and hence to investigate the spherical and chromatic aberrations of the system, although it would be necessary in investigating an objective alone carefully to allow for the corresponding aberrations of the system of the eyepiece and eye, by direct observations on very small "mercury globule" stars supported in the focal plane of the eyepiece. Such methods are sometimes called "Visual Hartmann Tests."

The difficulty attaching to this and allied methods is, however, very considerable in practice, and seems to arise through the irregularities of the refracting media of the eye. The cornea is traversed, in this test, by two very narrow beams which can easily suffer considerable deflection by a minor irregularity which would make little or no difference to a broad beam. For this reason it is usually difficult to estimate the correct focus of a particular zone; probably some persons would find the experiment much easier than others would.

The Foucault Test. The Foucault test was briefly described in Vol. I, p. 283; it has the merit that it avoids the criterion of a judgment of position, using rather an estimation of equality of intensity of light; the setting is photometric. Originally applied to test astronomical mirrors, and still mainly employed for that purpose, it can also be applied to "refractors" with useful results.[11]

A small source of light s, conveniently a pinhole backed by a flame or opal bulb lamp, sends light to the mirror (Fig. 226), which forms a corresponding image s'; the eye is placed immediately behind this image, and sees the mirror under the condition of the Maxwellian view in which it is filled with light, provided that all rays from the mirror enter the pupil unobstructed.

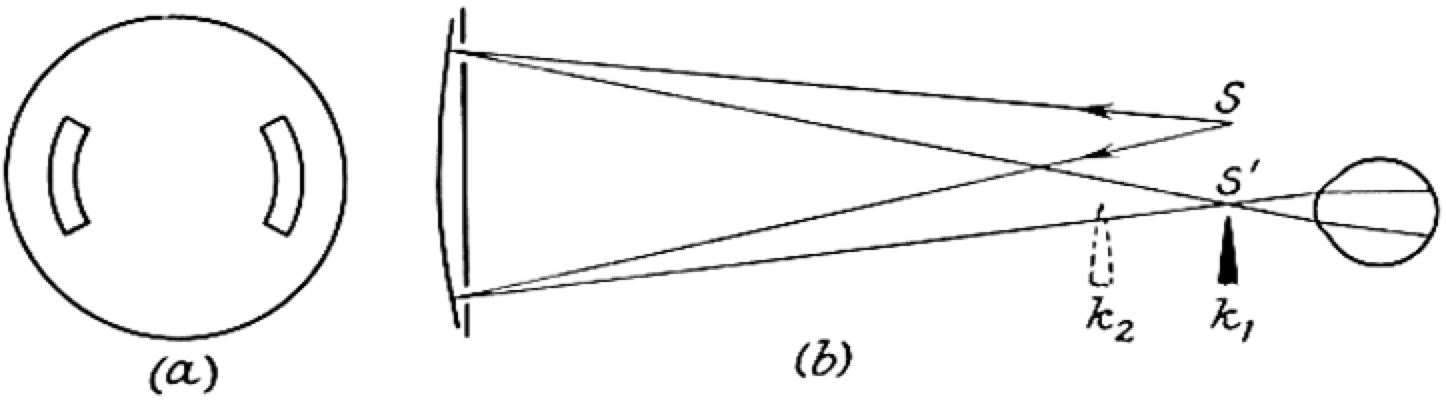

FIG. 226. ZONAL DIAPHRAGM FOR FOUCAULT TEST

If a knife-edge k_1 is brought upwards into the focus, the whole mirror darkens uniformly if the light from all zones is focused in the same point; but if spherical aberration is present, then characteristic distributions of light and shade appear on the surface of the mirror. The shadows naturally occur on those regions the rays from which are cut off by the knife-edge. The necessity of conducting the mirror test with image and object point away from the axis makes the characteristic shadow figures unsymmetrical with the reflector, but the light distribution always represents the gradient of the error rather than its absolute amount.

Experienced workers recognize the figures characteristic of parabolic, elliptical, hyperbolic, and other forms of reflecting surface, but the test is made quantitative by measuring the focus for successive zones by the use of diaphragms, such as shown in Fig. 226. It is clear from the figure that when the knife-edge intersects the beams either nearer or farther than the focus, the two patches exposed by the diaphragm will not darken simultaneously. Considerable precision is attained by this method if care is taken with the precautions usual in photometric work. The exact theory of the test has been examined by E. H. Linfoot[12] and others.

If the pinhole and the knife-edge are moved on the same carrier, the pointer attached thereto may be made to read the relative positions of the centres of curvature of successive zones; but if the knife-edge alone is moved while the pinhole remains stationary, the separation of the images will be clearly double that of the corresponding zonal centres of curvature. Thus, if the radius of the zone is y, and the approximate radius of curvature is r, the centres of curvature of the paraxial and zonal foci are separated by $\frac{y^2}{r}$ for a parabolic mirror.

The Foucault test is not only used for testing the main mirrors, but also the secondary mirrors of Cassegranian and Gregorian reflecting telescopes. According to Hindle[13] it is best carried out for the secondary mirrors independently with the aid of an auxiliary spherical mirror which has itself been tested alone. In this method one focus of the hyperbolic or elliptical mirror will lie at the centre of the spherical mirror, the other focus being at the point of observation. A useful modification of the Foucault test due to Platzeck and Gaviola is described below.

Of the other tests adapted for visual use, those of the shadow fringe methods developed by Ronchi and Jentsch (see below) may be mentioned, but if actual measurements are desired they may best be made with the help of photographic recording. The same applied to the general methods suggested by Chalmers, which have been revived in a more practical guise by Gardner and Bennett. We shall now proceed to describe several methods involving photographic recording.

The Hartmann Test.[14] The objective under test is directed to a suitable star or artificial star, and a diaphragm placed behind it has circular apertures, h_1, h_2, representing a particular zone of the lens. A photographic plate-holder is arranged so that photographic plates can be exposed when held perpendicular to the axis in two positions, within and without the focus, at a measured distance apart. The light from each aperture forms a small diffraction patch on the plate, the centre of which may be regarded as representing the track of the ray from the centre of the aperture. A suitable diameter for the holes is about $\frac{1}{200}$th of the focal length.

If A_1 and A_2 are the plate-holder scale readings corresponding to two plate positions, and l_1, l_2 are the distance of the centres of the two dots on each plate, then the scale reading A corresponding to the focus of the zone is (see Fig. 227)

$$A = A_1 + \frac{l_1}{l_1 + l_2}(A_2 - A_1)$$

The possibility of cylindrical error of the system has to be remembered however. It follows from the theory, Vol. I, p. 303, that

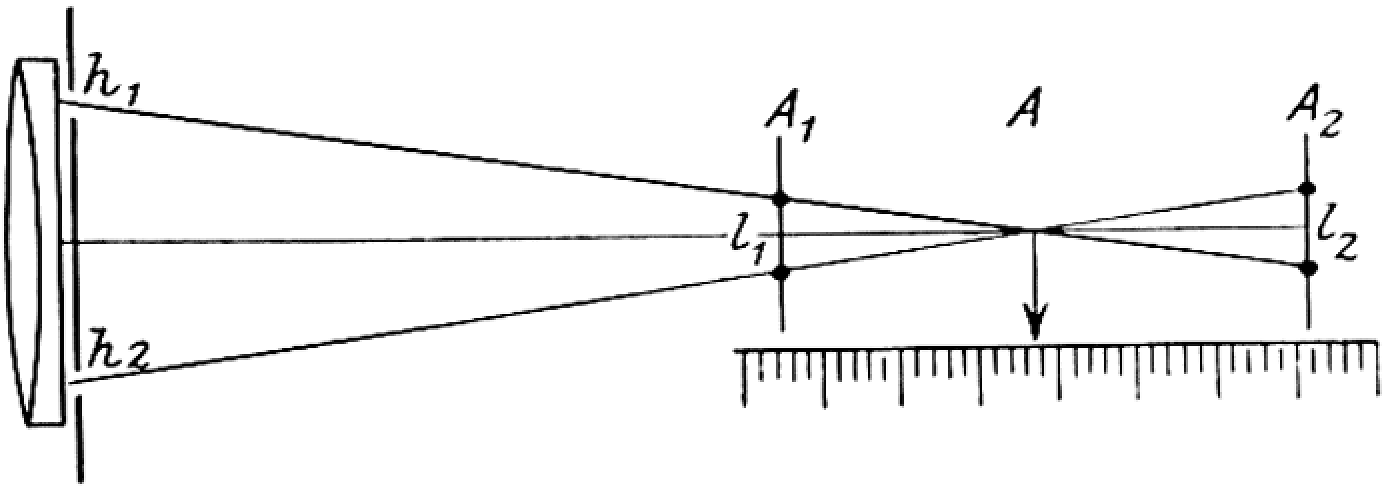

FIG. 227. THE HARTMANN TEST

the astigmatic *difference* of focus between the diameter of minimum focal length and the diameter at an angle of θ with the first will be given by

$$A_\theta = A_{minimum} + a \sin^2 \theta$$

where a is the "astigmatism." Then

$$A_{\theta + 90^\circ} = A_{min} + a \cos^2 \theta$$

and the mean

$$\frac{A_\theta + A_{\theta + 90^\circ}}{2} = A_{min} + \frac{a}{2}$$

We may, therefore, eliminate irregularities due to astigmatism, when taking results from apertures in different diameters, by always taking observations for one zone in two perpendicular diameters. There is thus no reason to limit the diaphragm to two holes. A specification for the arrangement of apertures in the diaphragm for a typical case is as follows—

Objective 80 mm. Diameter		1 metre Focal Length
Angles (θ)		Radii of Zones containing Apertures (in mm.)
0°	90°	10, 18, 26, 38
22·5	112·5	30
45	135	6, 14, 22, 34
67·5	157·5	30

The diameter of the holes are 4 mm. in the outer, and 3 mm. in the inner zones.

It is well to add a single hole in the diaphragm, so that the proper orientation of the plates may be recognized after development. The numerical part of the work lies in the systematic measurement of the values of l_1 and l_2 for the various zones and the calculations of the corresponding focusing points.

If the diaphragm itself is not too large it can be measured up on the measuring microscope, and one photograph may thus be dispensed with if the distance from diaphragm to plate is known, and also whether the plate is taken inside or outside the focus.

The main difficulty experienced in this test is the distortion of the dots arising from interference effects between the light from successive dots. This effect sets a limit to the reduction of the distances between the adjacent holes in the diaphragm, and also makes it imperative to take the negatives at a distance from the focus sufficient to avoid marked interferences.

With ordinary aperture ratios it is not difficult in this way to ascertain the relative differences between the foci of successive zones to within a few tenths of a millimetre. The way in which the geometrical aberrations so found can be interpreted in terms of phase differences of disturbances at the focus was shortly explained in Vol. I, p. 114. A full discussion of the application of the test to a microscope objective will be found in a paper by the writer.[15] Kingslake[16] has pointed out that the assumption made in the above theory that all rays cross the optical axis is not justified, and he prefers to obtain a three-dimensional trace of the rays from a large refractor. The co-ordinates for any ray in any chosen focal plane can hence be calculated.

Hartmann has suggested a criterion for the comparison of astronomical objectives; it is a magnitude expressing the "weighted mean diameter" (in hundred thousandths of the focal length) of the cones of light from the various zones in the plane where the circle of light containing all of the converging pencils is the smallest. Weights are given according to the light gathering power of the zones.

Since the area of a zone is $2\pi r\delta r$, the weight is proportional to the radius and the criterion (denoted by T) is

$$T = \frac{10^5}{f}\frac{\Sigma rd}{\Sigma r}$$

where d is the diameter of the circle of light from a zone of radius r.

Hartmann's original papers gave particulars of similar methods for the measurement of oblique aberrations. The subject has also been developed by Kingslake,[17] who uses the method of the determination of the foci for pairs of rays derived from the extremities of chords of a zone drawn parallel to the axis of tilt of the lens, using formulae developed by Conrady[18]. The method has proved very convenient in practice; reference should be made to the original paper for particulars.

Tilted Plate Method (Wetthauer). In cases where it is desired to measure the spherical aberration and to plot the tangential and sagittal image fields without specific determination of resolving limits, a procedure due to Wetthauer[19] may be employed. The testing bench employs a collimator and camera, the plate in the latter being held tilted at a steep angle, about 80° to the focal plane, Fig. 228 (*a*). Only the horizontal diameter of the plate is in the normal position. The whole camera can be rotated round a vertical axis as in Fig. 228 (*b*) so that the image falls into any required position of the off-axis field and a succession of photographs is thus taken.

The collimator object is pictured in Fig. 228 (*c*). When spherical aberration is to be measured, a series of annular diaphragms is used, each isolating one zone of the objective. A photograph taken with any one diaphragm resembles (owing to the steep tilt) a horizontal section of the convergent beams near the focus, and the position of the best focus can be estimated; thus the variation of focal positions with the variation of the zonal radius can be found from a series of photographs.

For the mapping of the off-axis fields, a single zone can be isolated if desired, while a series of photographs is taken at various angles; the provision of the grid of close horizontal lines in the "object" (they should be at such a frequency that they will just be clearly resolved at the sagittal focus) will enable the tangential and sagittal fields to be determined.

It should, of course, be remembered that where methods of this kind are employed to measure the spherical or chromatic errors of a lens, it is desirable to have a collimator of focal length two or three times at least that of the lens to be tested; with a reasonably well-corrected collimator the spherical error can then usually be neglected, and even the axial colour will produce little error; if an apochromatic collimator (triple lens, etc.) can be had, so much the better. In exacting cases it may be desirable to use real test objects, and in order to avoid any modification of the spherical correction of an objective due to the finite instead of an infinite conjugate distance, the object should be situated at a distance from the camera not less than thirty to forty times its focal length.

Interference Methods—Fizeau's Experiment.[20] Fizeau placed two parallel slits behind an object glass forming the image of a star. If the distance of the slits is d, Fig. 229, and the distance from slits to focus is f', the appearance found in the focal plane is a series of interference fringes of which the successive maxima are separated by the distance $\lambda f'/d$. The angular subtense, at the back nodal point of the lens, of the distance from a maximum to the neighbouring minimum is therefore $\lambda/2d$. If the telescope is directed towards a double star with components at this angular separation, the maximum of one fringe system will coincide with the minimum of the other system, and the pattern will become a uniform band if the two systems are of equal brightness. This is the basis on which was

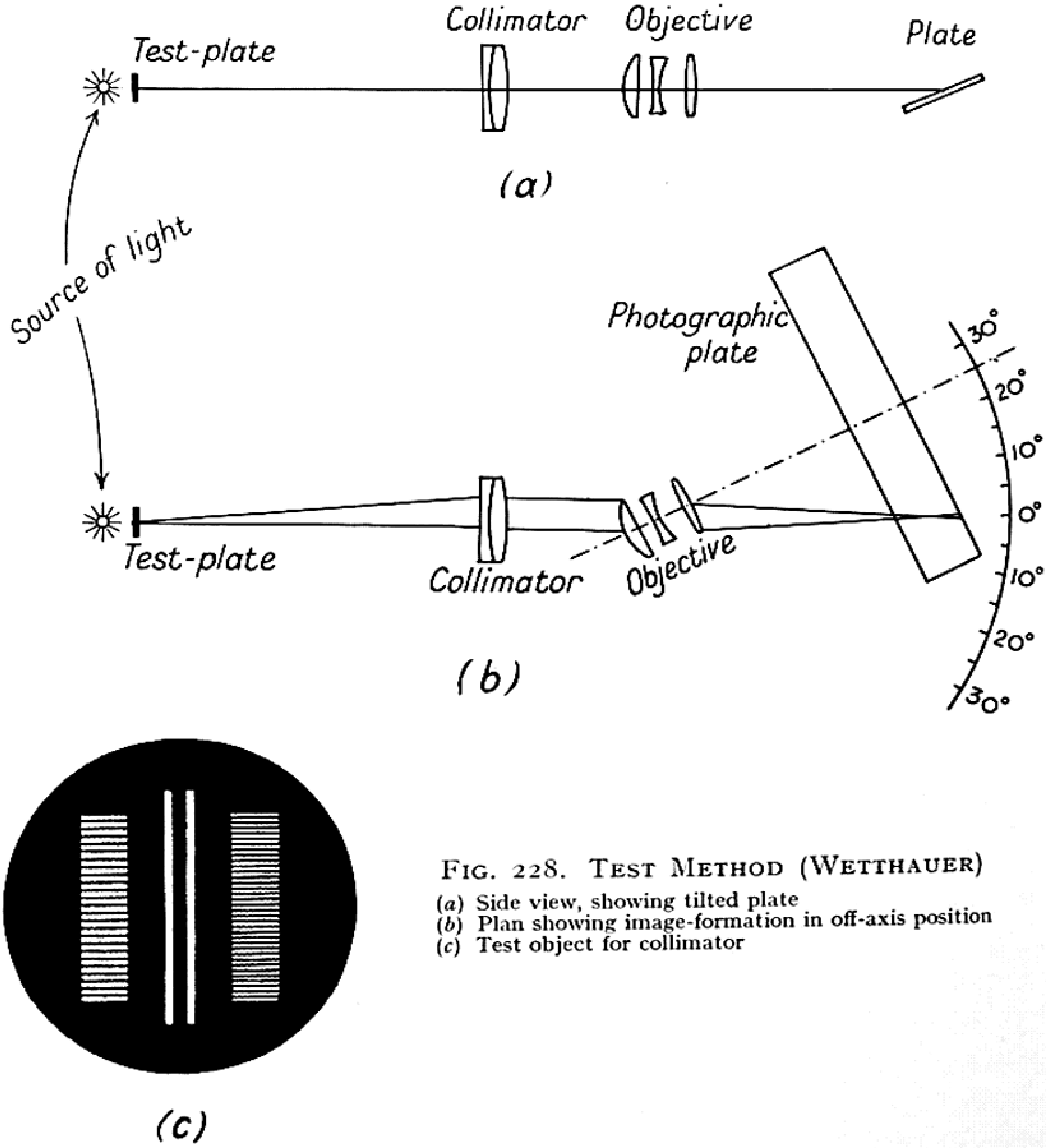

FIG. 228. TEST METHOD (WETTHAUER)
(*a*) Side view, showing tilted plate
(*b*) Plan showing image-formation in off-axis position
(*c*) Test object for collimator

developed Michelson's method of measuring the angular diameters of stars.

Chalmers' Tests. If the slits in Fizeau's experiment are no longer symmetrically disposed with respect to the axis, the central fringe of the interference system will still fall in the axial focal point if the lens is free from aberration, because the disturbances from all parts of the wave-front will reach this point in the same phase. But if spherical aberration is present, the central fringe will be, in general, displaced away from the axis to some position where the phase agrees. In Fig. 230 let A and B be the two apertures; let C be the axial focal point, and let D be the position of the central "maximum" of zero path difference.

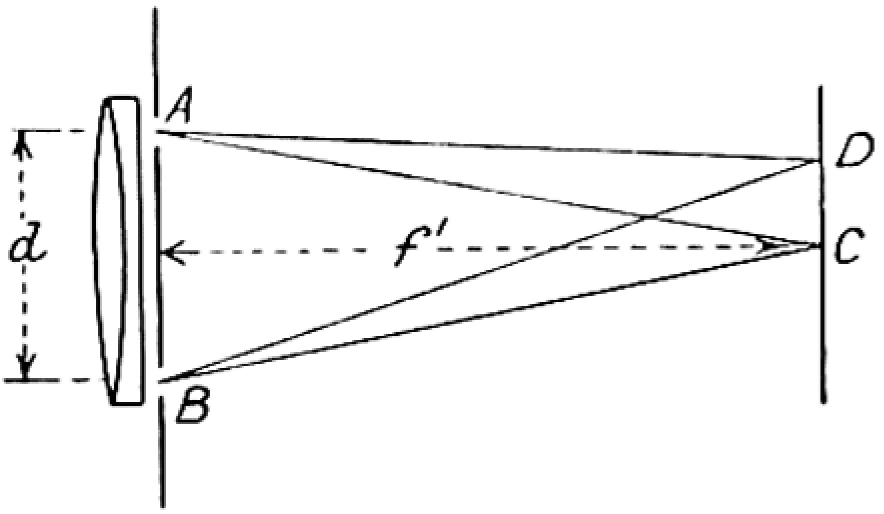

FIG. 229. FIZEAU'S EXPERIMENT

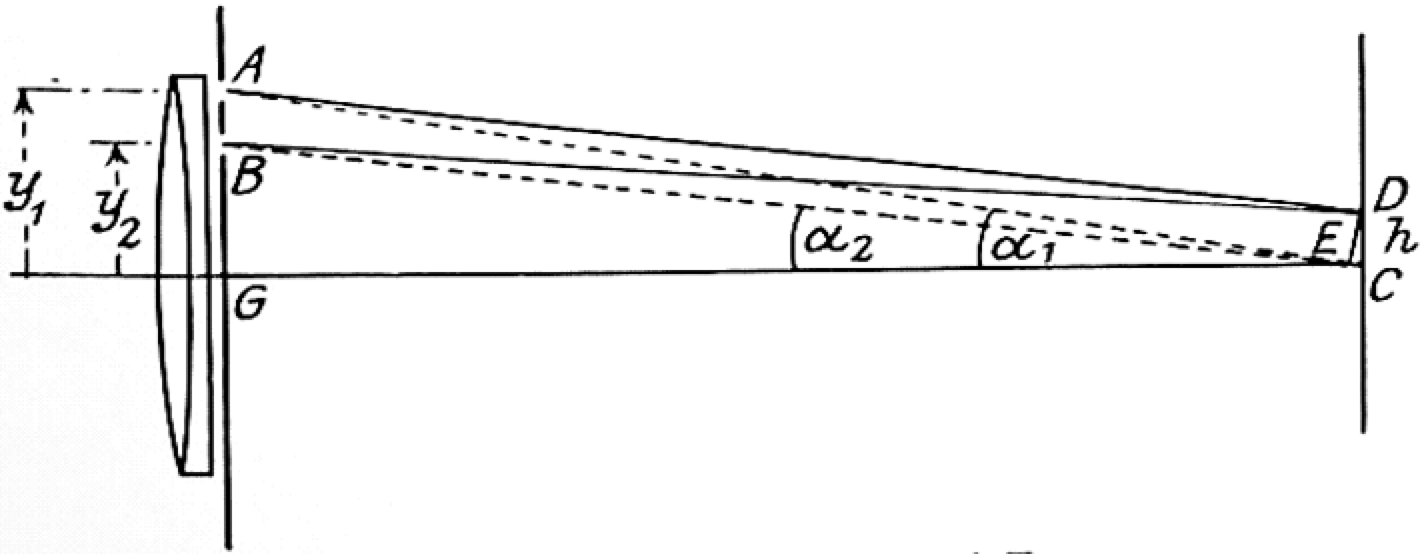

FIG. 230. THEORY OF CHALMERS' TEST

In the actual use of the lens, the image will be formed at the point C, and we are, therefore, concerned to write down the phase difference with which the disturbances from A and B arrive in the point C.

From the point D drop a perpendicular DE on the line AC. The angles $\widehat{ACG}$ and $\widehat{EDC}$ are equal, and will be written α_1. Now

$$\begin{aligned} AC &= AE + EC \\ &= AD + EC \end{aligned}$$

within a small quantity of the second order if AG and CD are small in comparison with GC.

Writing $$h = \text{CD}$$

we thus obtain $$\text{AC} - \text{AD} = h \sin \alpha_1$$

and, similarly, $$\text{BC} - \text{BD} = h \sin \alpha_2$$

Now (AC — AD) and (BC — BD) represent the net changes of optical path of the disturbances from A and B when meeting at C, rather than at D where we know that they have the same phase. Hence, writing y_1 and y_2 for the heights of the apertures A and B respectively above the axis, the path difference at C is

Optical path difference

$$= h (\sin \alpha_1 - \sin \alpha_2) = \frac{h}{f'} (y_1 - y_2), \text{ very nearly.}$$

By starting with one aperture on the axis and one at a distance y_1, we may, therefore, find the optical path difference for the first zone with respect to the paraxial zone. Now moving one step outwards, so that the inner aperture falls into the position previously occupied by the first one, and so on, we may find by addition the optical path differences for a number of successive zones with respect to the paraxial zone. Delicate measurements with an eyepiece micrometer are involved in the practical application of this test, which is not particularly easy to carry out quantitatively, although it makes a ready means of testing an objective by visual estimation; for this purpose the slits may be made in a diaphragm held in front of the objective, while the appearances in the focal plane are watched with a high power eyepiece.

Gardner and Bennett's Method.[21] This is essentially an extension of Chalmers' method in which a photographic record of the fringe positions is obtained for a number of zones simultaneously; it is taken away from the focal plane, thus resembling the Hartmann test; in this case, however, the interference effects which spoil the accuracy of the Hartmann record are turned to good use.

In Fig. 231, let A and B be two apertures; C is the focus of the lens for the central zones, and D the displaced position of the central fringe from A and B. If we take a record inside the focal plane, the fringe will be at E′ on the straight line joining D to the mid-point between A and B, whereas it would have been at E on the corresponding line to C if the aberration were zero. If EE′ is h_1 and the distance from diaphragm to the recording plate is l',

then $$\frac{h_1}{l'} = \frac{h}{f'}$$

If we measure the distance ω between successive maxima in the fringe system due to A and B, the interval will be (closely)

$$\omega = \frac{\lambda l'}{y_1 - y_2}$$

But in the theory of Chalmers' method we found that the optical path difference (O.P.D.) for disturbances arriving at C is

$$\text{O.P.D.} = \frac{h}{f'}(y_1 - y_2) = \frac{h_1}{l'}(y_1 - y_2) = \frac{h_1\lambda}{\omega}$$

$$\text{path difference in wave-lengths} = \frac{h_1}{\omega}$$

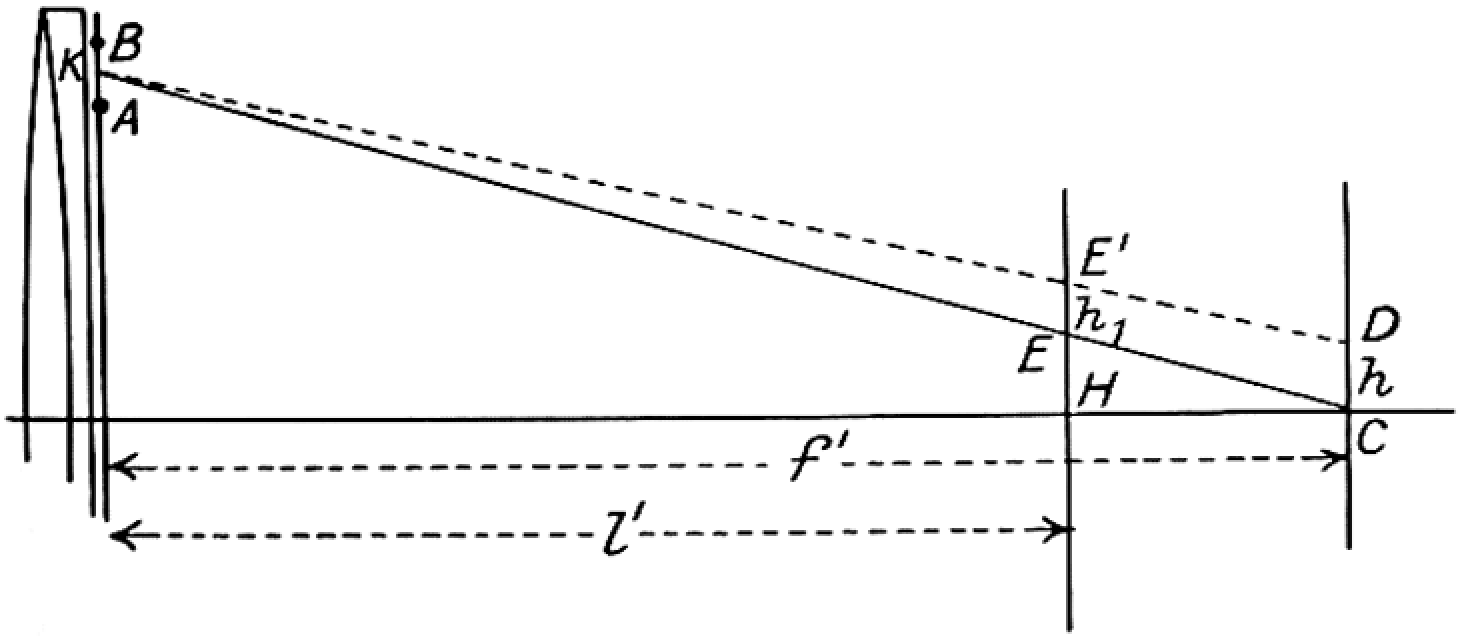

FIG. 231. THEORY OF GARDNER-BENNETT TEST

In practice, a diaphragm is prepared with a number of equidistant holes arranged symmetrically with the axis, and the photograph is taken sufficiently far inside or outside the focus to obtain satisfactory interference fringes in the pattern. The width of the fringes is measured, and also (for each pair of holes) the distance of the centre fringe from the axis, i.e. E'H in the diagram. Then

$$\text{EE}' = \text{E}'\text{H} - \frac{(f' - l')(y_1 + y_2)}{2f'}$$

So that the optical path differences can be obtained for any assumed focus C.

It is to be noticed that the "focus" is not an exactly ascertainable point. We can select a likely point, and then find the optical path differences of the disturbances meeting there; a good criterion for the performance of a system is the maximum residual path difference between any two zones, at the focus where these residuals are numerically least. These remarks apply also to the ordinary form of the Hartmann test described above.

The Shadow-fringe Methods. These methods have been developed by Ronchi[22], [23] and Jentsch,[24] who employ gratings of various descriptions, chiefly straight line gratings. Ronchi chiefly recommends rulings with 10–20 lines per mm., while Jentsch uses commercial process screens with 4–8 lines per mm. The gratings are placed in the convergent beam from the test lens which forms an image of an artificial star. If the focus is a perfect one, the projection pattern of the grating has one point, i.e. the focus, as perspective centre, and the shadow pattern in any plane is an undistorted series of dark and light bars; but if the lens suffers from spherical aberration, then the more oblique rays have a different centre, and distortion results in the shadow pattern. Ronchi develops the theory on the basis of the interference of light, which is a more suitable method for dealing with the effects under the conditions he uses. A brief geometrical treatment due to Jentsch may be reproduced here.

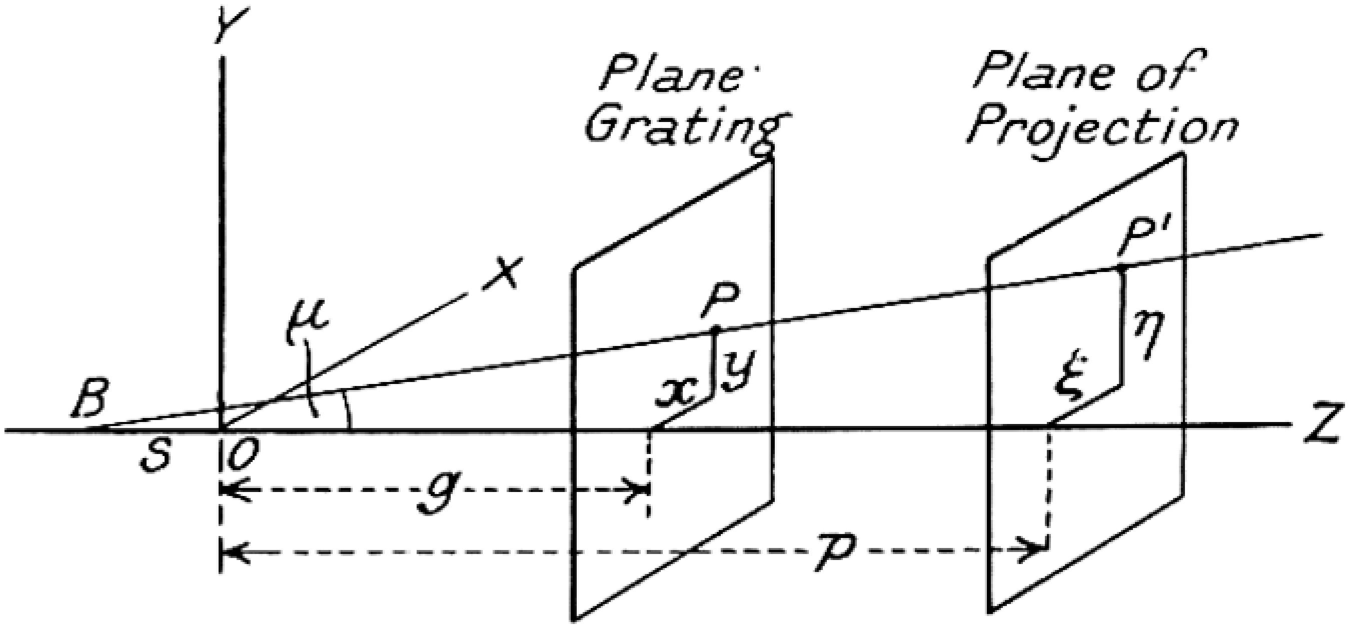

FIG. 232. THEORY OF JENTSCH'S TEST

Refer to Fig. 232. A ray diverges from the point B at an angle u with the Z axis of co-ordinates, and projects the shadow of the point P in the grating plane as P′ in the projection plane. The origin of co-ordinates O is the paraxial focus, and the intercept OB $= s$ is the axial spherical aberration associated with the angle u. The co-ordinates of the points P and P′ are x, y, g, and ξ, η, p, respectively.

From the geometry of the figure,

$$\frac{\xi}{x} = \frac{\eta}{y} = \frac{p-s}{g-s}$$

and

$$\tan u = \frac{\sqrt{\xi^2 + \eta^2}}{p-s}$$

In the simplest type of spherical aberration $s = \mathrm{A} \tan^2 u \ . \ . \ .$, say, sufficiently nearly.

By elimination of s and $\tan u$ in the above equations we obtain

$$\mathrm{A}\ (\xi - x)^3(\xi^2 + \eta^2) = (g\xi - px)(p - g)^2\xi^2 \quad . \tag{7.04}$$

If the aberration is zero, $\mathrm{A} = 0$, and the equation reduces to

$$(g\xi - px) = 0$$

$$\xi = \frac{px}{g}$$

So that if the grating is rectilinear with parallel bars, then the projection will show a series of parallel bars.

Again, when A is finite we may write $x = \mathrm{N}\delta$, where δ is the grating interval, and N is a constant which may be assigned a succession of positive and negative integral values; thus giving the curves of the various shadows in the projection plane.

For a central bar, $x = 0$, and equation (7.04) splits into two parts.

i.e. $\xi = 0$ (a straight line), and

$$\xi^2 + \eta^2 = \frac{g}{\mathrm{A}}(p - g)^2$$

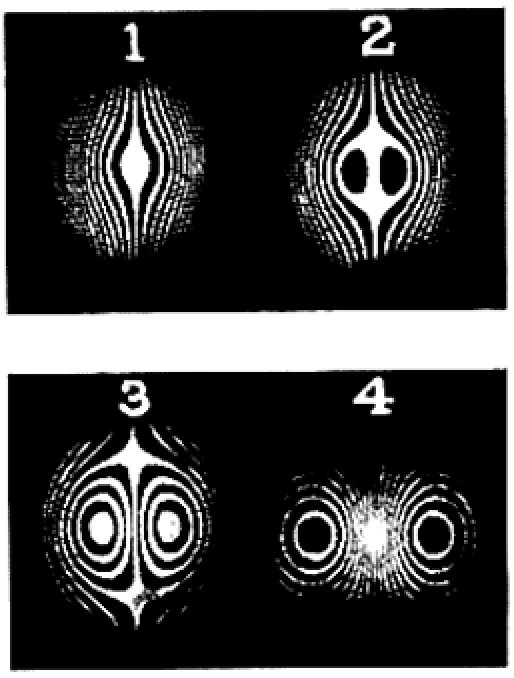

FIG. 233. JENTSCH'S GRID METHOD

1 and 2 = Shadow fringes outside the caustic

3 and 4 = Shadow fringes inside the caustic

The appearances follow in the progression 1, 2, 3, 4

this latter curve is a circle, but it can only be real if g and A have the same sign, i.e. it will only appear on one side of the focus. Fig. 233 shows typical shadow fringes inside and outside the focus of a lens suffering from aberration.

For more exact quantitative measurements, Jentsch advises photographic recording of the shadow fringes and measurement of the plates along the X direction; the calculation will be carried through in ways similar to those already described.

Modification of the Foucault Test. Platzeck and Gaviola. One of the principal difficulties in the Foucault test is caused by the edge-diffraction of the zonal apertures; the boundary waves (p. 140) produce rings of light around the holes which tend to prevent the critical judgment of photometric equality. Platzeck and Gaviola[25] point out that a knife-edge used in the ordinary way will notcoincide with the true zonal focus of more than one zone of the system. The locus of the zonal foci is the caustic curve of the bundle of rays; if a fine wire is moved laterally inwards so as to intersect the caustic, a shadow first appears on the surface of the system (seen without any diaphragm) at a point corresponding to the origin of the ray first intersected. Hence the first part of the test is to map the caustic by this means, the wire being mounted so that its movements

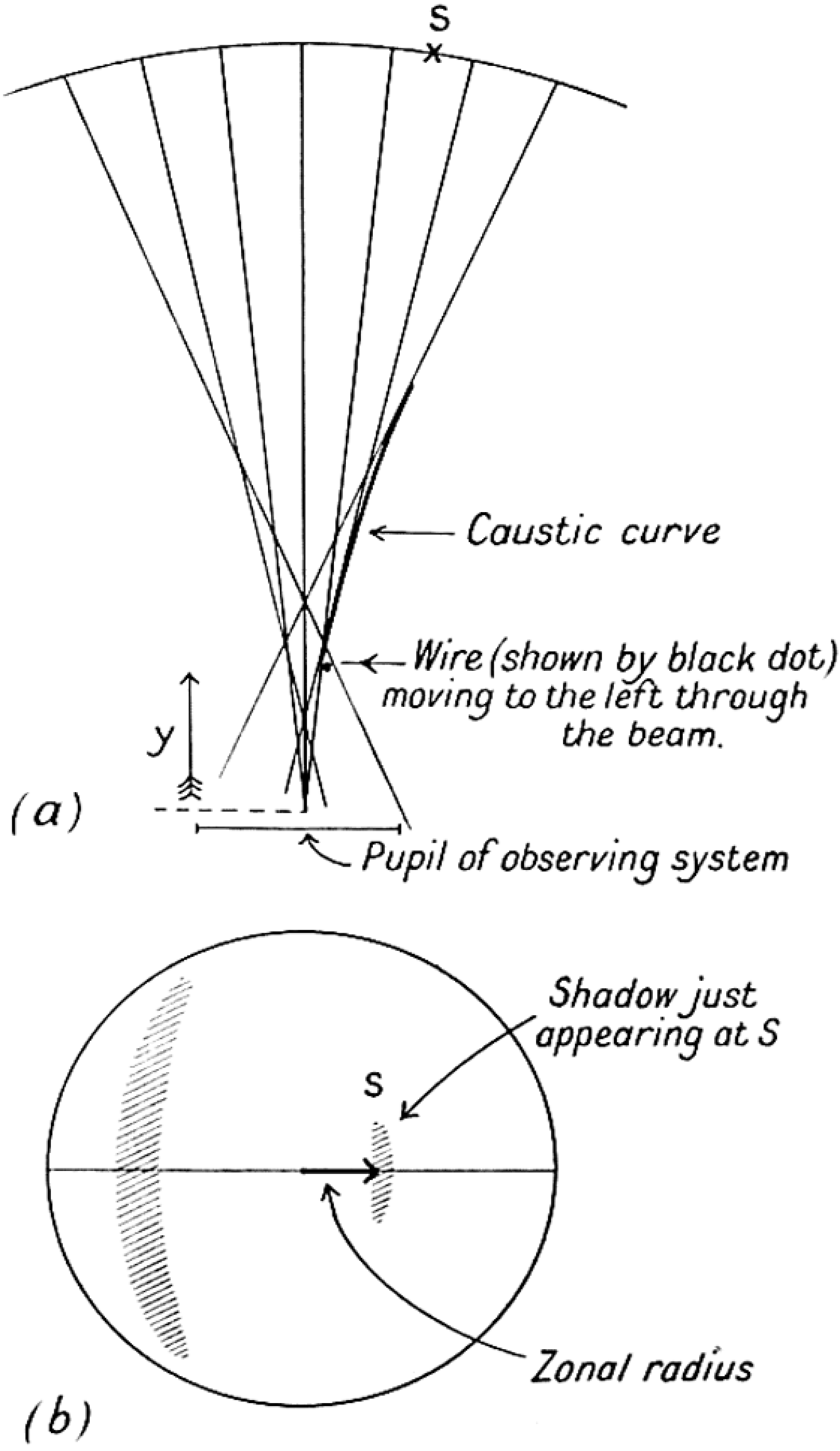
S
Caustic curve
Wire (shown by black dot) moving to the left through the beam.
y
(a)
Pupil of observing system
Shadow just appearing at S
S
Zonal radius
(b)

FIG. 234. PLATZECK AND GAVIOLA'S TEST (1ST PART)

To find the axial displacements (y) for the zonal foci of the wave front. A case of spherical under-correction is shown

(a) The caustic curve. (b) The appearance in the field

can be exactly measured. The observation can be assisted by the use of a telescope; a *very* fine pinhole is necessary for the source of light. We can thus first determine approximately the "axial" displacement for the foci of successive zones. Fig. 234 suggests the conditions. A vertical wire, shown by a black dot, is supposed to have been moved in from right to left; a curved shadow has appeared on the left of the field where the outer rays are cut off and a shadow just *begins to appear* at S, the ray from which touches the caustic curve on the point represented by the wire.

The next part of the test is to isolate by a diaphragm a square or circular element of the surface in a known zone. The wire is moved axially by the appropriate amount, and then brought by lateral movement as exactly as possible into the focus of the element; this can be checked by observing the image disc, partially occulted by the wire, with the aid of a microscope. The diaphragm set-up is now changed so as to expose the diametrically opposite element of the system on the other side of the axis, and the necessary shift of the wire is measured. In this way both the x and y co-ordinates of the caustic curve can be found with respect to the zonal foci and the aberration of the objective accurately determined.

The method of evaluation of the optical path differences from the x (lateral) and y (axial) co-ordinates of the points of the caustic will be apparent on reference to Fig. 235. In Vol. I the measure of the angular aberration (Vol. I, p. 113) was shown to be given by $(\delta p/\delta\sigma)/n'$, where δp is the increase of optical path incurred in an increase $\delta\sigma$ of the zonal radius. In Fig. 235, the point O is the centre of a spherical reference surface of radius R while W is the trace of a wave emerging from the system; the points A and B are situated on the normals to the wave from the zone indicated by p_1 and p_2 (they lie also on the caustic). The "angular aberration" in radians of the normals with respect to the radii of the spherical surface is given by the close approximation

$$\frac{1}{n'}\left(\frac{\delta p}{\delta\sigma}\right)=\frac{y\sin\phi-x\cos\phi}{R+y\cos\phi},$$

where ϕ is the angle between the radial line P_1O and the axis of symmetry; n' will be unity as the image is in air.

Whence
$$\delta p=\delta\sigma\frac{y\sin\phi-x\cos\phi}{R+y\cos\phi}$$

The sign convention used in this equation will be obvious; it differs from that of the original paper.

The apparatus for measuring the displacements of the wire should measure the y co-ordinates within 10^{-2} mm., and the x co-ordinates with an accuracy approaching 10^{-4} mm. The total ranges of movement desirable for a typical mirror are one to two cm. axially and two to three mm. laterally. A careful analysis of the possible accuracy of the results is given in the original paper; it is concluded that if seven zones are used in the test the probable error in the estimation of the optical path difference for any one zone must not exceed one hundredth part of a wave-length if the result for the whole mirror is to be taken as a trustworthy indication as to whether the errors of the system are within the Rayleigh Limit.

Interferometer Tests on Optical Apparatus. The interference of two beams reflected from the surfaces of a thin film was discussed in Vol. I, p. 185, *et seq.* It was shown that if the faces of the film are parallel, and the angle of reflection inside the film is i', then the relative optical retardation of the second disturbance is $2n't \cos i'$, where n' is the refractive index of the medium of the film. Remembering the phase change of π consequent on reflection at one face (that which would involve a step from a lower to a higher refractive index for the transmitted light), the condition for destructive interference is

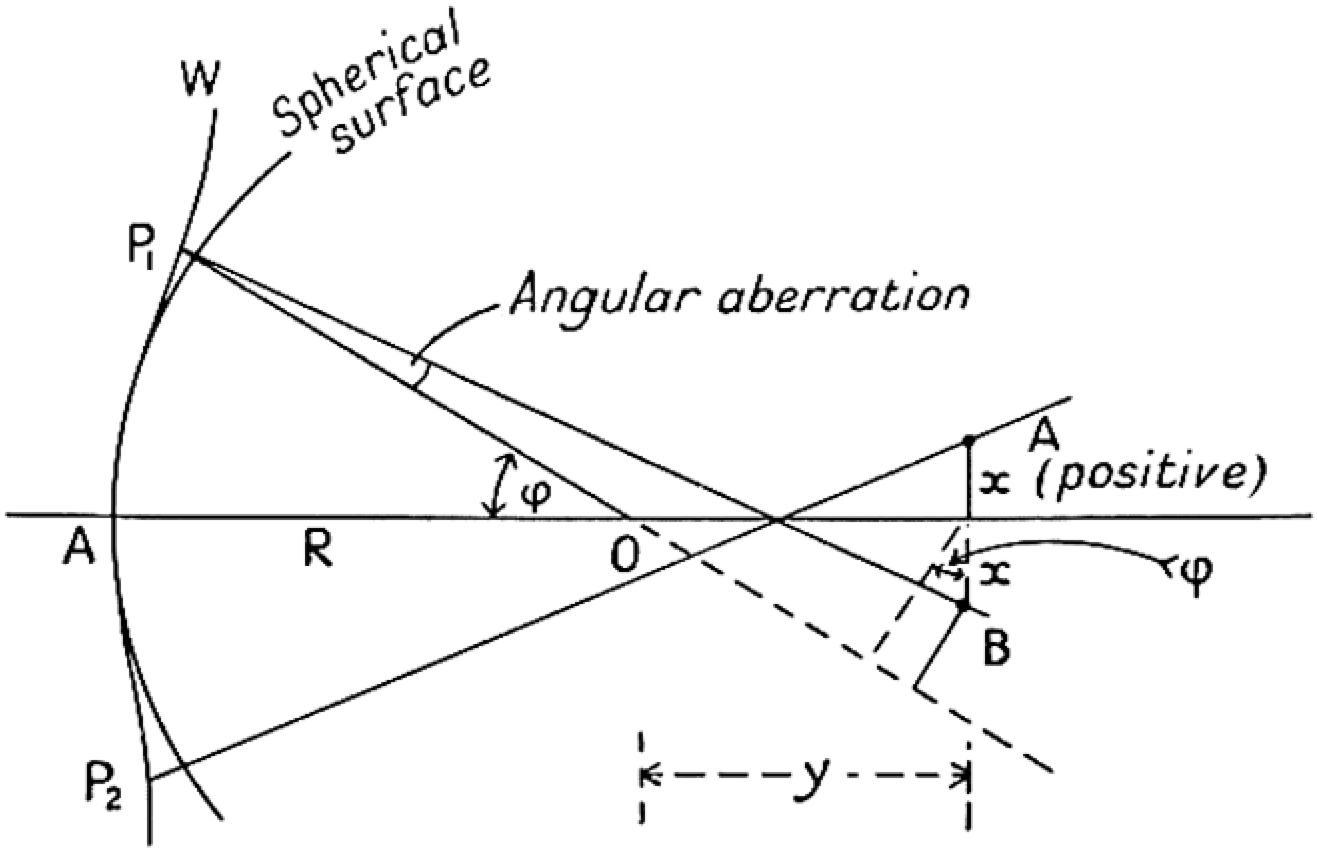

FIG. 235. PLATZECK AND GAVIOLA'S TEST
A likely value of y determined by the first part of the test is taken, and the exact value of x corresponding thereto is then measured. This diagram suggests over-correction

$$2n't \cos i' = m\lambda$$

where m is 0 or an integer.

When the faces of the "film" are truly parallel, interference effects can be seen at separations much greater than those characteristic of the soap bubble, in fact for quite large separations if light of good spectral homogeneity is employed. The limits of optical path difference up to which interference effects can be observed are not certain, but Michelson's experiments showed that they are not less than half a metre if the illuminating light is "monochromatic" and sufficiently homogenous.

One very useful method of testing employs the so-called Haidinger's fringes. Referring to Fig. 236, we see that the thin glass plate reflects the green light from a mercury vapour lamp (preferably a low pressure glass lamp) downwards on a piece of glass G with plane parallel faces. The parallel components of the reflected light are brought to a focus on the retina of the eye accommodated for infinity, and the condition for destructive interference is still given by the above equation. The optical path difference for the disturbances following the exactly normal path is $2n't$, and it is clear that $\cos i'$ will be constant for any circle surrounding the normal. The fringes therefore appear as circles with the normal as centre.

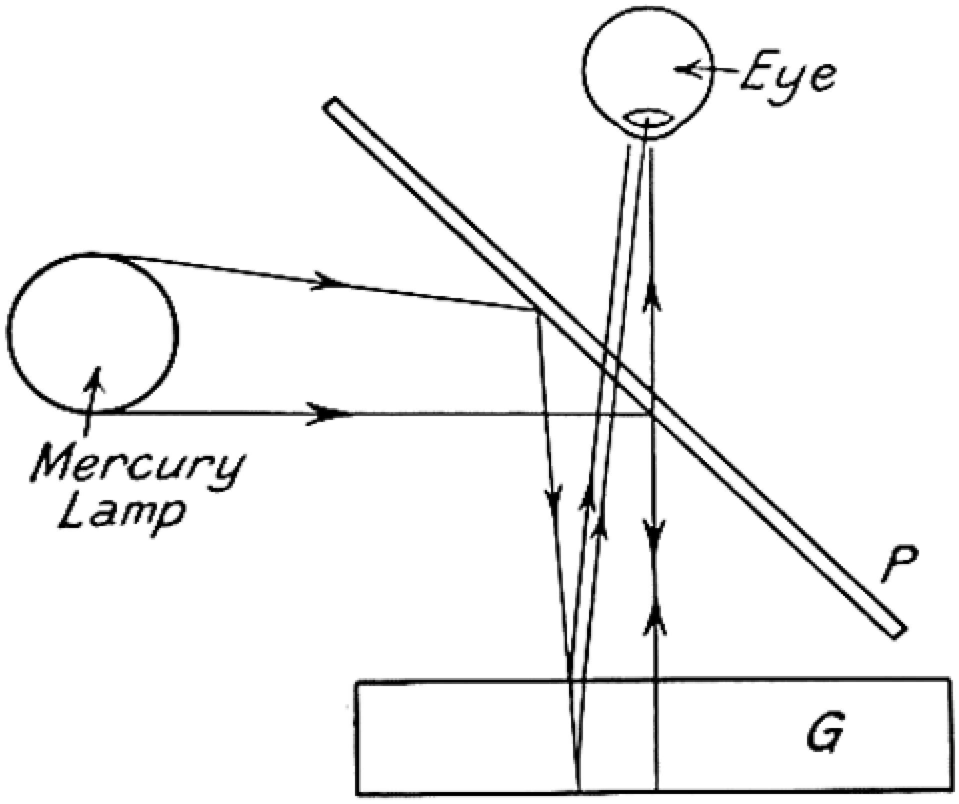

FIG. 236. PRODUCTION OF HAIDINGER'S FRINGES

If the plate G is moved about and the thickness is slightly irregular, we may find a region in which t increases. Now one definite fringe corresponds to a fixed amount of retardation. It will, therefore, move to a place where $\cos i'$ diminishes in order that $2n't \cos i'$ may remain constant. This means that i' must increase, and hence the fringes move outwards from the centre, fresh fringes developing at the central point and expanding outward; the development of one fringe corresponds to a variation of one wave-length in $2n't$. The reverse effect takes place if the thickness diminishes. Hence by systematic movements of the plate it is possible to examine the variation of thickness over the whole area.

It may be mentioned that the interferences arising in a very similar way at two parallel surfaces are found and examined in the

Fabry-Perot interferometer, but in this case two half-silvered glass surfaces held exactly parallel separate a film of air, and the circular fringes are viewed by transmitted light. This system has been used by Barnard to test the accuracy of the slides of a microscope. One plate is mounted on the stage of the instrument, the other in place of the objective; the fringes remain central while the plates are truly parallel.

Fizeau's Apparatus. Fizeau used a device by which the angle of incidence of the light is constant over the reflecting surfaces, but a lens is then necessary in order to bring the light from the different parts into the eye. The optical parts of an arrangement similar to Fizeau's are shown in Fig. 237, which is adapted for sensitive interferometric tests of an optical surface.

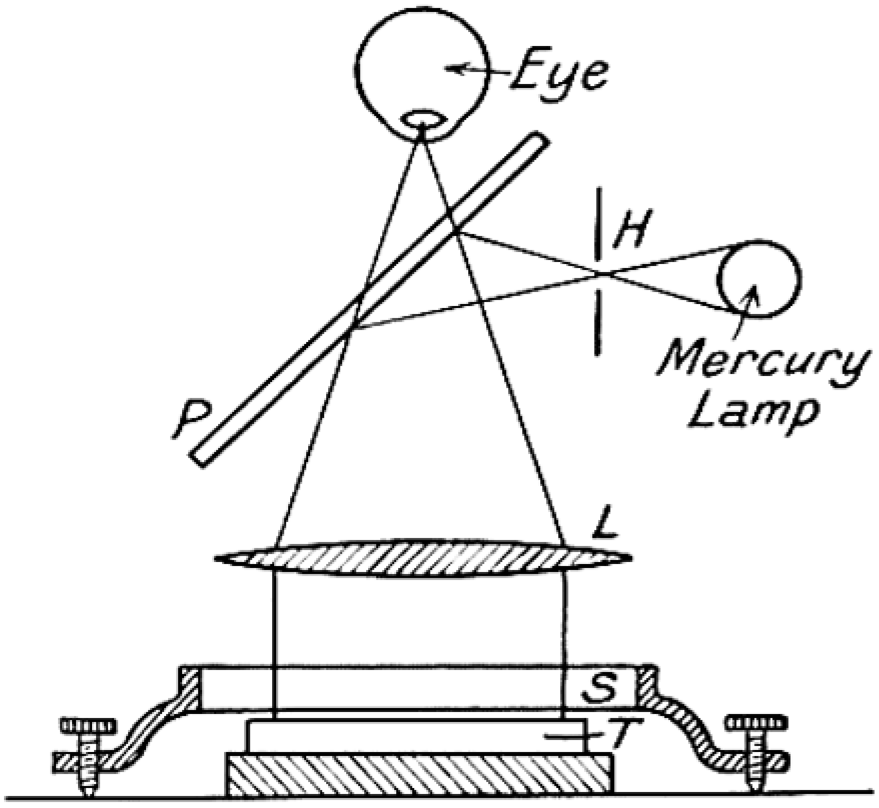

FIG. 237. FIZEAU TYPE OF INTERFEROSCOPE

The glass plate S has a truly plane lower surface and is supported by a stand with levelling screws, so that this surface can be brought closely parallel to the upper surface of plate T, which is to be tested for its "figure." The mercury lamp illuminates the pinhole H, which gives a diverging beam reflected downwards by the plate P. The light is then rendered parallel by the lens L. The components reflected from the adjacent glass surfaces of S and T show interferences, the geometrical path difference at any point being twice the thickness of the air film, since the incidence is everywhere normal. The light on its return journey is brought to a focus in the eye, so that a Maxwellian view* of the interference field is obtained, but showing a distribution of fringes which represent, in fact, a contour map of optical thickness.

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* So called after Maxwell's arrangement in his "colour box" where the eye, placed in the focus of a lens, sees the whole aperture of the lens uniformly illuminated.

If the plate S is slightly prismatic, reflected light from its upper face may be avoided; the lower surface of T may be blacked or greased.

In cruder applications of the test the surfaces S and T may be carefully cleaned and then slid or wrung together in contact. Any irregularities of the test surface are then again revealed by the variation in thickness of the separating film. The disadvantage in this method lies in the possibility of the deformation of one or both plates.

The Michelson Interferometer. The optical system of the Michelson interferometer is analogous to the arrangement for the production of Haidinger's fringes. It is illustrated in Fig. 238. Light from an extended source meets the mirror M_3, and is half transmitted and half reflected by the lightly silvered back surface. The components proceed to the mirrors M_2 and M_1 respectively, from which they retrace their path, combining again at the surface of M_3. The eye placed in the position E can view the interferences with or without the aid of a telescope; the formation of two interfering components in an oblique direction is very closely analogous to that of Fig. 236 (Haidinger's Fringes), but there is the immense advantage that one surface does not "get in the way" of the other. The system is equivalent to an air-plate formed between M_1, and the image of M_2 in M_3.

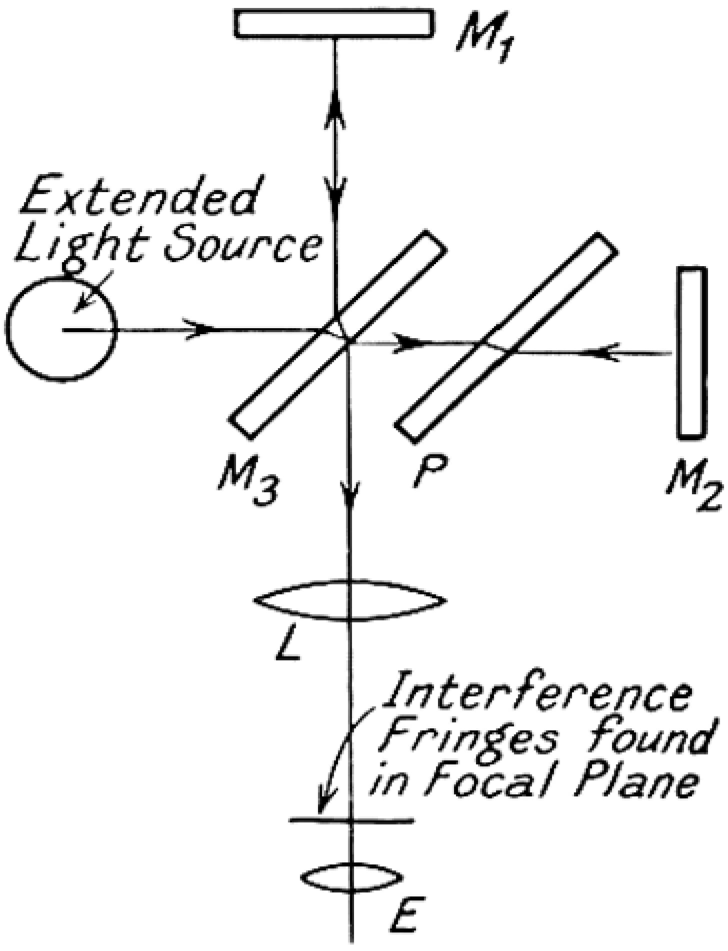

FIG 238
MICHELSON INTERFEROMETER

The interferences are ascribed to the variation of relative retardation with the obliquity of the light, and the fringes are grouped around the normal as before. The component reflected towards M_1 has an extra double transmission through the plate M_3; this is compensated for the M_2 component by the insertion of the plate P equal in thickness to M_3; the fringes are then circular.

Interferences are only visible with white light in very thin films; the colours in Newton's rings soon fade out as the thickness of the film increases, owing to the superposition of the systems of different sizes in different wave-lengths. Equally, the fringes are only found with white light in the Michelson interferometer when the "equivalent plate" is very thin; i.e. when the effective lengths of the "arms" of the apparatus are very nearly equal. With such a source as a high-intensity mercury arc (with a green filter) the path-lengths must be equalized to within a few millimetres.

The Prism Testing Interferometer. As the Michelson interferometer is analogous to the Haidinger fringe apparatus, so the lens and prism testing interferometer, the earliest form of which was patented by Twyman and Green in 1916,[26] is analogous to the Fizeau arrangement. A simple form of the apparatus is shown in Fig. 239. As in the Fizeau apparatus, the beams are brought to a focus by a lens which gives a Maxwellian view of the interference field; this is in effect a contour map of the optical thickness of the "equivalent plate" of the system.

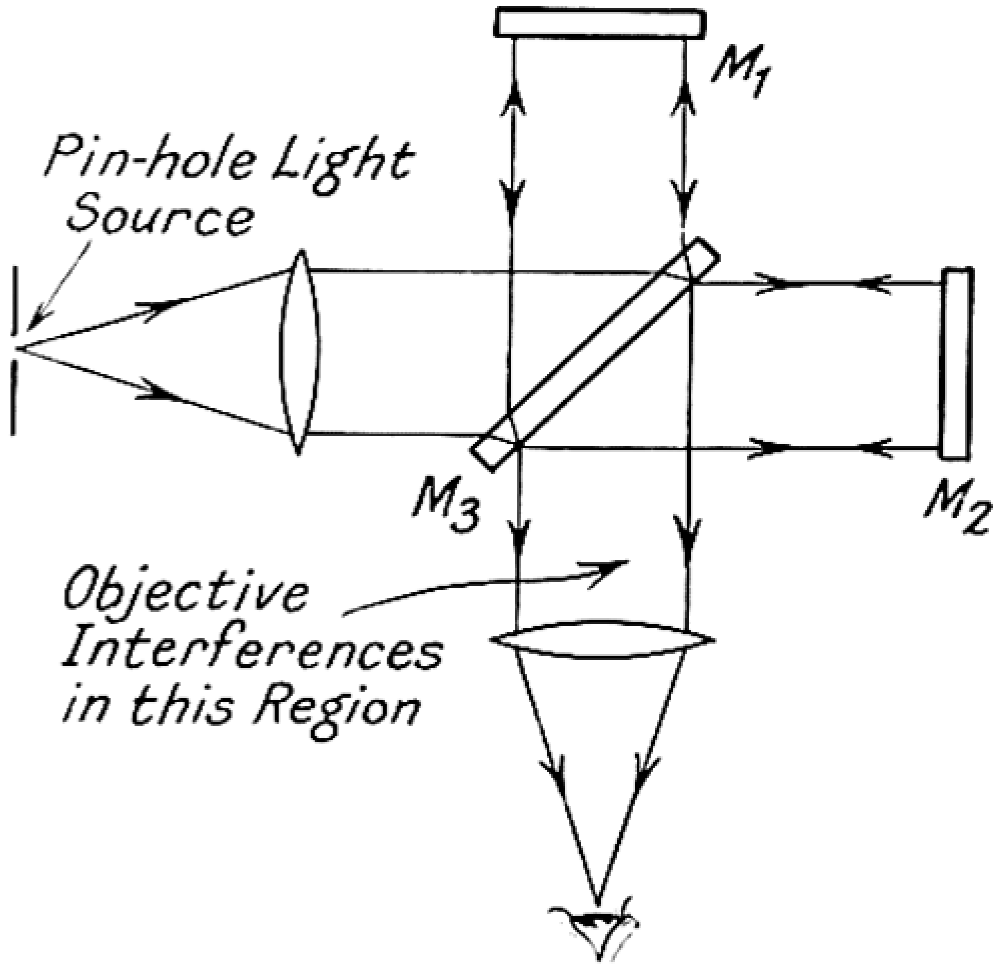

FIG. 239. TWYMAN AND GREEN INTERFEROMETER

We may obtain a simple view of the action by considering a point source and a perfect lens giving two wave-trains of plane waves which unite after passing through the system (Fig. 240). If their directions are inclined at a small angle α and the wave-length is λ, it can easily be seen that a steady interference system of maxima and minima distributed in planes equally inclined to each group of waves will result. The distance between two "maximum" planes is strictly $\lambda \operatorname{cosec} \alpha \cos (\alpha/2)$, but for all practically interesting cases this becomes λ/α since α is always very small. The objective existence of these maxima and minima in the region of superposition can easily be demonstrated with the aid of a piece of ground glass on which the fringes can be found, or by exposing a photographic plate shielded from extraneous light. When the value of α is zero, a single dark or light fringe broadens so as to fill the whole field.

Imagine now a very slightly wedge-shaped piece of glass to be placed in the path of the M_2 beam, the thicknesses at the two ends of the plate being t_1 and t_2 respectively. The increase of optical path owing to the introduction of a thickness t of a medium of refractive index n into a space previously filled with air is $(n - 1)t$. Hence the relative path difference between disturbances passing through the extremities of the plate is $(n - 1)(t_1 - t_2)$. The double transmission in the interferometer will make the actual optical path difference $2\,(n - 1)(t_1 - t_2)$, and the number of resulting fringes will be $2\,(n - 1)(t_1 - t_2)/\lambda$. Note now that the same plate tested by the Fizeau method would show $2n\,(t_1 - t_2)/\lambda$ fringes, so that if $n = 1{\cdot}5$ we shall only find one-third the number of Fizeau fringes when using the interferometer; though it is less sensitive for this particular purpose, there are other advantages which more than compensate.

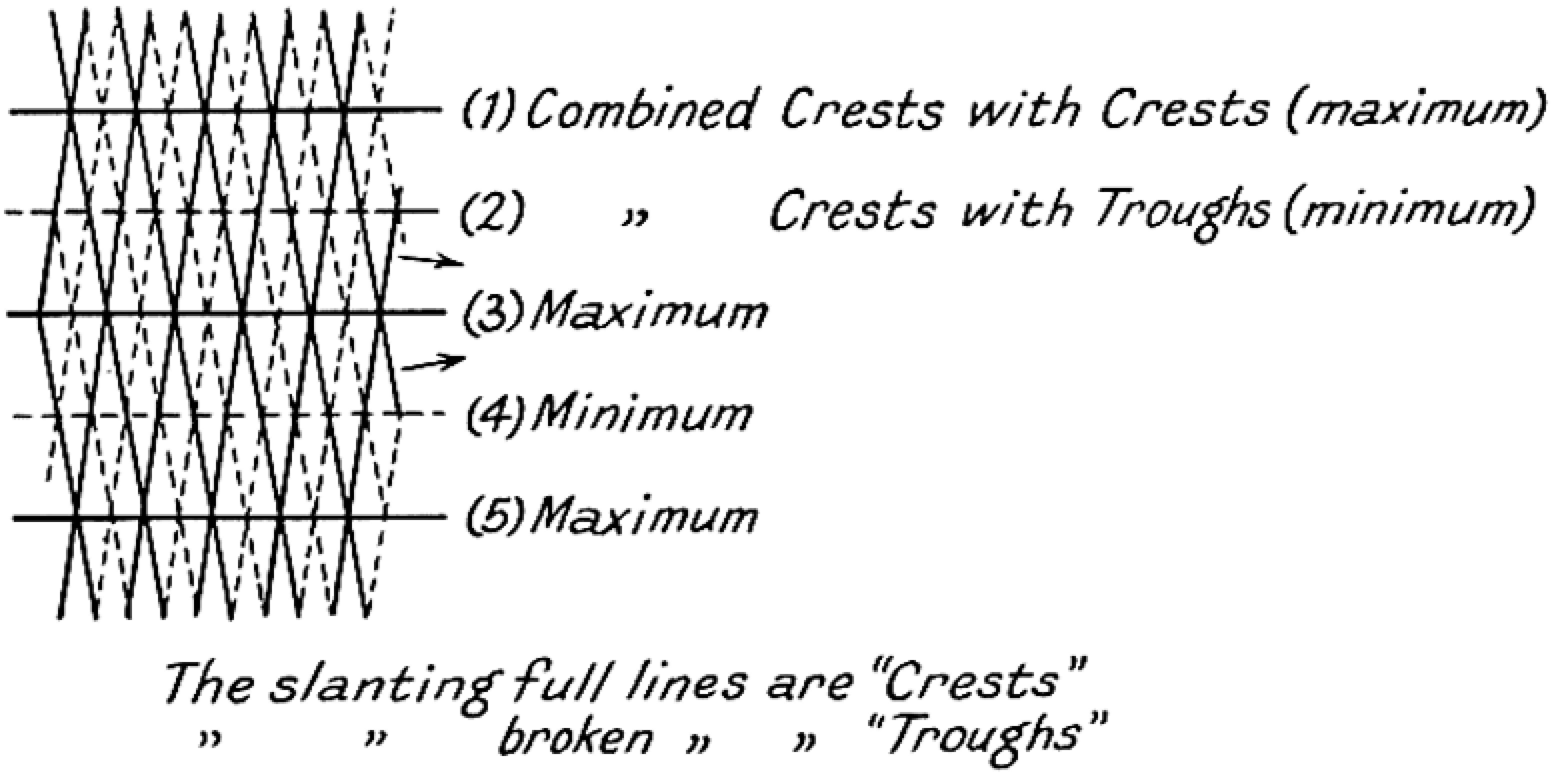
(1) Combined Crests with Crests (maximum)
(2) " Crests with Troughs (minimum)
(3) Maximum
(4) Minimum
(5) Maximum
The slanting full lines are "Crests"
" " broken " " "Troughs"

FIG. 240. THE SUPERPOSITION OF PLANE WAVE-TRAINS

If, for example, a prism of considerable angle is placed in one arm, the mirror can be rotated so that it is normal to the incident light, which therefore retraces its path. The optical path differences along the various ray routes will now be zero if the surfaces of the prism are uniform and the glass is homogeneous; but local irregularities in the surfaces, or variations of refractive index of the specimen, will cause variations which are seen in the contour map of optical thickness.

Slight pressure on the iron bed of the instrument on which the mirrors are mounted can be applied so as to elongate or diminish the path in one arm. The fringes will therefore (moving so as to retain the same retardation) proceed to places of lesser or greater path in the specimen, thus giving a criterion of the sign of the irregularity.

Suppose, for example, that a prism for use in a spectroscope has a local increase of refractive index in one region; this part may be marked out on the surface of the prism by a brush or grease pencil, and subjected to local rubbing by a chamois leather pad charged with rouge. In this way sufficient glass may be removed to compensate for the local variation of index and perfect the interference pattern. In such ways as this it is possible greatly to improve the action of prisms and optical parts, although the compensation will only be exact for one direction of the light, and nothing can wholly compensate for any lack of homogeneity in the glass.

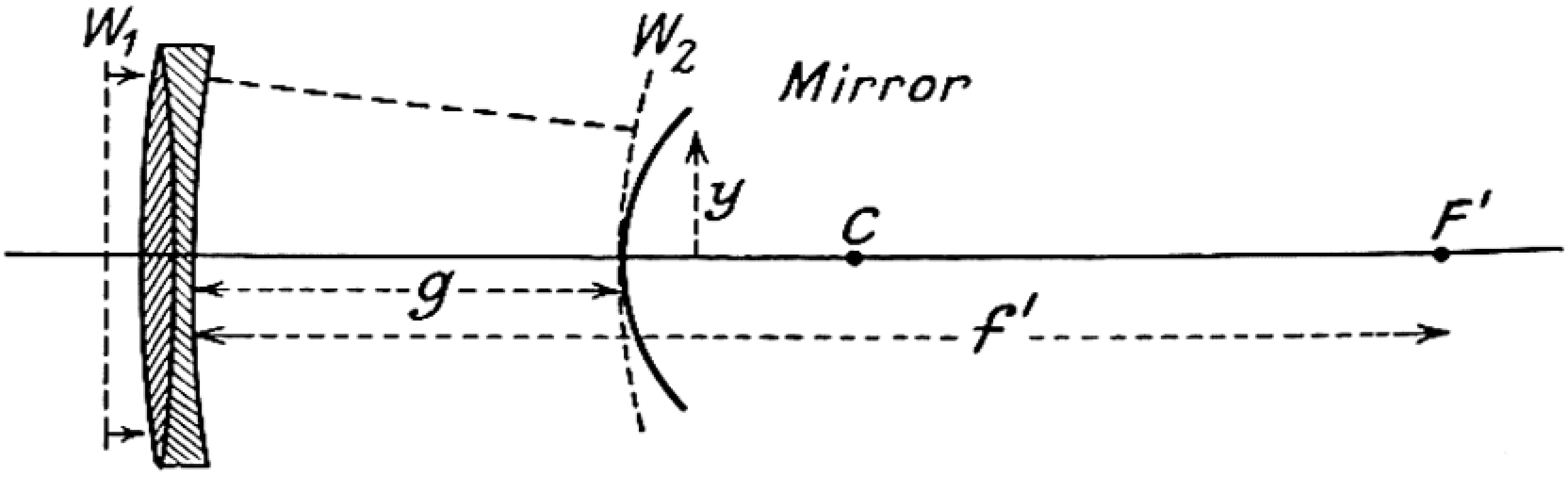

FIG. 241

The method of testing lenses for axial aberrations will be understood from Fig. 241. The convex mirror, which must be of reasonably perfect figure (correct to within $\lambda/8$), should be so disposed that its centre of curvature coincides with the focus of the lens. This system now replaces the plane mirror M_2 of Fig. 239. With a perfect lens the "rays" will all fall normally on the surface of the mirror, and will return along their own paths; consequently, no difference of direction can arise for any part of the wave-front, and a uniform interference pattern will be obtained.

Aberration Path Differences with Lens and Mirror System. Referring still to Fig. 241, the lens is shown with principal focus F′, and the mirror with centre of curvature C. These will not in general coincide at first, although the apparatus will be so constructed that C falls on the axis of the lens, and in cases of practical interest C will be very close to F′, so that rays will return very nearly along their own paths. The wave-front has the radius $(f' - g)$ on reaching the apex of the mirror, f' being the vertex focal length of the lens, and g the separation of the adjacent apices of the lens and mirror.

Assuming as the usual basis of Fermat's theorem (Vol. I, Chapter IV) that we can calculate optical paths along known ray routes, the extra geometrical path for the marginal over the paraxial ray will be represented by double the marginal gap between the circles centred in C and in F respectively; that is, if the lens is free from aberration.

$$\text{Marginal path} - \text{axial path} \approx 2y^2 \left\{ \frac{1}{2r} - \frac{1}{2(f' - g)} \right\}$$

The optical path difference will be proportional to y^2, and the frequency of the circular fringes will increase from the centre outward, but the central fringe will broaden out to fill the whole field when $r = (f' - g)$, i.e. when the mirror is drawn back so that C and F coincide.

However, proper collimation is not the only condition to be fulfilled; the radius of the convex mirror must be chosen so that the conditions for maximum contrast and visibility of the fringes are not unduly transgressed. (See p. 323 below.)

Measurement of Aberration. If the lens suffers from spherical aberration, the optical path difference for marginal and paraxial rays at the paraxial focus can be represented by a series (Vol. I, p. 119).

$$\text{Marginal path} - \text{axial path} = c_2 y^4 + c_3 y^6 + \text{etc.}$$

Hence, in general, the optical path difference with lack of coincidence of the centre of the mirror with the paraxial focus, and in the presence of spherical aberration, will be represented by

$$c_1 y^2 + c_2 y^4 + c_3 y^6 + \text{etc.}$$

This equation may be discussed in the manner used in Vol. I, but it will suffice here to point out that if the focus for any particular local zone of the wave surface coincides with the centre of curvature of the mirror, there must necessarily be a local zonal maximum or minimum of optical path. Therefore, on applying the test above (slightly lengthening one branch) the fringes will gather towards or spread from this zone in both radial directions; a fringe at such a zone will be evanescent if the other fringes crowd towards it.[24] This criterion affords a ready means of setting the radius of curvature of the mirror into coincidence with the foci of successive zones of a lens suffering from spherical aberration, and hence allows the axial aberration to be measured if the mirror is furnished with a suitable micrometer screw. At the same time, the optical fringe system is a "contour map" showing the optical path differences with which the disturbances from particular zones meet in the focus corresponding to the centre of curvature of the mirror.

A recent apparatus[27] for correlated interferometer and resolving-power tests on camera lenses avoids the earlier difficulty of associating a particular interferometer pattern with the exact focus at which the resolution test is made.

While the interferometer is at the service of the professional optician, the older methods will still be fruitful in the hands of those who may only have occasional necessity for the accurate measurement of aberrations.

Sensitiveness. The Rayleigh limit of $\lambda/4$ for the allowable differences of path at the best focus for a telescope objective would, if present in a lens tested on the interferometer, give optical path differences of $\lambda/2$ owing to the double transmission. Taking a case when this arises through "first order" spherical aberration, the fringe system might then be (say) at its brightest for a mid-zone where the phases of the re-united waves agree, and shade off to the minimum both at centre and margin where the path differences would reach half a wave-length. In practice, the difficulties in making the surfaces, especially the convex surface of the mirror, accurate within say $\lambda/8$ are considerable, and the above would represent something like the lower limit of aberration which it is possible to measure with accuracy. The most serious defects of lenses, when the theoretical aberrations are reduced below about $\lambda/2$ or so, are generally faults of regularity, centring, etc. Local path differences of very much smaller amounts can readily be detected by the relative variation of intensity in the interference field.

Chromatic Aberration. When the centre of curvature of the mirror coincides with the principal focus of the lens for any zone, the combination of lens and mirror is clearly equivalent to a plane reflecting surface, at least in the zone considered. This consideration is the basis of the method of measuring the chromatic aberration on the interferometer; it is, however, advisable to employ a chromatically corrected lens to collimate the light. With the aid of a hydrogen tube and a mercury lamp with a monochromator, or with suitable colour filters of the Wratten series, it is possible to set the mirror to the focus of the lens for a series of wave-lengths throughout the spectrum.[25] With a photographic lens it is possible to study the chromatic variations for one particular zone in observing the evanescence of the fringes, but with more highly corrected lenses it will be necessary to study the pattern with the minimum number of fringes for the lens as a whole.

Measurement of Oblique Aberrations. The earlier simple forms of the lens testing interferometer used for testing telescope objectives have been followed by other more complex arrangements by the aid of which the aberrations of other lenses, such as camera lenses and microscope objectives, may be ascertained. The camera lens interferometer[28] allows of numerical tests of the oblique aberrations, and Kingslake[29] has shown how the numerical coefficients of the Seidel aberrations may be deduced from an interferogram.

Microscope Interferometer. Fig. 242 illustrates the principle of one form of the microscope interferometer. It will be seen that the plane mirror in one arm is replaced by a negative lens followed by the microscope objective. The virtual image of the pinhole source formed by the negative lens must lie at the proper working point fixed by the appropriate tube length of the objective. The spherical mirror in this case is sometimes represented by a globule of mercury. These globules, when sufficiently small, become sufficiently spherical owing to the great surface tension pressure. Details will be found in a paper by Twyman.[30]

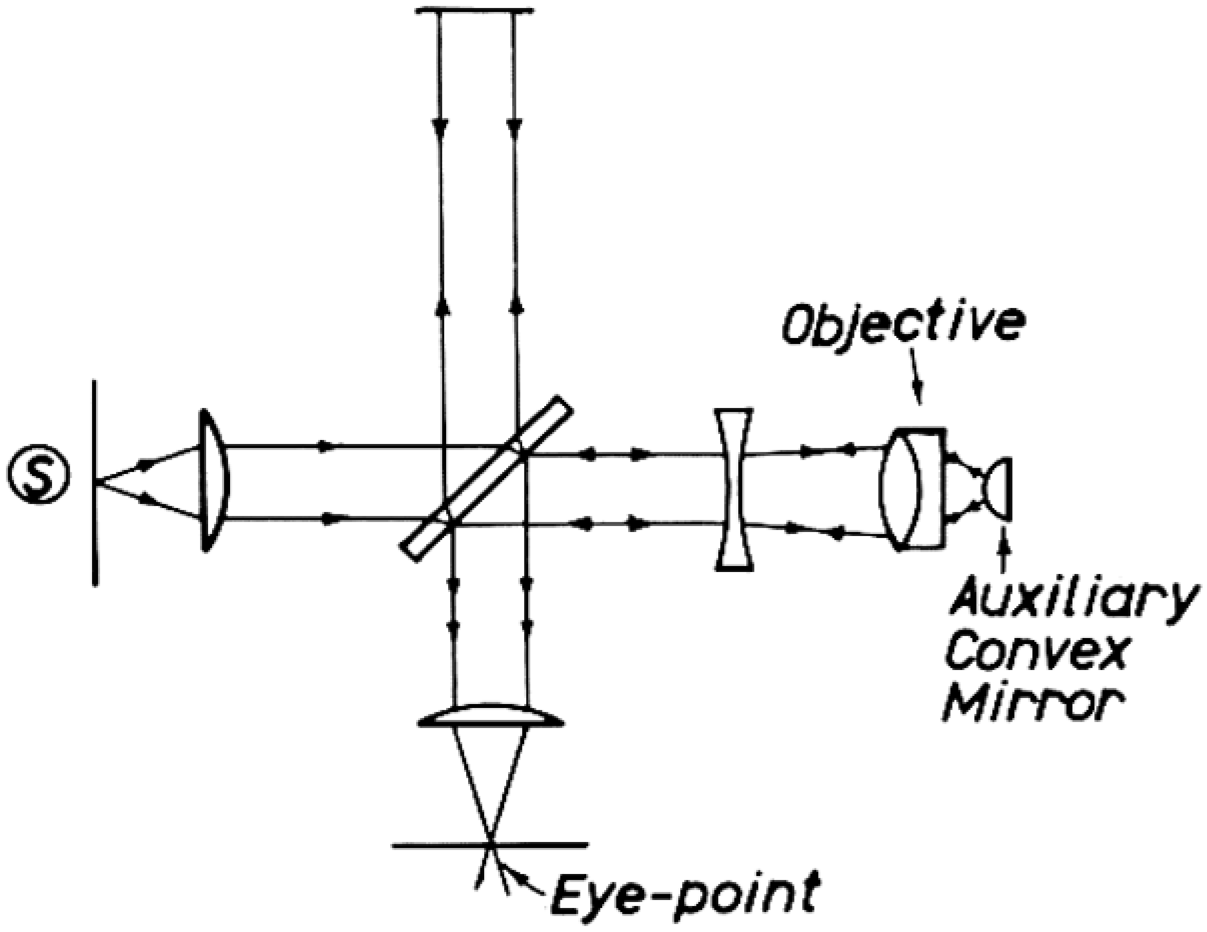

FIG. 242. MICROSCOPE-OBJECTIVE INTERFEROMETER

Visibility of Fringes in the Twyman-Green Interferometer. Hansen[31] has examined the question of the optimum brightness and contrast of the fringes in the Twyman-Green interferometer. In the Michelson arrangement in "normal" adjustment with unlimited source, the fringes are "rings" "at infinity" and are imaged in the focal plane of the lens L, Fig. 238. When the source is, in effect, severely restricted in angular size by the use of the pinhole (Fig. 239) only a very small part of the "ring" system will be formed, in the pinhole image. If the equivalent "air-plate" is very thin, the rings are broad and the pinhole may only cover a part of the centre ring. Now if one mirror is tilted slightly, straight fringes will appear parallel to the axis of tilt; these straight "Fizeau" fringes can be regarded as formed in the neighbourhood of the air-plate, and they will be seen in good contrast if the angular direction of the light is sufficiently restricted, say, by a pinhole at the principal focus of L (Michelson arrangement) but the circular fringes seen with a broad source disappear if the path difference between the mirrors varies by more than about $\lambda/4$ over the whole aperture; they will re-appear, however, if one of the mirrors is covered except for a narrow slit parallel to the axis of tilt. If the pinhole used is sufficiently small, and the straight slit is seen through it as the pinhole is moved about in the focal plane, the slit aperture will appear light or dark according to the particular part of the circular fringe system exposed; the pinhole must therefore not be too large or the contrast in the Fizeau fringes will suffer. When the equivalent air plate has a thickness Δ and the aperture at the focal plane subtends a semi-angle ϕ, there will be a relative difference of retardation as between the wave-fronts focused (*a*) in the centre of the aperture, and (*b*) in the edge of the aperture; this relative path difference will be

$$2\Delta - 2\Delta \cos \phi \approx \Delta\phi^2$$

If this relative difference does not amount to more than about $\lambda/4$, there will be no appreciable diminution in the contrast of the Fizeau fringes. If the diameter of the aperture is p, and the focal length of the collimator lens is f, $\phi \approx p/2f$, so that the thickness compatible with good contrast fringes is

$$\Delta = \frac{\lambda f^2}{p^2}; \text{ thus } p^2 \ngtr \frac{\lambda f^2}{\Delta}$$

The pinhole must not, however, be too small, as its finite aperture is required in order to form a *bright* image of the Fizeau fringe system referred to the equivalent air plate.

Effect of Plane Parallel Plate Placed in One Arm of the Interferometer. Consider the case when a plane parallel plate is placed, normal to the light, in one arm (A) of the (previously adjusted) apparatus; its refractive index is N and the thickness D. Let the plane mirror in the other arm (B) be withdrawn through a distance Δ normal to itself. The angle of incidence of a slightly oblique set of parallel rays being ϕ, the path difference G between the two beams, at this angle, in the two arms of the instrument is given by

$$\frac{G}{2} = D\{(N^2 - \sin^2\phi)^{\frac{1}{2}} - \cos\phi\} - \Delta\cos\phi$$

Expanding this in terms of ϕ

$$\frac{G}{2} = \{(N-1)D - \Delta\} + \left(\frac{N-1}{N}D + \Delta\right)\frac{\phi^2}{2} + \text{terms in } \phi^3, \text{ etc.}$$

Since ϕ will be very small with the limits set by a diaphragm of modest size we need not consider terms of order higher than the second in ϕ.

Now if $(N-1)D = \Delta$ the shift outward, Δ, of the mirror B exactly compensates the increase of optical path due to the plate; but there will be a path difference proportional to ϕ^2; thus a circular fringe system will appear in the focal plane of L even though the arms are exactly equal; on the other hand, if the shift of the same mirror is *inward* and equal to $-\{(N-1)/N\}D$, the term in ϕ^2 disappears and the optical path difference in the two arms is (though appreciable) independent of the value of ϕ. Hence no rings will be visible in the focal plane of L, and it follows that the visibility of the Fizeau fringes will not suffer if a wider pupil is used to gain more light.

The plane of the mirror B in the case when the optical paths in the two arms are exactly equal is called the "null plane"; its plane in the second case clearly coincides with the virtual image of the mirror formed by the plane-parallel plate (paraxial rays), and is called the "virtual mirror plane." The distance P between these two planes is

$$P = D\,(N^2 - 1)/N.$$

The requirement for maximum contrast with imperfectly homogeneous light is that the optical paths in the two arms should be exactly equal; on the other hand the condition for maximum contrast and brightness of the Fizeau fringes is that the mirror B shall coincide with the virtual mirror plane; these requirements are mutually incompatible unless the plate introduced into arm A is compensated by a similar plate in B.

Hansen's paper goes on to examine the incompatability arising in other cases of the use of the Twyman-Green interferometer. For example, when a convex mirror is used behind a lens under test to form an afocal combination, the virtual image of the vertex of the mirror, formed by the lens, is easily shown to lie at a distance

$$\frac{(f'-r)^2}{r} - h$$

from the mirror-vertex, where r is the radius of the mirror, f' is the focal length of the lens, and h is the distance between the principal planes of the lens. As before, high-contrast fringes demand optical path equality in the two arms; the need for bright fringes demands that the compensating mirror shall occupy a place corresponding to the virtual mirror plane. It is possible in this case to fulfil both conditions by a suitable choice of the radius of curvature of the mirror.

In this case, the optical path in the two arms will be equalized when the optical path in arm B corresponds exactly to that encountered in arm B up to the apex of the convex mirror; the equivalent air plate now extends between this point and the virtual mirror plane. If this distance is d we have (neglecting h above for a first approximation in the case of a test-lens of sufficiently long focal length)

$$d = \frac{(f'-r)^2}{r}$$

As before, the allowable diameter of the aperture used with the collimator lens of the interferometer is given by putting $d = \Delta$ in the equation in the middle of p. 324 above. We get

$$p \not> \frac{f'}{(f'-r)}\sqrt{(r\lambda)}$$

In a later paper[32] Hansen goes on to examine the conditions applicable in testing a number of different lens systems. He also examines the visibility of the Fizeau fringe system under certain conditions. This may be looked upon as dependent on the mutual coherence of the radiation reflected from the surfaces of the equivalent air film; a fuller discussion of the question has been given by Hopkins.[33]

Common Path Interferometers. The direct nature of the information, regarding the aberration of a system, yielded by an interferometer test is sufficient to justify the use of such expensive and delicate apparatus (including optical parts of the highest possible accuracy) as the various forms of the Twyman-Green instruments. However, it is sometimes possible to use simpler means to obtain a similar result although it has been found hitherto that they need a high degree of technical skill.

The Twyman-Green interferometer furnishes, in effect, a coherent un-aberrated wave to interfere with the wave showing aberration, and thus exhibit the relative phase variations in terms of the intensity of the interference pattern. The essential problem is therefore the production of the auxiliary coherent wave with the necessary curvature. In 1934 F. Zernike[34] made the suggestion that a coherent spherical wave can be generated by diffraction from a phase-changing disc of sufficiently small diameter if the latter is placed in the focus through which an aberrated wave is converging. Referring to Fig. 226 (*b*), if the aperture of the mirror is left unobstructed and the pinhole S is made exceedingly minute, the image at S′ corresponds to the Airy disc as modified by the imperfections of the wave. However, if the imperfections are small most of the energy of the wave is concentrated through an area smaller than that within the first dark "Airy" ring. In this place, instead of the knife edge, a phase retarding or accelerating disc, suitably mounted on plane parallel glass and having approximately the same diameter is now mounted; it should produce a phase change of about 90° in the transmitted light. The eye placed behind S′ now sees a phase contrast pattern revealing the wave-aberration by interference with the coherent spherical wave which can be regarded as derived from the diffraction by the disc. The argument follows lines closely similar to that used in discussing the phase-contrast microscope (see p. 135). The various techniques required in testing actual systems (e.g. astronomical mirrors) have been discussed by C. R. Burch,[35] and it is the usual experience that the experimental difficulties are formidable owing mainly to the extremely small size of the permissible source of light and the extreme necessary rigidity of the whole apparatus if the "Zernike disc" is to be kept still and stable in the mirror focus.

Scatter-fringe Systems. Less stringency regarding stability is required if the comparison wave is not derived from a single "Z disc" as above, but the light passes through a piece of glass with a suitably roughened surface which causes a little of the energy to be scattered around the beam. It was suggested in 1953 by J. M. Burch[36] that an optical system typified by the mirror M (Fig. 243) might be tested as follows. A very narrow beam of light, derived from a high intensity source screened by a small circular aperture, is projected by a small-aperture lens (not shown) which forms an image of the aperture at B′ on the mirror surface. The part of the mirror covered by the image is so small that it causes no aberration in the reflected beam. At the point A, close to the centre of curvature C of the mirror, the incident beam passes through the scattering surface, which produces some rays diverging to illuminate the whole mirror aperture. The scattered "wave" derived from any single element of the surface A is labelled S_D in the figure; after reflection from the mirror the scattered wave is turned into the convergent wave S_{CM} and passes through the conjugate A′. Here, however, the light encounters a replica A′ of the scattering surface, so mounted that its details coincide with their own images. The greater part of the convergent aberrated wave S_{CM} passes through it "unchanged" by the plate and becomes the divergent aberrated wave S'_{CM}. However, the direct light beam also passes through the replica scattering element and therefore produces a spherical divergent wave, S'_D. Owing to the narrow aperture of the incident and reflected beams there will be a high degree of coherence across their apertures, so that interference effects will still be visible even if the corresponding elements of the replicas A and A′ do not precisely coincide. However, if these elements (which are the

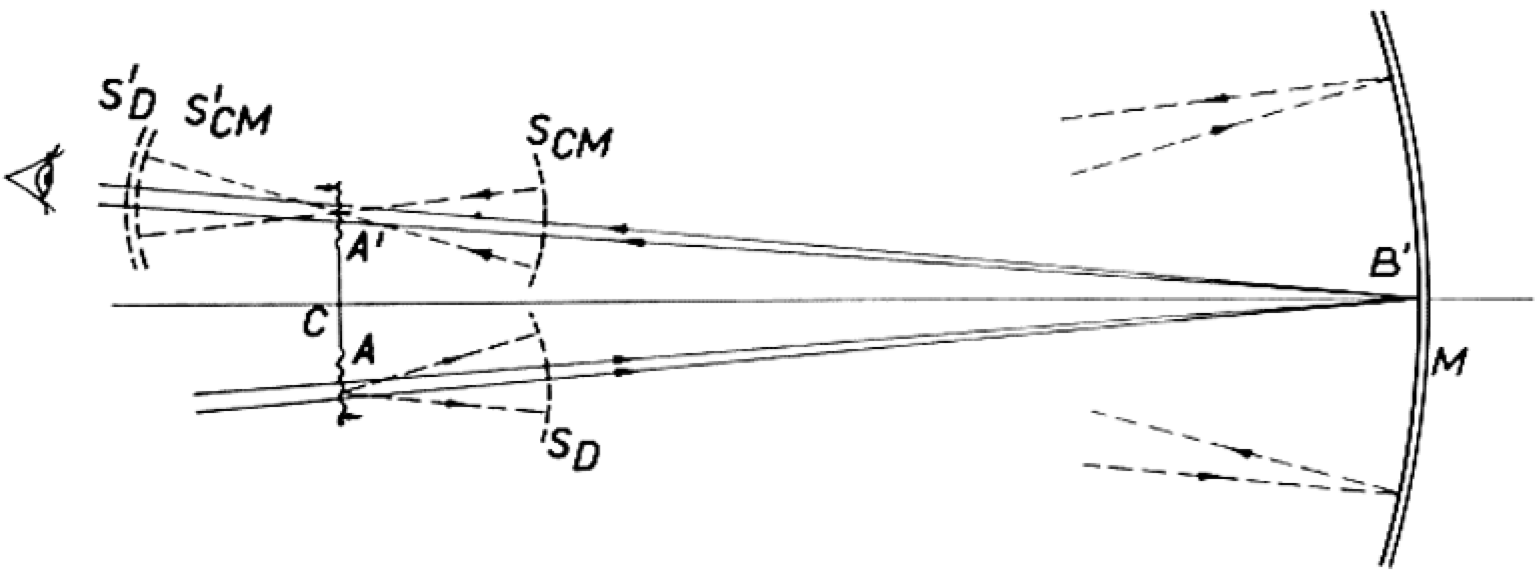

FIG. 243. SCATTER FRINGE TESTING
(J. M. Burch)

respective centres of S'_D and S'_{CM}) are displaced laterally the pattern will show straight fringes; if they are displaced longitudinally it will show circular fringes; it is only when they coincide that the pattern directly represents the aberration of the wave. Owing to the multiplicity of elements (of which the separate effects and their similar contributions to the fringe pattern) there is no lack of light in the field, so that one of the main disadvantages of the Zernike test is overcome, but the pattern is still sensitive to vibration. The eye behind A′ receives a sharp image of the mirror surface, and therefore sees an interference pattern referred to that surface. However, the comparatively bright image of the source

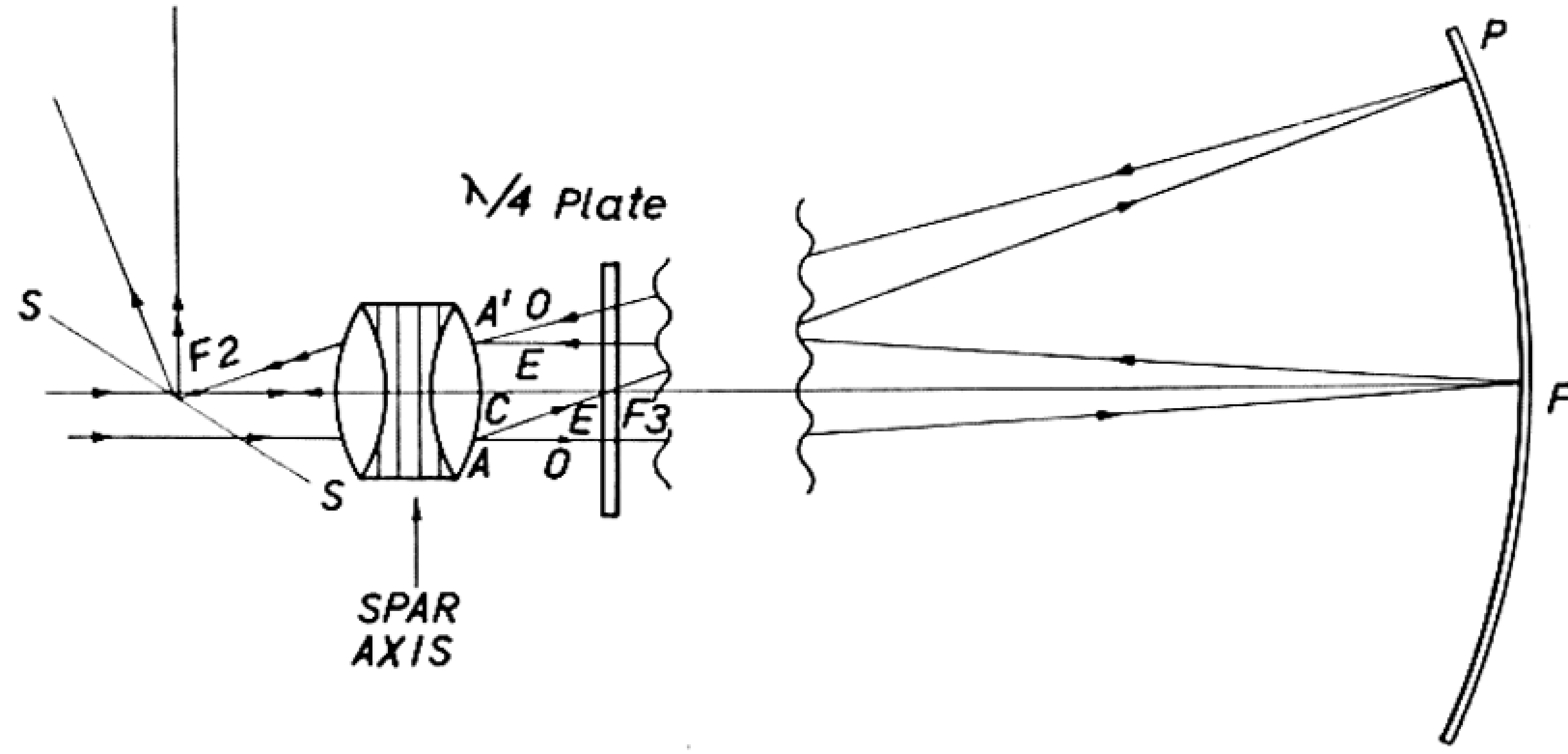

FIG. 244. PRINCIPLE OF DYSON'S COMMON PATH INTERFEROMETER (FIRST TYPE)

seen at B′ is prominent in the field and renders the observation of the fringe system somewhat difficult. (It should be possible, however, to add an auxiliary observation system in which this bright image is masked by a stop.)

Dyson's Common Path Interferometers. In an attempt to avoid some of the difficulties of the Scatter-fringe method, Dyson[37] proposed to use a system which again separates the light into two parts, one using a negligible aperture of the optical system under test, the other the full aperture so that it is subject to the aberration effects. The preservation of equality of optical path and the maintenance of full coherence have to be ensured. One system demonstrated by Dyson is shown schematically in Fig. 244. The separating system consists of a triple lens with outer components of glass and the inner negative lens of Iceland spar cut so that the crystal axis is perpendicular to the geometrical axis of symmetry.

Thus the lens has zero power for the ordinary ray and appreciable positive power for the extraordinary ray. (See also F. H. Smith, p. 151.) If a "parallel" bundle of rays, originally polarized so that the direction of vibration is at 45° to the crystal axis, passes through the lens the ordinary and extraordinary beams have the same intensities. The "parallel" bundle is proceeding to the test-system (in our case a mirror) at a comparatively great distance; it forms an image of the source on the mirror surface and then returns to the triple lens, meeting it again as a nearly parallel bundle. (The triple lens is at the centre of curvature of the mirror.) On the other hand, the original extraordinary beam is focused by the lens through F_3 and diverges to fill the full mirror aperture; thus the rays diverging from the lens at A are re-united by the mirror at A′ in the figure, but the rays typified by the O ray have suffered no aberration.

No interference would be possible on re-union unless the vibrations are parallel, but this can be secured by the use of the quarter-wave plate placed in the position shown; this changes the incident ordinary ray into a circularly polarized ray which proceeds to the mirror and returns to the plate again where the second transmission changes it into linearly polarized light vibrating at 90° with its original direction; thus it reaches the triplet lens again as the extraordinary beam for which the lens has a finite focal length and focuses the beam at F_2 in the figure; likewise the original extraordinary ray E reaches the triplet again as O for which the lens has zero power; it also passes through F_2.

The light diverging from F_2 has to be deflected for observation by an oblique partly-reflecting surface S, which may conveniently be a polarizing interference filter; at any rate an analyser is necessary to view the parallel interfering components of the final beams, and a suitable polarizing filter can fill the functions of both polarizer and analyser. The eye receiving the reflected light sees the Maxwellian image of the mirror aperture as far as it is exposed by the aperture of the triplet, and the field will contain fringes corresponding to the optical path characteristics of the mirror as in the Twyman-Green case. In case a transmitting system has to be tested it must be made auto-stigmatic by the addition of a suitable mirror.

The aberrations of the bi-refringent lens impose a restriction on the aperture of the system which can be tested. Dyson therefore proposes an alternative system for the beam-splitting device, for a description of which the original paper should be consulted.

Such systems as those just described are easy to use without special precautions. The fringes are very bright and do not need the use of a darkened room or elaborate precautions against vibration.

Shearing Interferometers. As an alternative to testing the errors of a wave-front by interference between one wave (derived perhaps from a system subject to aberration) and an auxiliary coherent wave free from aberration, it is possible to test a wave-front having revolution symmetry by producing a division of the wave-front by a partially reflecting surface (or etc.) and recombining the parts with a suitable lateral shear (or mutual displacement produced by rotation about a suitable axis). The apparatus must allow this to be done without significant changes of path length. Interferometers involving semi-reflecting surfaces have been described by Bates[38] and by Drew[39]; a system in which the shear is produced by a

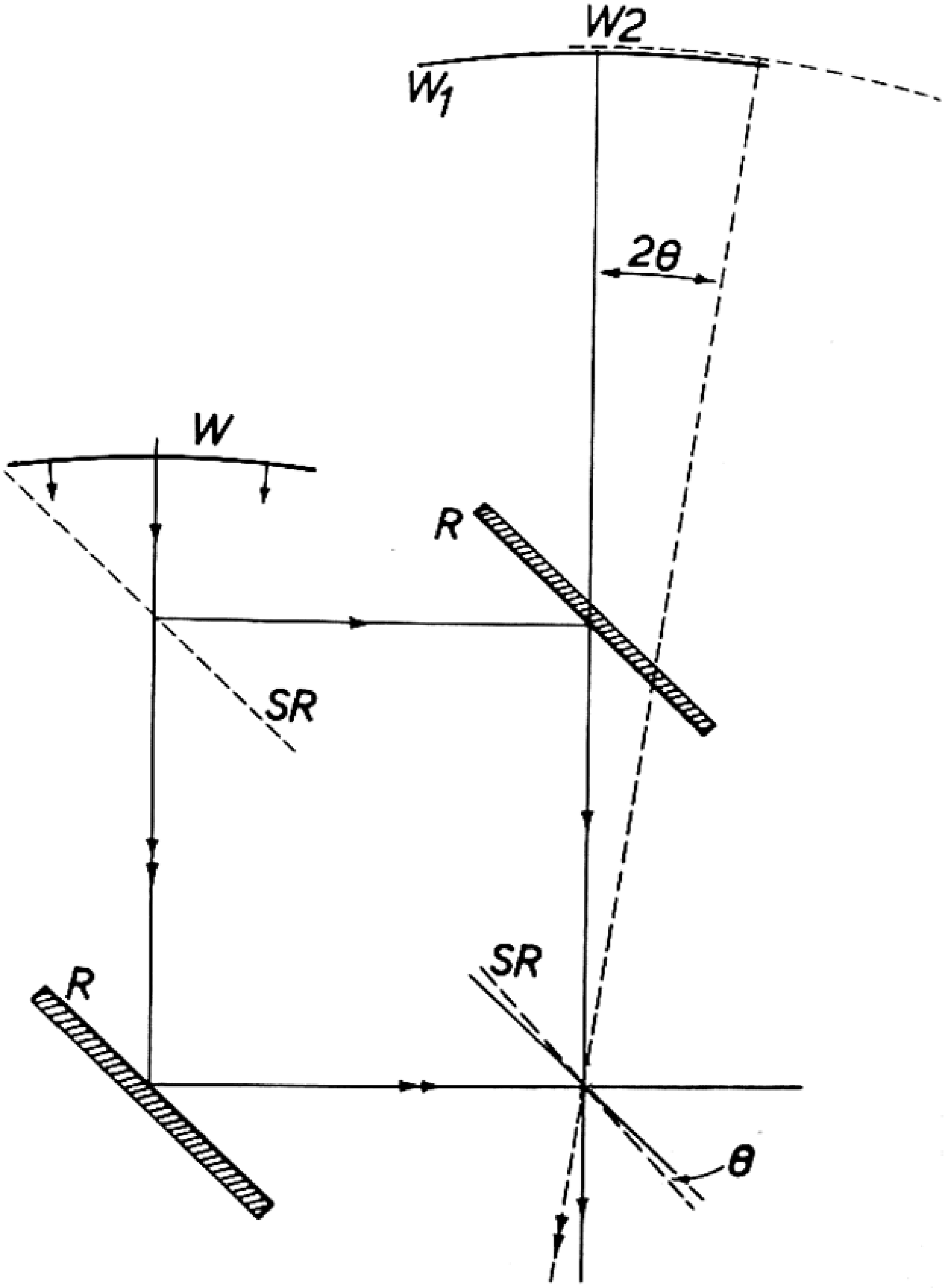

FIG. 245. PRINCIPLE OF SHEARING INTERFEROMETER (BATES)
R = reflecting surface;
SR = semi-reflecting and transmitting surface

polarization device (Savart plates) is described (as employed in a polarization microscope) by Françon, but which is capable of other uses; other shearing devices using polarization are described by F. H. Smith.[41]

Bates points out that even if the combined coherent waves have dissimilar asphericities, the resulting interference pattern can lead to a determination of the asphericity of both provided that *each has revolution symmetry*; it is easily seen, for example, that if the centre of one wave is located in the interference pattern, any circular locus around this point has a constant optical path for one wave only; hence the path differences for the other wave reckoned around this locus (determined by fringe-counting, preferably with the help of a suitable compensator) yields the optical path differences for an eccentric zone of the second wave, and this aberration can be further determined by step-by-step methods. Added information can be obtained if in addition to the shear, one wave-front can be rotated about its principal ray. The Bates interferometer (Fig. 245), consists essentially of two semi-reflecting surfaces (SR) and two mirrors R. The second reflector intersects the focus of the convergent wave W, and a tilt (θ) of this produces an angular displacement (2θ) between the two coherent images (W_1 and W_2) of the wave-front. Naturally the semi-reflectors have to be supported on glass plates, and the use of suitable (rotatable) compensating plates is required.

The mirror interferometer used by Drew avoids the requirement of mechanically controlled compensating plates; in this instrument a fairly thick glass plate (about 8·0 mm.) produces a division of the beam and its reconstitution in the manner shown in Fig. 246. Fully silvered mirrors are placed at B and C, the former coinciding with the focus I_2 of the wave after one reflection; thus rotation of the mirror B does not cause a change of path-length along any ray-route through the plate. When B and C are symmetrically placed, and parallel to the plate, the images of the wave-front coincide; a shear is introduced by the rotation of B. Lateral separation of the images (producing vertical fringes) is produced if (say) the mirror C is made to change its distance from the plate, so that the image formed by reflection at C and D is separated from the position I. In order to obtain horizontal fringes, Drew rotates the main plate and the mirror C about horizontal axes in such a way that the symmetrical relative dispositions of B and C with regard to the plate are preserved but there is a vertical separation of the two images at I. Such rotations need only be very small, and the resulting failure of exact compensation is discussed in the paper.

The shearing interferometer described by Françon[40] makes use of the lateral separation between ordinary and extraordinary rays which may be produced by transmission through a plate of double-refracting material. In a pair of "Savart plates," Fig. 247, the crystal optic axes are inclined at 45° to the end faces, and they lie in mutually perpendicular principal sections. Thus the extraordinary ray in the first plate becomes the extraordinary ray in the second; on emergence both component rays have suffered a displacement (though in perpendicular directions); moreover, provided the plates are properly cut and are equally thick, both rays have finally suffered the same retardation. Such a pair of plates can therefore be used as a shearing interferometer to produce a (fixed) lateral shear between the separated parts of an image. A polarizer controls the amount of light originally directed into the two beams, and the final analyser brings the transmitted components into parallel vibration directions so that interference can take place.

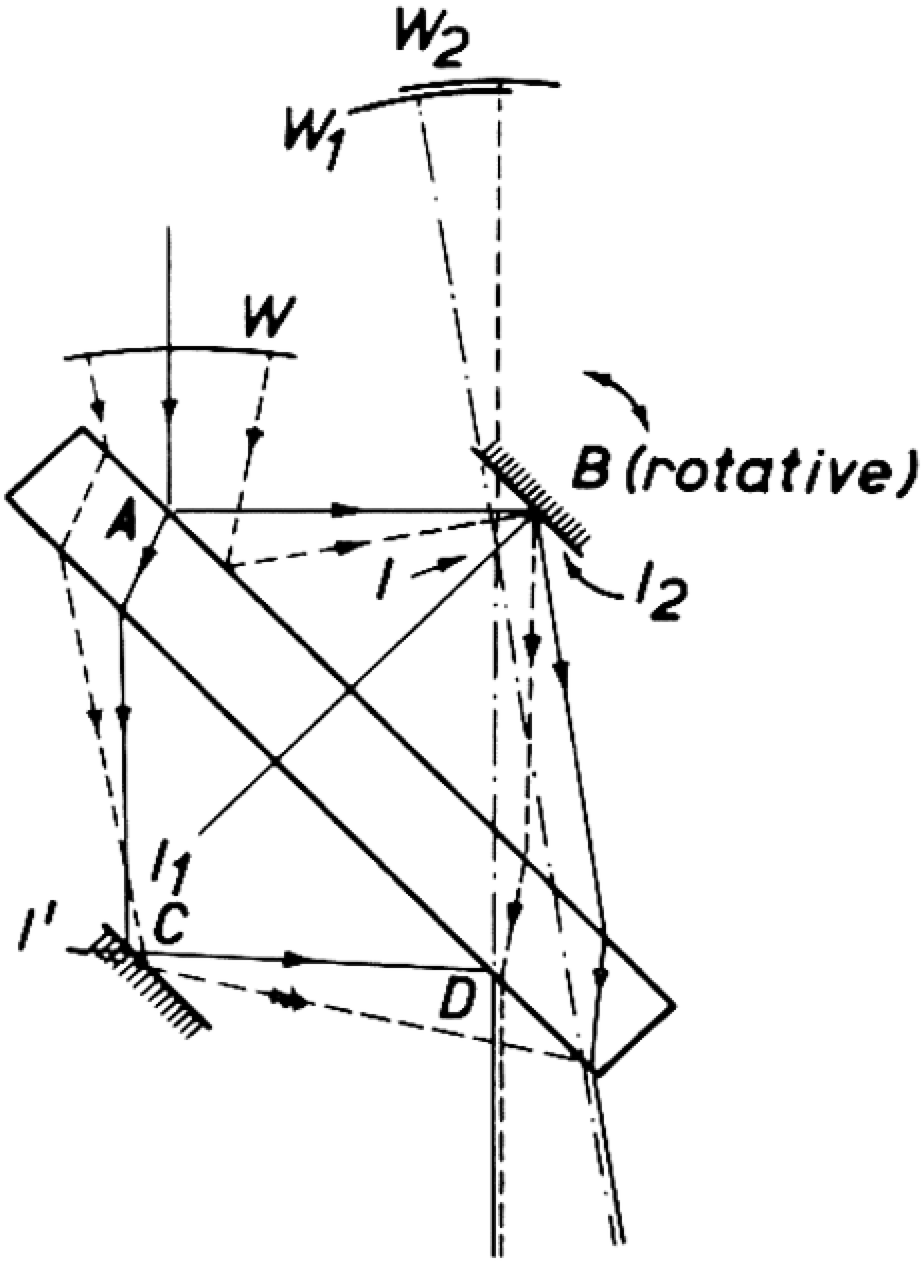

FIG. 246. DREW'S SHEARING INTERFEROMETER

The actual use of this device has so far been confined to the interference microscope. In this case white light can be employed, and the path variations are exhibited as differences of colour. A very convenient feature of polarization instruments is the way in which mutual path differences can be delicately controlled by a suitable form of compensator (Babinet or other type, Vol. I, p. 211). Small path differences are effectively registered as colour changes by the use of a sensitive-tint plate (Vol. I, p. 260). The shearing interferometer due to F. H. Smith is shown as used in a microscope in Fig. 109.

Interpretation of Pattern (*Shearing Interferometer*). The "height error h" in a surface having rotational symmetry is expressible as a function of the radius.

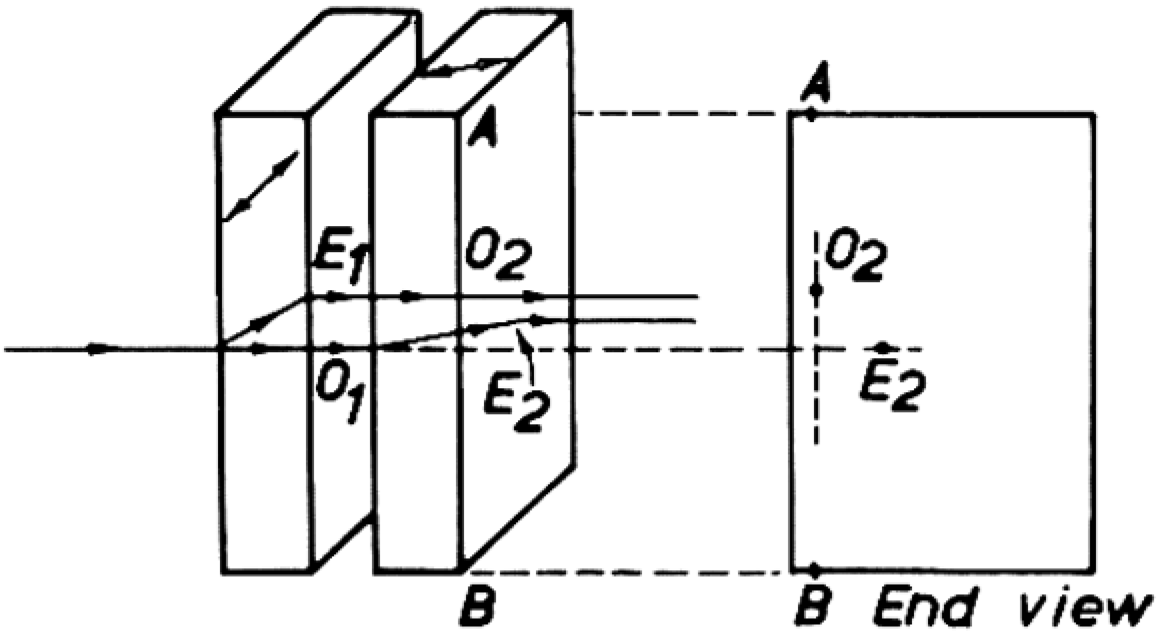

FIG. 247. SHEARING ACTION OF SAVART PLATES OF POSITIVE UNIAXIAL CRYSTAL

(Deviations much exaggerated)

$$h = f(r)$$

If the shear is $\triangle r$ the height error at r for one component meets the height error at $r + \triangle r$ for the other. Hence by Taylor's theorem

$$f(r + \triangle r) = f(r) + \triangle r \frac{dh}{dr} + \frac{(\triangle r)^2}{1.2} \frac{d^2h}{dr^2} +, \text{etc.}$$

The interference pattern can be made to show the difference of the height errors; then if $(\triangle r)^2$ is negligible in comparison with $\triangle r$

$$h_{r+\triangle r} - h_r \approx \triangle r \frac{dh}{dr}$$

That is, if the shear is small and the rate of change of dh/dr is not large. Now the interference field can be arranged so that when the horizontal shear is zero there is only a vertical shear giving perhaps eleven straight horizontal fringes across the field, corresponding, say, to a range of positive and negative retardations of 5λ, respectively (not forgetting the central fringe). When the horizontal shear is introduced, the fringe pattern is antisymmetrical about the centre point of the interference aperture. We may take it that if the wave-front has revolution symmetry, dh/dr is zero when r is zero, and $h_{r+\triangle r} - h_r$ is also zero. But a given fringe corresponds always to the same retardation. If the difference grows, the centre fringe "climbs" to the point where the vertical displacement tells us the retardation is so many wave-lengths ($x\lambda$) on the scale of the original pattern. Hence the "slope" of the fringes is given by

$$\frac{dh}{dr} = \frac{h_{r+\triangle r} - h_r}{\triangle r} = \frac{x\lambda}{\text{shear}}$$

and

$$h_r - h_0 = \int_0^r \left(\frac{x\lambda}{\triangle r}\right) dr$$

"Height" differences are therefore obtained by an integration of the area under a curve showing the number of fringes per unit distance in the radial direction. If both waves were spherical, and one wave differed only in radius from the other, the gap between them when both are coaxial would be closely proportional to h. Hence if we take a reference line at an angle to the shear direction for the fringe-spacing count it is equivalent to referring the aberration to a new focus. The process is equivalent to the similar integrations which are required in other methods of testing. (See Platzeck and Gaviola[25].)

It should be added that some of these shearing interferometers can be effectively used in an auto-collimating system, where the system under test forms a "parallel" beam returned through it by a plane mirror; the light can traverse the interferometer on both the outward and return paths (though naturally a semi-reflecting mirror is necessary to intercept a part of the return beam so that the fringes may be viewed). Both the Bates and Drew interferometers require a very narrow slit source if the illuminating beam is to be sufficiently coherent over a wide aperture in the ordinary usage; however, if used with auto-collimation the extreme restrictions on source-size disappear, as in the Twyman-Green interferometer.

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CHAPTER VIII

ASPHERIC SURFACES OF REVOLUTION

In spite of the great difficulties hitherto associated with the production of axially symmetrical aspheric surfaces to optical accuracy, their use is extending, and it is desirable to give a short account of their possibilities, main present uses, and methods of production.

The Elimination of Spherical Aberration. A plane wave AB is incident on a refracting surface AP, Fig. 248 (*a*) separating media of refractive indices n and n' respectively. Let A be the origin of co-ordinates and let AF′, the normal to the wave, be the axis of x. Let x, y be the co-ordinates of P, then if BP is also normal to the wave the condition that all such paths as BPF′ may be constant is (writing AF′ as l')

$$n'l' = nx + n'\{y^2 + (l' - x)^2\}^{\frac{1}{2}}$$

On re-arrangement, this equation takes the form

$$y^2 + x^2\{1 - (n/n')^2\} - 2\{1 - (n/n')\}l'x = 0,$$

which may be compared with the usual equation of an ellipse referred to its vertex, i.e.

$$y^2 + x^2\,(b^2/a^2) - 2\,(b^2/a)x = 0.$$

The trace of the surface is an ellipse (provided that $n' > n$) with eccentricity e, where, in the case which interests us here,

$$e = \{1 - (b^2/a^2)\}^{\frac{1}{2}} = n/n'$$

and the semi-major axis is $l'n'/(n' + n)$.

We could clearly make a lens free from spherical aberration by using a second refracting surface with centre F′, (Fig. 248 (*b*)).

If $n' < n$, the equation represents an hyperbola, the equation being referred to the vertex of the left-hand branch, i.e.

$$y^2 - x^2(b^2/a^2) + 2(b^2/a)x = 0$$

The refracting curve, Fig. 249 (*a*), now lies on the left-hand side of the y axis, and the optical focus is easily shown to be the geometrical focus of the right-hand branch of the curve.

A lens free from spherical aberration can be obtained by adding a plane front surface which would not introduce any curvature into the incident plane wave.

When both conjugate distances are finite, a similar investigation shows that the necessary section of the refracting surface will be a curve of the fourth degree; it is known as a cartesian oval. The theory was first worked out by Descartes.

It may be noted that lenses of the kinds mentioned above can be used in symmetrical combinations (Fig. 250) which will be free from spherical aberration, coma, distortion, and lateral chromatic aberration, though not from axial colour, field curvature, or oblique astigmatism. The hemi-symmetrical principle could be employed for systems to work at unequal conjugates.

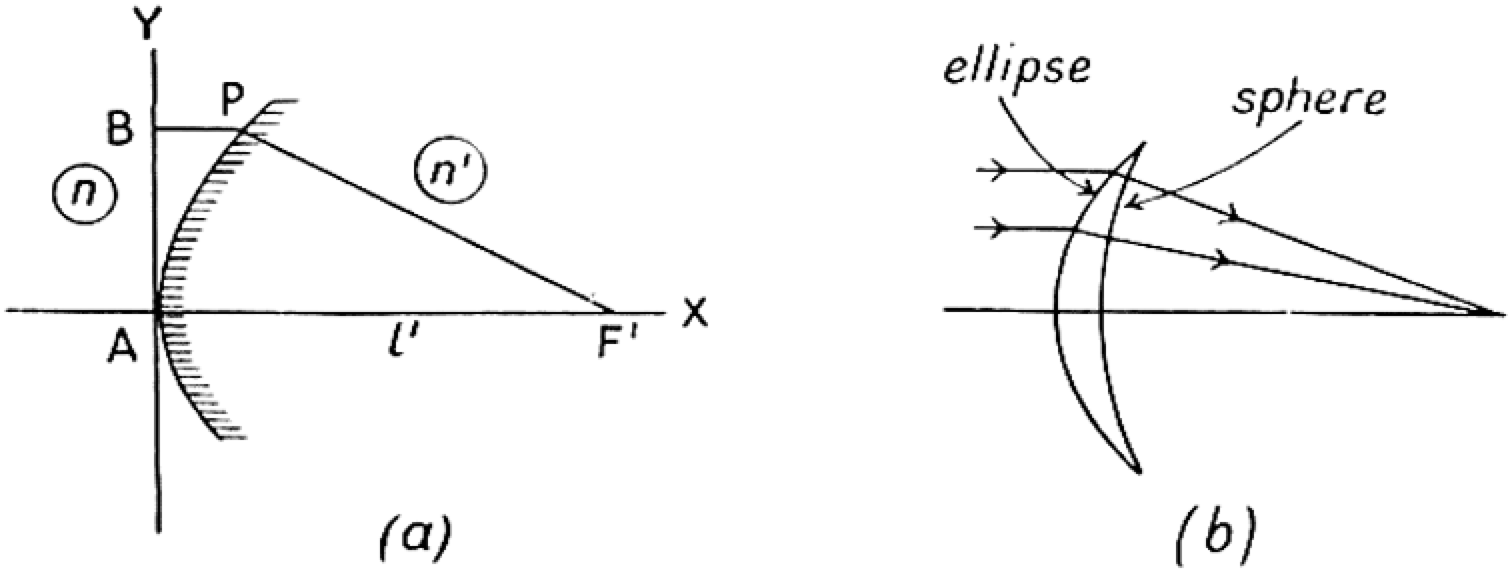


FIG. 248

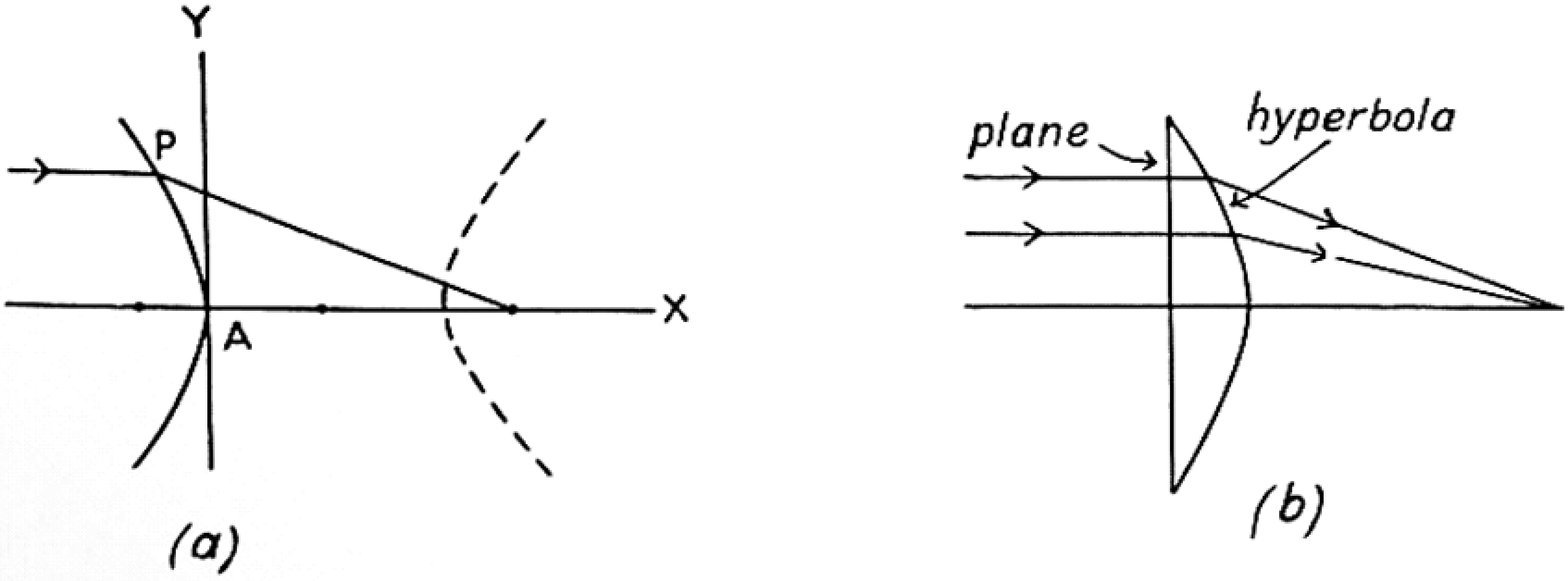


FIG. 249

Huygens[1] gives the following construction for finding the form of a final surface to eliminate the spherical aberration of a system. Having given the relative positions and shapes of the first $k - 1$ surfaces, and the position A_k of the pole of the final surface together with its paraxial radius, a number of rays are traced through the $k - 1$ surfaces from the axial object point; one of these emerges from the point P_{k-1}, Fig. 251, and cuts the axis in the point B'_{k-1}. We can of course find also the final paraxial image point B'_k by tracing the paraxial ray through the whole system. We now compute the optical path, O_m, say, from the object point to the point P_{k-1}, and the path, O_p, from the object point to B'_k along the paraxial route; finally computing the value of $(O_p - O_m)/n_k$

where n_k is the medium of the last lens. This length is marked off from P_{k-1} on the path $P_{k-1}B'_{k-1}$, thus giving the point B. Now drop a perpendicular B'_kF from B'_k on the line $P_{k-1}B'_{k-1}$; measure its length and calculate the value $(B'_kF)\, n'_k/n_k$ which is used as the radius for an arc struck with centre B as shown. A tangent is now drawn from B'_k to this arc, cutting the line $P_{k-1}B'_{k-1}$ in C, which is a point on the surface required. Sufficient points to construct the whole surface may thus be found. It will be seen that $n_k \cdot CB = n'_k \cdot CB'_k$, so that the optical path for the marginal route via C is now equal to the paraxial path.

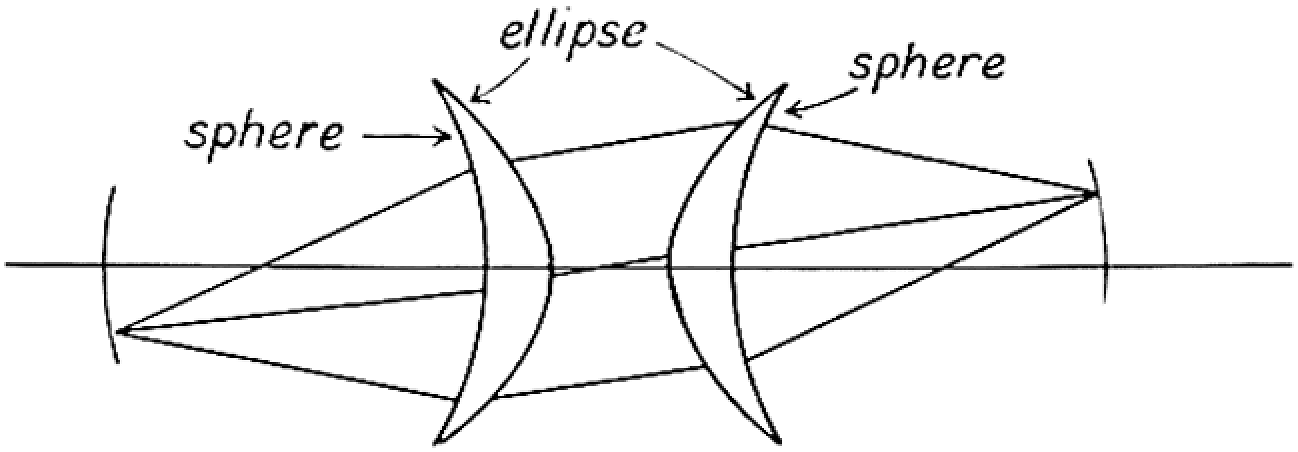


FIG. 250. SYMMETRICAL APLANATIC SYSTEM

It is frequently possible by graphical means, using "optical path" theory, to obtain a rough approximation to the required shapes of systems. Thus Huygens found the shape BK, Fig. 252, for the required second refracting surface of a lens having given the refractive index (3/2), the conjugate points L and F, the shape AK of the first spherical refracting surface and the thickness AB. Usually, however, the graphical method is not sufficiently accurate, and exact calculation is required. However, in aplanatic and other simple systems the conditions of optical path equality, together with the geometrical statement of the sine condition (the incident and final

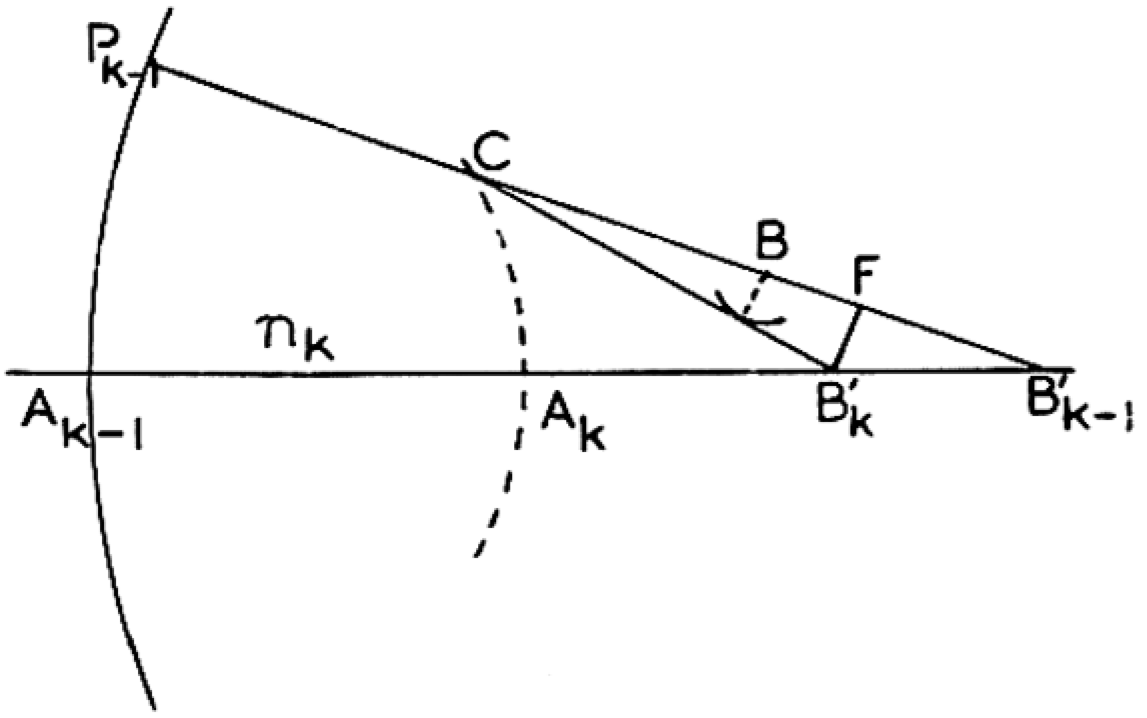


FIG. 251. HUYGENS' CONSTRUCTION

ray directions intersecting in a circle), or the tangent condition for freedom from distortion, provide the basis for the calculation.

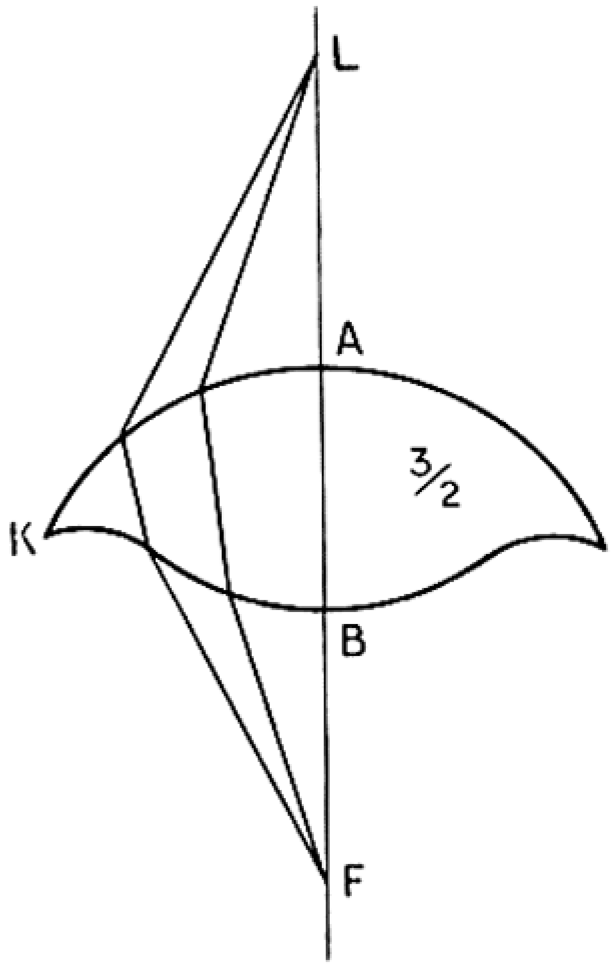
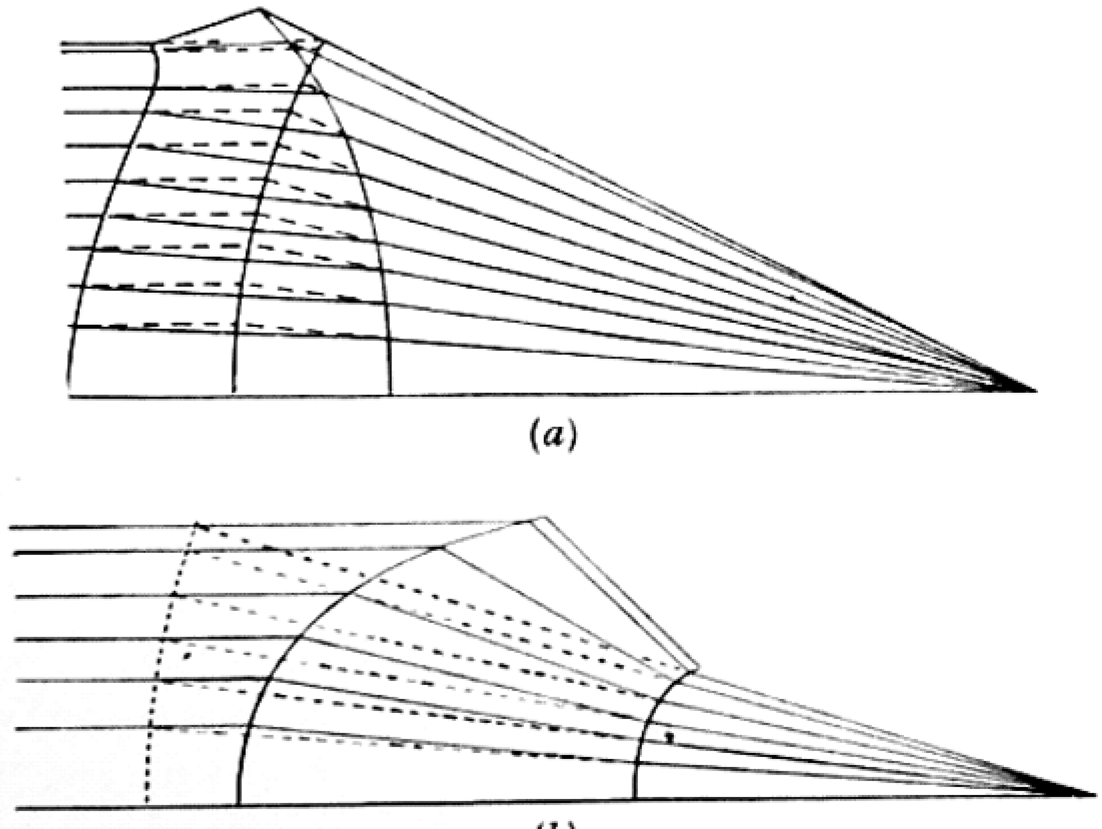


FIG. 252. HUYGENS' LENS

Schwarzschild,[2] and (independently) Chrétien[3] worked out the curves for an aplanatic two-surface reflecting system. In Chapter II, p. 85, it was shown that if the main mirror of a Cassegrain reflector is paraboloidal, the second one must be hyperbolic to give spherical correction; but the system will not be aplanatic. In order to attain this both mirrors must have another figure. Reflectors of this kind have been constructed by Ritchey. Schwarzschild's proposed systems would employ a concave second mirror, but they have been considered less convenient.

M. Linneman,[4] C. R. Burch,[5] and the present author[6] have calculated at various times the data for aspheric aplanatic single lenses. Fig. 253 (*a*) and (*b*) show lenses of this kind. As has been shown by Burch it is possible to calculate singlet lenses which are anastigmatic and aplanatic, though of course they suffer from axial chromatic aberration. Lateral chromatism can sometimes be corrected. The aspheric lens of Fig. 253 (*a*) will be seen to resemble



(*a*)

(*b*)

FIG. 253. APLANATIC LENSES

that of Huygens; though both of its surfaces are aspheric the second one does not depart so greatly from the spherical form.

Reflecting Microscopes. Mirror telescope systems modified for one relatively long and one short conjugate can obviously be used as microscope objectives if constructed on a sufficiently small scale, but the production of an aplanatic system corrected for spherical aberration and coma requires, in general, the use of aspheric surfaces, except in cases where very low numerical apertures ($< 0{\cdot}18$) are required. The great difficulties in the production of such systems were successfully overcome by C. R. Burch,[7] who produced two-mirror objectives having *NA* 0·65 and focal length 3·0 mm. They have a performance equal to refracting systems of the same numerical aperture. The wavelength for illumination can be chosen at will, ultra-violet, or visible, or infra-red, so that the maximum contrast can be obtained, through differential absorption, with any particular objects. A long working distance, of 13 millimetres, allows of the use of the high *NA* with objects which could not be investigated with an ordinary objective, i.e. hot objects, etc.

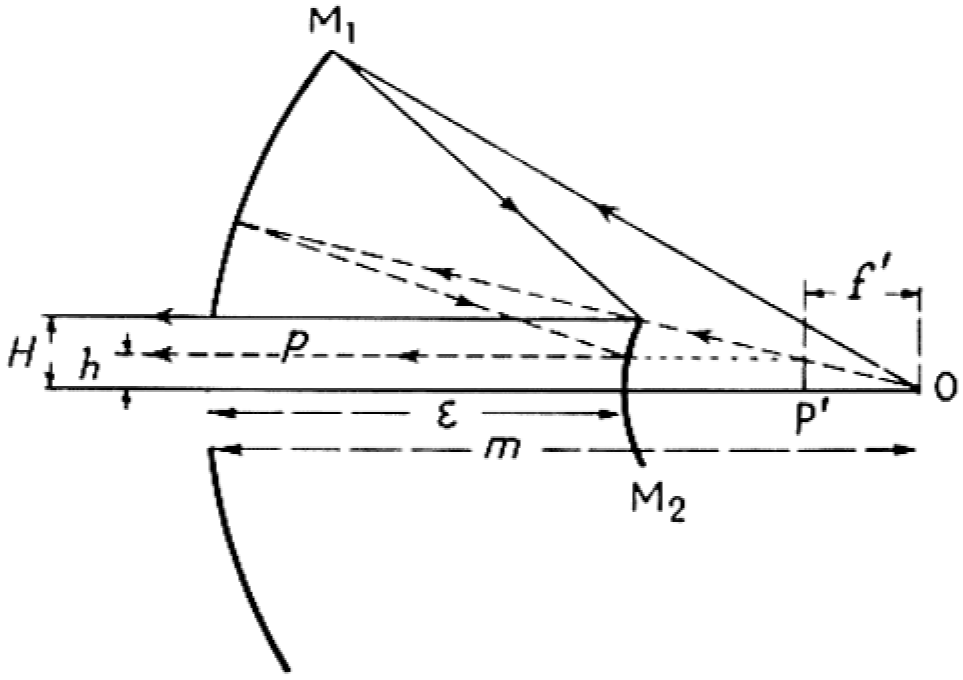


FIG. 254. REFLECTING MICROSCOPE OBJECTIVE

The addition of a *hemispherical* front lens used with homogeneous immersion allows the *NA* to be increased while still retaining the advantages of choice of wavelength.

Burch showed that, using a monocentric system, so that both mirrors have the same centre of curvature, an aplanatic system can be produced with spherical mirrors, though at the price of a high "obstruction ratio." Fig. 254 shows a reflecting microscope system, the main mirror M_1 receiving the divergent light from the axial object O while the second mirror M_2 renders the rays parallel to the axis. The separation of the mirrors is ε, while the distance of O from the pole of the main mirror is m. The height of the edge of

M_2 from the axis being H, it is seen to obstruct rays from O so that the parallel ray (marked p) nearest the axis is at a height h. Now imagine all the rays reversed and p produced to intersection with its final path which it intersects in a plane at a distance from the principal focus O of this system equal to f' the focal length (see Vol. I, p. 42). From similar triangles in the figure we now see that

$$\frac{h}{\mathrm{H}} = \text{the obstruction ratio} = \frac{f'}{m-\varepsilon}$$

This result must be slightly modified if the final rays from M_2 are convergent, but it is a useful approximation. In the Burch microscope $h/\mathrm{H} = 0{\cdot}14$.

Seeds and Wilkins,[8] using an experimental approach, showed that systems of practical utility though with generally inferior field corrections and higher obstruction ratio could be produced by using spherical mirrors and departing from the aplanatic disposition. This work has led to the production of commercial reflecting objectives (Messrs. R. and J. Beck, Ltd.).

The effect of the central obscuration is to produce a slightly smaller Airy disc with more light in the surrounding diffraction rings. Barer,[9] in reviewing the modern situation, is of the opinion that obstruction ratios exceeding 0·1 produce noticeable deterioration of contrast. Reflecting objectives, however, facilitate other experimental techniques such as microspectrophotometry. Bouwers[10] has used Meniscus Cassegrain and other systems, and Grey[11] has designed other forms of catadioptric objective. The achromatized Mangin Mirror systems by Johnson[12] obviate the difficulty of central obstruction by the use of a suitable beam-splitter.

The development of such systems has taken place so quickly within recent times that their final place in microscopic technique can hardly be assessed at present.

The Schmidt Camera. The use of a parabolic form for the mirror of a reflecting telescope entails the appearance of coma in the extra-axial parts of the image as well as astigmatism. However, if a diaphragm is placed at the centre of curvature of a spherical surface, the principal ray is normal at all points, and both coma and astigmatism are eliminated, the field being still round, as already seen. It was suggested by Schmidt[13] in 1932 that a figured plate might be placed in the plane of the diaphragm so as to control the spherical aberration associated with the spherical mirror, while largely retaining its other advantages.

The elementary theory of the arrangement may be discussed with the aid of Fermat's principle. A wave-front WMP, Fig. 255, is proceeding towards a spherical concave mirror R; S is a plate which may first of all be considered absent; P and M are points on the incident paths of paraxial and marginal disturbances. Now we know the ray M will (in the absence of the plate) exhibit under-correction; it will cross the axis short of the paraxial focus F. However, when the plate is introduced the glass may be given a prismatic action which will deviate this ray outwards so that it will, after reflection, pass through F.

The form of the plate can conveniently be discussed with the aid of the principles of equal optical paths. The sum of the straight intercepts "M to Q" and "Q to F" is less than the paraxial track PL + LF by an amount which we can calculate from equation (411 *A*) of Vol. I, p. 110. We put, as usual for the case of reflection, $n' = -1$ and $n = 1$; also $l' = r/2$, $l = -\infty$. Then we find for p_m the "marginal *minus* axial path difference,"

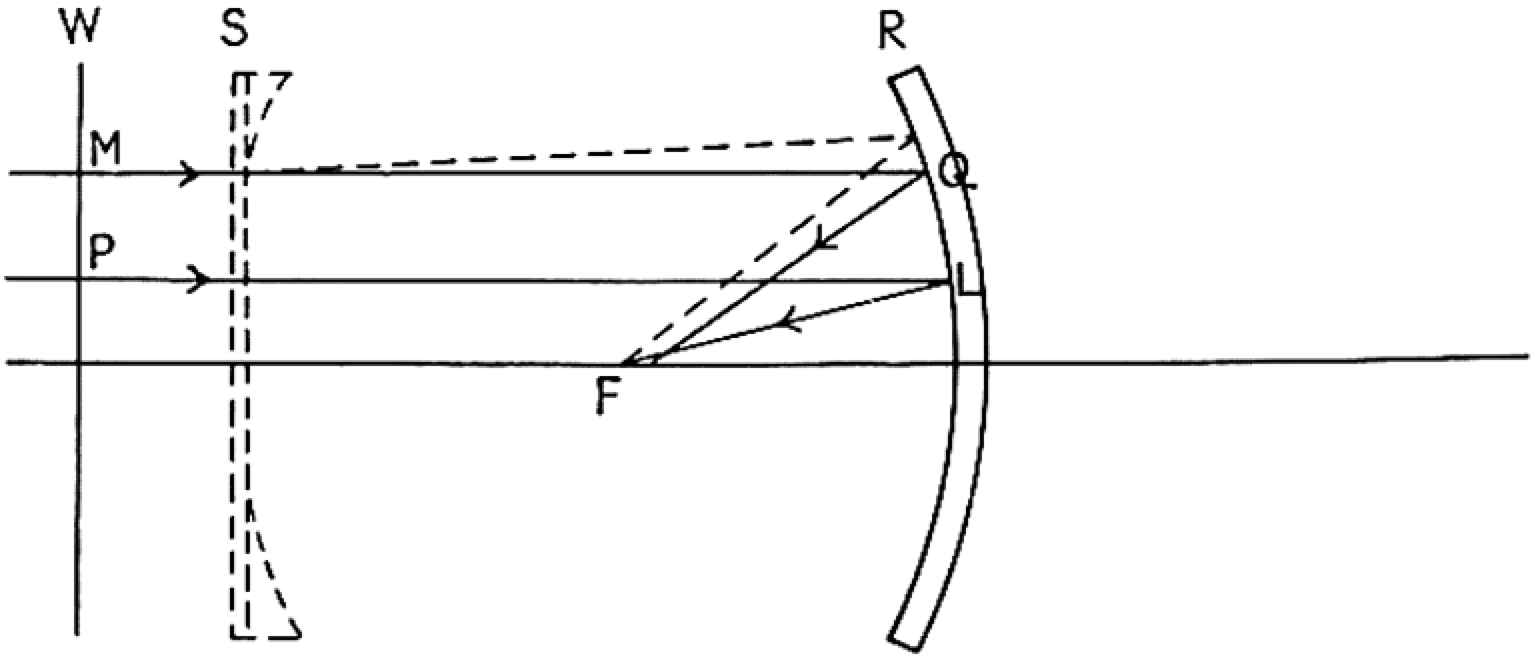


FIG. 255. THEORY OF SCHMIDT PLATE

$$p_m = \frac{y^4}{4r^3} - \frac{y^6}{8r^5} + \text{etc.}$$

Since r will be numerically negative, we have p_m negative (under-correction) in the cases of interest to us. Remembering Fermat's theorem, we may argue that the above geometrical path MQ + QF will still lie close to the actual ray path even after the introduction of the plate; thus the above expression is still (even when the plate is present) equal to that part of the optical path difference *not* due to the plate; we may, therefore, add the path difference p_p due to the plate (which we will give a thickness d_y at height y). Then

$$p_m + p_p = (n - 1)d_y + \frac{y^4}{4r^3}$$

is an expression for the total path difference valid within the realm of the primary aberration theory. The difference will be zero if we make

$$d_y = -\frac{1}{(n-1)}\frac{y^4}{4r^3}$$

If f is the numerical value of the focal length of the mirror, taken to be intrinsically positive, we can write this

$$d_y = \frac{1}{(n-1)}\frac{y^4}{32f^3}$$

The form of plate indicated would have a thickness which, starting from an axial value of zero, increases proportionally to the fourth power of the zonal radius. We might "add" a plane plate of constant thickness and so make a system possible to construct in practice, but the marginal deviation would be large, and would thus involve a considerable amount of chromatic aberration.

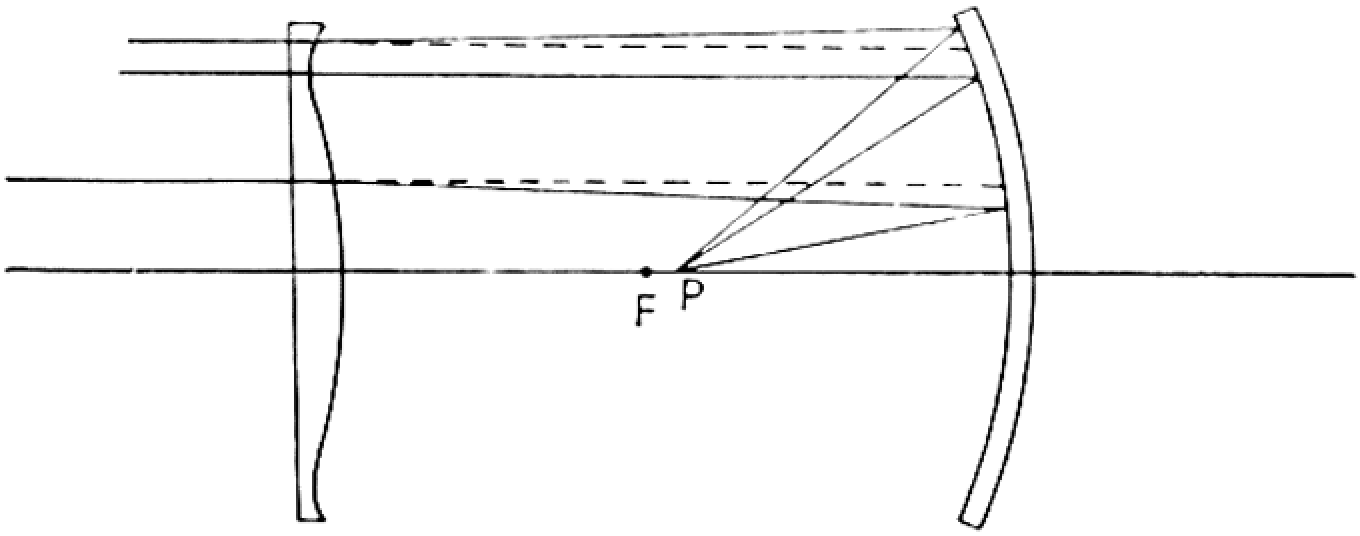


FIG. 256. ORDINARY FORM OF SCHMIDT PLATE
(Exaggerated)

It is better to "add" to the above type of plate a very thin plano-convex lens so as to make a plate of the general shape shown in Fig. 256. Such a lens will obviously shorten the focusing distance of the paraxial rays, and the plate will have a neutral zone towards the margin, so that a ray incident parallel to the axis will be undeviated.

Reverting for a moment to the case of the uncorrected mirror, the greater amount of the aberration for mirrors of modest aperture will be of the primary character, and the distance from the paraxial focus to the marginal focus will thus be $y_m^2/8f$, (see Vol. I, p. 115) and the disc of least confusion is distant from the paraxial focus by $\frac{3}{4}$ of this amount, i.e. $3y_m^2/32f$. If we bring all rays to this focus with our plate, the maximum necessary angular deviation for any of the rays will be a minimum and therefore the chromatic aberration will be a minimum also. It will be necessary to consider the compensation of optical paths at this new point.

On travelling the last-mentioned distance towards the mirror from the paraxial focus the *relative* increase in a non-axial path as compared with the *axial* is (Vol. I, p. 106)

$$\frac{1}{2}\left(\frac{3y_m^2}{32f}\right)\left(\frac{y}{f}\right)^2$$

and this expression was shown (Vol. I, p. 106) to be accurate within the required limits (the terms in y^4, etc., being usually negligible in cases of practical interest); thus the total optical path at this focus for the system of mirror and plate is,

(Marginal — axial)

$$= (n-1)\,d_y + y^2\left\{\frac{(3/2)y_m^2}{32f^3}\right\} - y^4\left(\frac{1}{32f^3}\right);$$

and the path difference will be zero if

$$d_y = \frac{1}{32\,(n-1)\,f^3}\,(y^4 - \tfrac{3}{2}\,y_m^2 y^2).$$

The actual plate will have a thickness found by adding a suitable constant amount to the above, and thus acquiring the shape suggested in Fig. 256. The radius of the neutral zone is $y_m\sqrt{(3/4)}$.

This plate will have a marginal thickness somewhat less than the axial; it is possible to design plates which have the supplementary imaginary lens of somewhat lower power; the neutral zone falls closer to the axis, and the marginal thickness is about equal to the axial. Such plates offer the advantage of requiring less "grinding out" if they are made by starting with a plane parallel plate of glass, and the chromatic aberration is not greatly increased. The foregoing equations will suffice for an approximate roughing out of a plate if the mirror to be corrected has a modest aperture—say $f/4$, but if larger apertures are to be employed, of the order of $f/1{\cdot}5$ perhaps, a more exact trigonometrical design of the plate will be undertaken.

Higher Aberrations of Schmidt Camera. The introduction of a suitably computed and figured plate of the type of Fig. 256 can thus cure the spherical aberration of all orders; we may, however, briefly consider the off-axis aberrations. For this purpose it is necessary to consider the deviation of a ray by a prism of very small angle, and its dependence on the obliquity of the ray.

A ray lying in the principal plane of a prism of refractive index n and of *very* small angle α, and incident at a small angle ϕ on the first face is deviated by the angle δ, where

$$\delta = (n-1)\alpha\left\{1 + \frac{n+1}{2n}\phi^2 + \text{etc.}\right\}$$

There is no component of the deviation which is proportional to ϕ and therefore the plate will give no primary coma. Referring to Fig. 257, let us consider the deviation of the top and bottom rays incident at an angle ϕ; the deviations are given by the above formula; it is evident that the distance of these rays from the central ray is $y_m \cos\phi$ (measured perpendicular to the rays) and the appropriate deviation *required by the mirror* is therefore smaller. The angle of the plate is $\delta d_y/\delta y$, i.e.

$$\alpha = \frac{1}{32(n-1)f^3}(4y^3 - 3y_m^2 y)$$

and the maximum angle, when $y = y_m$, is

$$\alpha_{max} = \frac{y_m^3}{32(n-1)f^3},$$

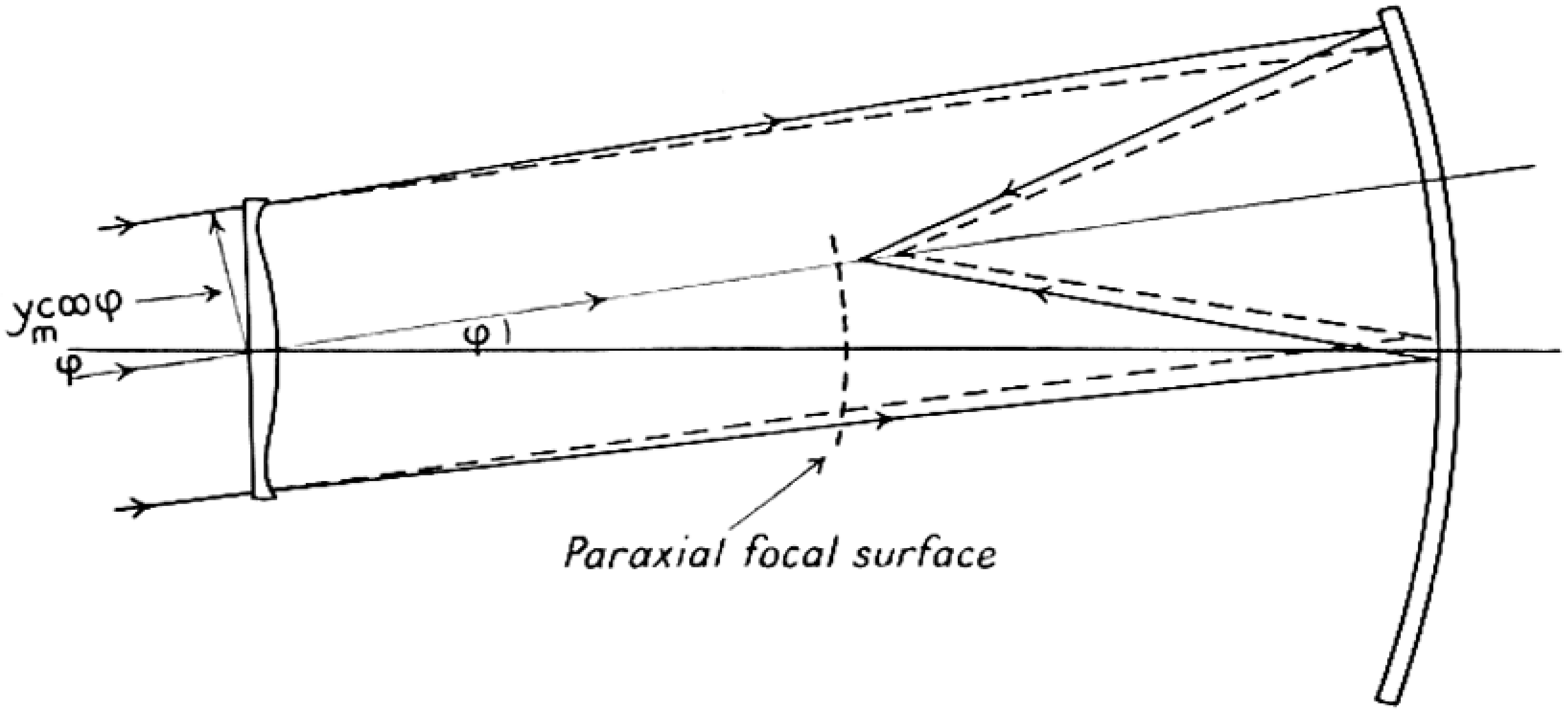


FIG. 257. THEORY OF SCHMIDT CAMERA

So that the angle of a zone at a height $y_m \cos\phi$ is easily found to be (using the usual trigonometrical expansions for $\cos\phi$, $\cos^2\phi$, etc.)

$$\alpha_{(y_m \cos\phi)} = \frac{y_m^3}{32(n-1)f^3}\left(1 - \tfrac{9}{2}\phi^2 + \text{etc.}\right)$$

The corresponding deviation for a normally incident ray is obtained by multiplying this expression by $(n-1)$, thus obtaining the required deviation "$\delta_{required}$."

$$\delta_{(required)} = \frac{(n-1)y_m^3}{32(n-1)f^3}\left(1 - \tfrac{9}{2}\phi^2 + \text{etc.}\right)$$

However, our top and bottom rays pass through the extreme margins of the plate, and the deviation will be,

$$\delta_{max} = (n-1)\,\alpha_{max}\left\{1 + \frac{n+1}{2n}\phi^2 + \text{etc.}\right\} = \frac{(n-1)\,y_m^3}{32(n-1)f^3}\left\{1 + \frac{n+1}{2n}\phi^2 + \text{etc.}\right\}$$

Hence $\delta_{max} - \delta_{required} = \dfrac{y_m^3\phi^2}{32f^3}\left\{\dfrac{n+1}{2n} + \dfrac{9}{2}\right\}$ + terms in ϕ^4 etc.

$= \dfrac{y_m^3\phi^2}{64f^3}\left(\dfrac{10n+1}{n}\right)$, sufficiently nearly when ϕ is not very large, We now remember that the plate is self-conjugate in the Gaussian sense. Hence any pair of rays diverging from a point on the plate will meet again at a distance of $2f$ from the mirror. The distance of these rays in the "focal plane" will now be $f\delta$ where δ is the divergence. Consequently the separation of the ideal ray and the actual ray in the "focal plane" of the mirror will be

$$\frac{y_m{}^3\phi^2}{64f^2}\left(\frac{10n+1}{n}\right)$$

This is clearly a separation of a radial character which will be symmetrical with regard to the principal ray; it corresponds to "oblique spherical aberration" (see p. 203).

It can be shown that the extreme rays in the sagittal diameter produce a sagittal aberration displacement also proportional to $y_m{}^3\phi^2$ but much smaller in amount. The results of a fuller analysis indicate that we may regard these residual errors as a combination of "oblique spherical aberration" with higher or "fifth order" astigmatism.

An expression for the angular tangential coma of a very thin lens with its first face plane and with incident parallel light (which can easily be derived from Conrady's expression, *Applied Optics and Optical Design*, p. 324) is

$$\text{Angular tangential coma} = -\tfrac{3}{2}n(n-1)\phi\alpha_m{}^2$$

where ϕ is the inclination of the principal ray and α_m is the maximum inclination of the tangent to the curved lens-surface. In our case of the plate

$$\alpha_m = \frac{y_m{}^3}{32(n-1)f^3}$$

and hence

Angular tangential coma for the plate

$$= -\frac{3\phi}{2(32)^2}\,\frac{n}{(n-1)}\,\frac{y_m{}^6}{f^6}$$

The actual coma displacement in the focal plane will therefore not exceed

$$-\frac{3}{2048}\,\frac{n}{(n-1)}\,\frac{\phi y_m{}^6}{f^5}$$

It will be negligible in the majority of cases.

The most objectionable defect, i.e. the curvature of the field (see p. 84), still remains, and is little changed by the plate. It may not be objectionable for some purposes, e.g. projection systems in which a television image on the curved surface of a cathode-ray tube is projected (using the system reversed) or in spectrographs where a curved film can be used to record the image, but it is inconvenient for ordinary photography.

The Petzval equation, Vol. I, p. 137, gives, when the object field is plane,

$$\frac{1}{n'R'} = -\Sigma\frac{n'-n}{nn'r}$$

In order to correct the result for a case of reflection at one mirror (Vol. I, Appendix I) we put $n = 1$, $n' = -1$ whence

$$R' = r/2$$

If the mirror is concave with negative radius, the image is concave also. If we now add a second mirror of radius r_2 to reverse the light once more, we must add a term to the Petzval sum in which $n = -1$, $n' = 1$. The n' on the left of equation above refers to the final image space, and is now $+1$. Thus

$$\frac{1}{R'} = -\left(\frac{-2}{-r_1} + \frac{2}{-r_2}\right) = 2\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

The field can therefore be flattened by reflection at a second spherical surface of the same numerical radius (convex if the first mirror is concave).

Again, a field-flattening refracting surface can be used as suggested by F. E. Ross[14]; $n = -1$, $n' = -N$ (say) where N is the numerical refractive index of the medium. Then

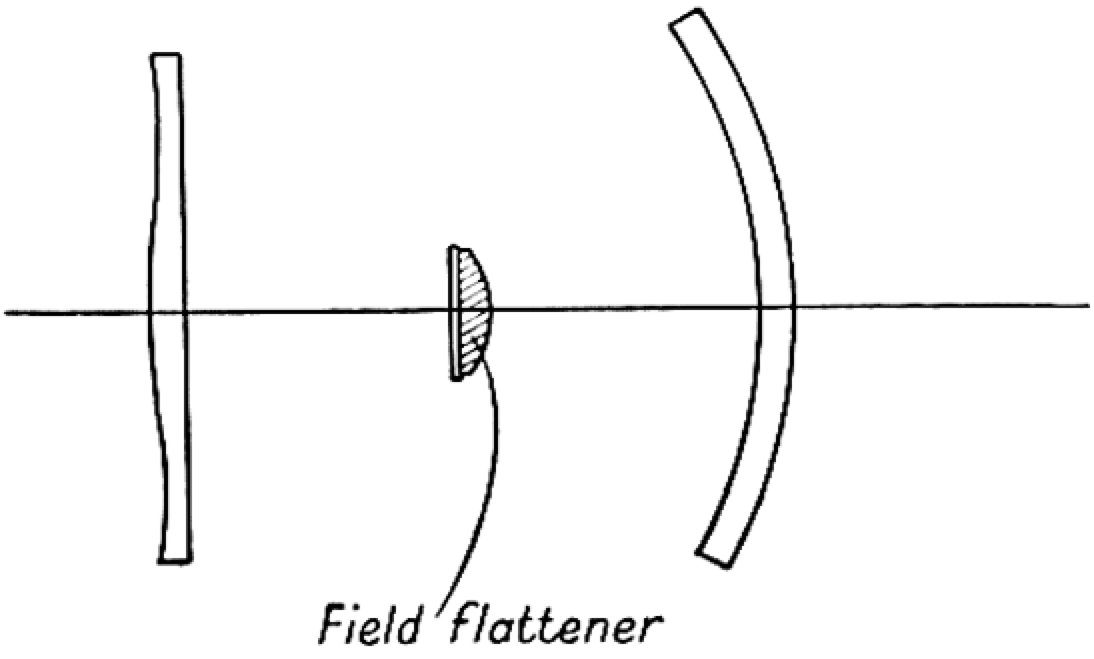


FIG. 258. SCHMIDT SYSTEM WITH FIELD-FLATTENING LENS

$$\frac{1}{-NR'} = -\left\{\frac{-2}{-r_1} + \frac{-N+1}{Nr_2}\right\} \text{ or}$$

$$\frac{1}{R'} = N\left\{\frac{2}{r_1} - \frac{N-1}{Nr_2}\right\}$$

Hence in order to make $1/R'$ zero

$$r_2 = r_1(N-1)/2N$$

The refracting surface (if of glass) should thus have a radius of about one sixth of the first mirror and must turn its convex face towards the light which it receives, Fig. 258. An experimental camera containing a lens of this type made by E. H. Linfoot has been given a nominal aperture $f/1$, which is restricted, by the screening of the main mirror, to an effective aperture of $f/1{\cdot}35$; the total field was 20° and the focal length 8 in.

Baker Camera. The use of a convex reflecting surface to produce a flat-fielded system is illustrated by the camera proposed by J. G. Baker[15] in 1940, Fig. 259. The possibilities associated with such systems and further developments have been discussed by C. R. Burch,[5] E. H. Linfoot,[16] and others. In the figure, M_1 is a large concave mirror, while M_2 (the convex) has very nearly the same radius of curvature; the correcting plate P may be near the centre of curvature of M_1. One of the mirrors must be aspherical, and it is then possible to produce a flat-fielded and anastigmatic system which will perform well over fields up to about 14° in diameter, and can be given an aperture of $f/2$ or even more. If both mirrors are made aspheric it is possible to correct the distortion, though the added difficulty is not usually considered worth while.

It seems probable that systems of such types will find their most notable applications in astronomical photography.

If the correcting plate is brought to a position near the focus of the main mirror in the Schmidt camera, suitable figuring of both plate and mirror will give freedom from spherical aberration and coma, together with a flat tangential field; in this case the residual oblique astigmatism sets a limit to the useful field but the optical system is much shorter, and thus in several ways more convenient to use than the ordinary Schmidt camera. Some of the Baker cameras have the same advantage of relative shortness, a very important matter in systems intended for use in astronomical observatories.

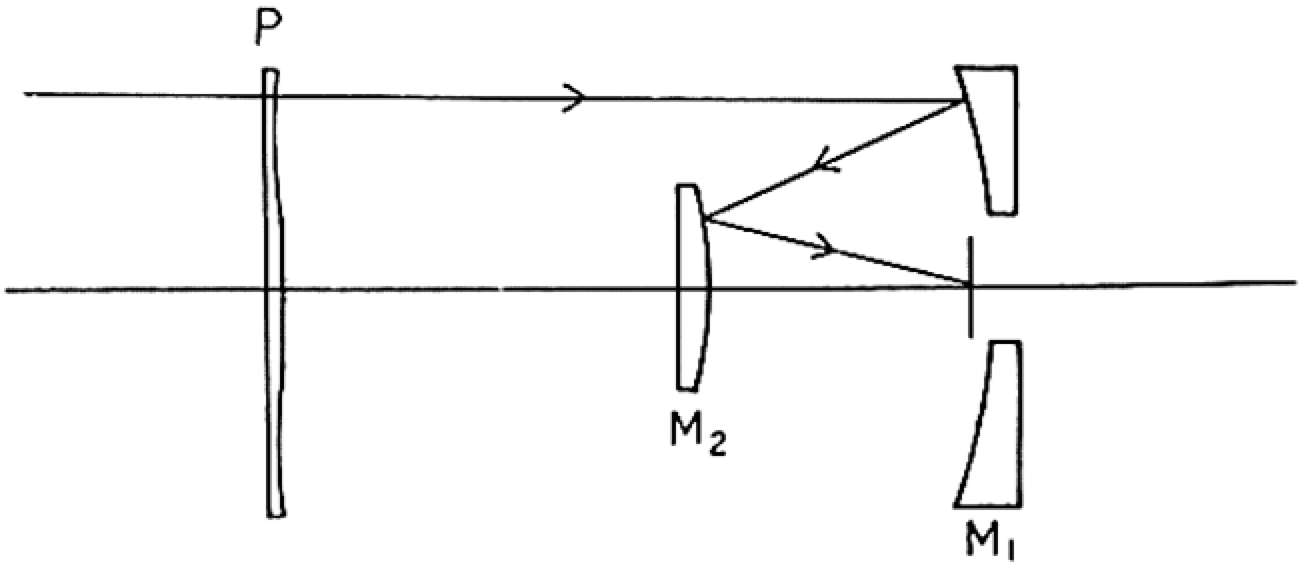


FIG. 259. BAKER CAMERA

The development of these aspheric correcting plates has led to the conception that similar corrections could be given approximately by optical elements with spherical surfaces and having small or negligible paraxial power, but compensating spherical aberration. Gabor[17] designed a high aperture system with a mean correction of spherical aberration and coma (Fig. 260) which was intended for projection purposes. Bouwers[18] proposed to retain some of the wide-field advantages of the Schmidt plate by using a *concentric* meniscus as corrector to the spherical mirror. Fig. 261 shows a system suggested by Maksutov,[19] which may be regarded as a modified Cassegrain system, the meniscus in this case being self-achromatic; such systems may be employed for astronomical photography. Still better correction can be obtained if the meniscus is supplemented by an aspheric plate to remove the residual errors.

Houghton[20] suggests the use of combinations of spherical lenses in place of the aspheric plate. Such combinations can be given negligible power, but appreciable aberration residuals.

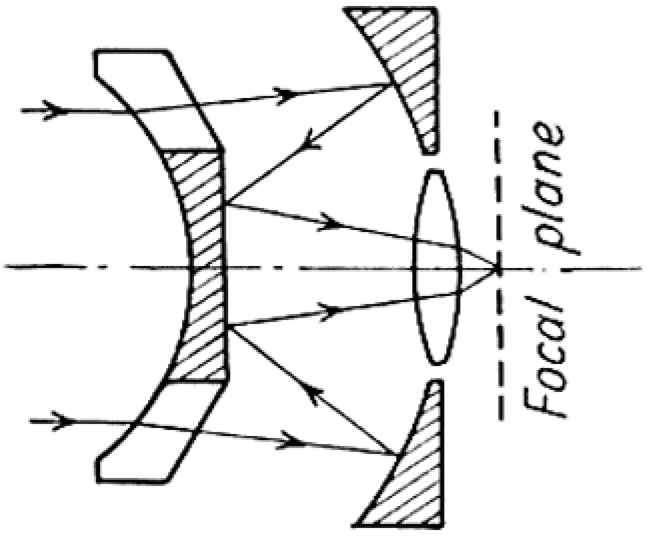


FIG. 260. GABOR CAMERA

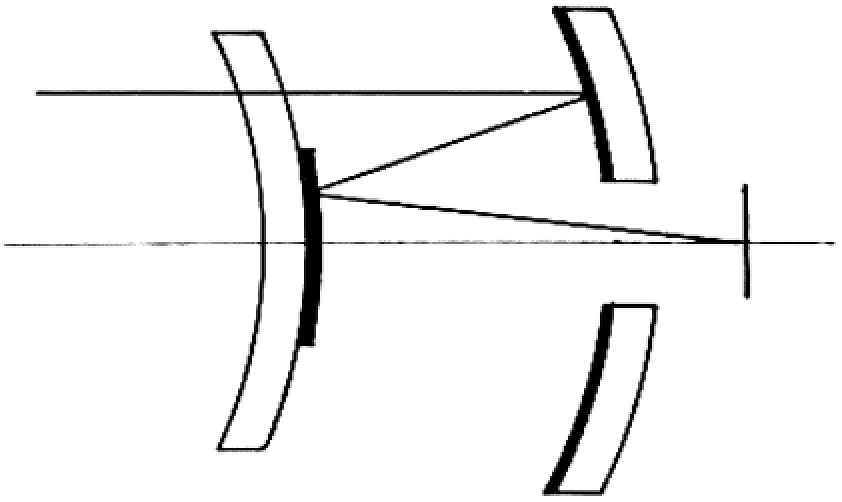


FIG. 261. MAKSUTOV CAMERA

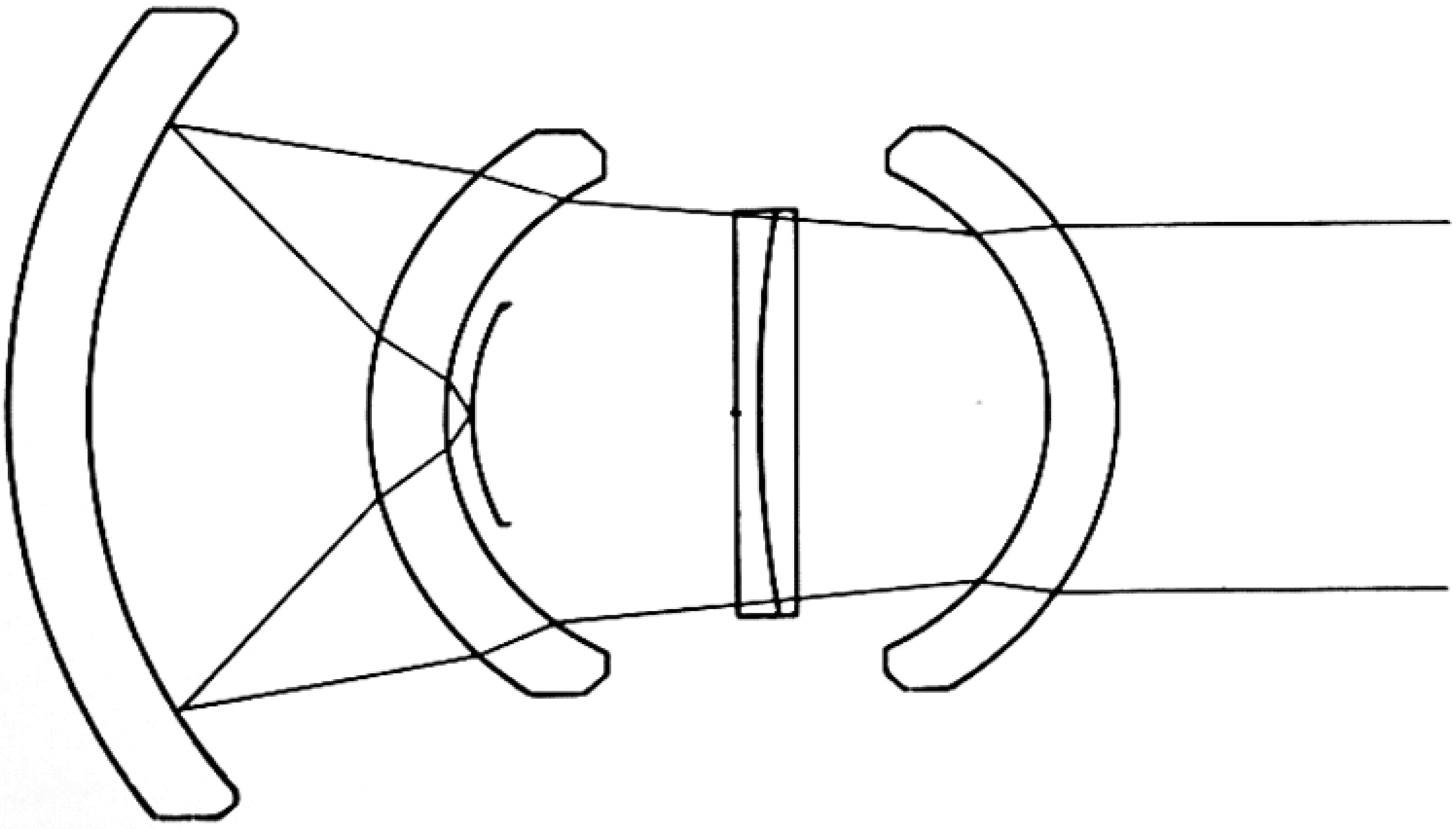


FIG. 262. CONCENTRIC MENISCUS CAMERA
(University of Bristol)

In 1945 Hawkins and Linfoot[21] suggested a system in which the major part of the spherical error is corrected by a strictly concentric meniscus, while the residual chromatic aberration due to the meniscus is corrected by a doublet achromat of very low power at the centre of curvature. Similar proposals were made by Bouwers. Later a team at Bristol University successfully made "meteor cameras" of high aperture (Fig. 262) with two concentric menisci and a central correcting doublet (figured) to work at about *f*/0·78 (allowing for obstruction) effective aperture with a 56° field on a (concentric) spherical plate. This entailed the construction of thick menisci of diameter 19 in. and a mirror of 26 in. diameter, the diameter of the effective stop being 12·9 in.

Somewhat similar cameras have been designed and produced by J. G. Baker, and manufactured by the Perkin Elmer Corporation. The requirement of such high apertures is due to the very small amount of light available in meteor photography. The limit to the size of such systems is probably set by the difficulties in the production and moulding of the thick glass menisci with the necessary optical perfection. Schmidt systems of various types have become somewhat important in regard to the construction of high aperture spectrographs for recording the spectra of very feeble sources of light. It is not possible within the limits of space available to give any adequate account of the above developments and others such as achromatized plate systems; reference should be made to the original papers. Further development of catadioptric systems is due to Bouwers[10], C. G. Wynne,[22] and others.

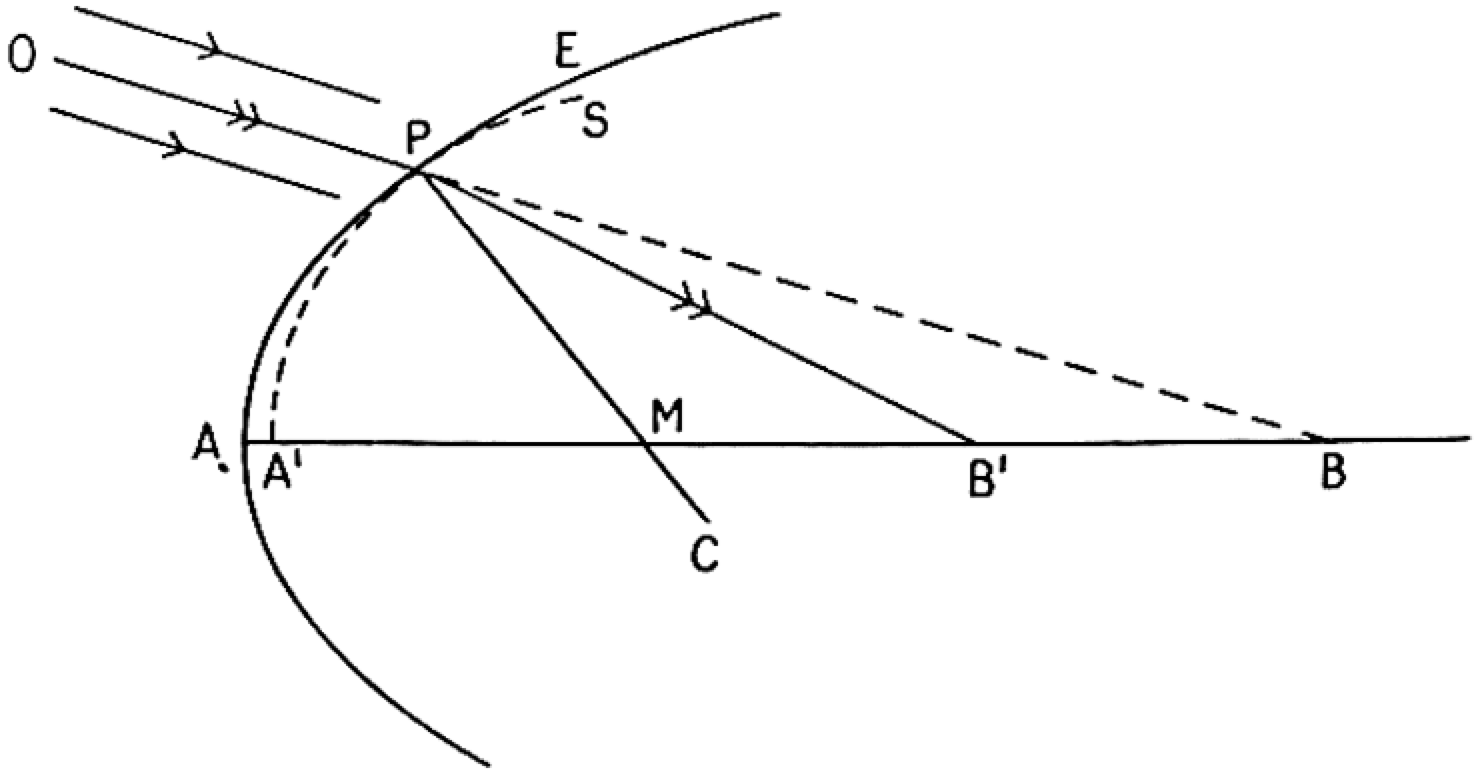


FIG. 263

The General Use of Aspheric Surfaces in Optical Systems. If an aspheric surface of revolution symmetrical about the axis is substituted for a spherical surface having the same axial radius of curvature, in an optical system, then if the surface coincides with the pupil position (and if the departure from the sphere is slight) the spherical aberration of the transmitted light will be changed, but no primary coma or astigmatism will be introduced.

Consider, however, the case of Fig. 263, where the incident principal ray OP is directed towards the entrance pupil B and is refracted by a spherical surface A′PS shown by a broken line. The normal at P is PM, and the length PM is the radius of curvature for all meridians of the refracting surface.

Now let the spherical surface be replaced by an ellipsoidal surface of which the elliptical trace is APE (shown by the full line), and is such that the elliptical and circular sections touch each other at P; it is evident that the ellipsoid and the sphere touch in the circle generated by rotating P around the axis of symmetry.

The centre of curvature of the element of the ellipse at P is now C, beyond M; the tangential refracting power of the new surface is diminished in this case as compared with the sphere. However, the angles of incidence of sagittal rays are precisely the same as before, so that the length PM marks the sagittal radius. It is obvious that primary astigmatism is introduced. We can therefore make effective use of aspheric surfaces not coincident with the pupil to modify the oblique astigmatism of a system; examples of this are found in modern eyepieces.

Consider, moreover, two parallel incident rays associated with the principal ray, Fig. 263. The change of optical path for the two rays associated with the replacement of the sphere by the ellipse will evidently *not* be equal, the increase being larger for the upper ray. Hence primary coma will appear.

In the case of Fig. 264 the axial radius of curvature is changed when the ellipse is substituted; however, it will be apparent that the use of axially symmetrical aspheric surfaces in place of spheres of the same axial radii will modify all the primary monochromatic aberrations of a system with the exception of the Petzval curvature, which depends only on the paraxial curvatures.

The theory of the calculation of the aberrations can be given a fairly simple form in terms of the "Plate Diagram." See Burch,[5] 1943. A great deal can be accomplished in the initial stages even if only surfaces of the second degree are contemplated and some of the simpler conceptions may be usefully set down here.

Second Degree Surfaces. The equation of a circle referred to the vertex as origin, taking the axis as the direction of the x co-ordinates, is

$$y^2 - 2rx + x^2 = 0 \quad . \quad . \quad . \quad (8.01)$$

In comparison with this, a family of second degree curves is represented by

$$y^2 - 2rx + px^2 = 0 \quad . \quad . \quad . \quad (8.02)$$

The curvature at the vertex is obtained by Newton's formula (Limit, $x = y \rightarrow 0$, of $y^2/2x$). Thus since

$$\frac{y^2}{2x} = r - \frac{px}{2},$$

we see that when x is negligible the vertex curvature is identical for all members and equal to r. Fig. 264 shows the various types of conic section curves which result from giving various values to p.

If we solve (8.02) for x we have

$$x = \frac{2r \pm (4r^2 - 4py^2)^{\frac{1}{2}}}{2p}$$

and, taking the square root, in this expression, relevant to this case we have

$$x = \frac{y^2}{2r} + \frac{py^4}{8r^3} + \frac{p^2y^6}{16r^5} + \text{etc.}$$

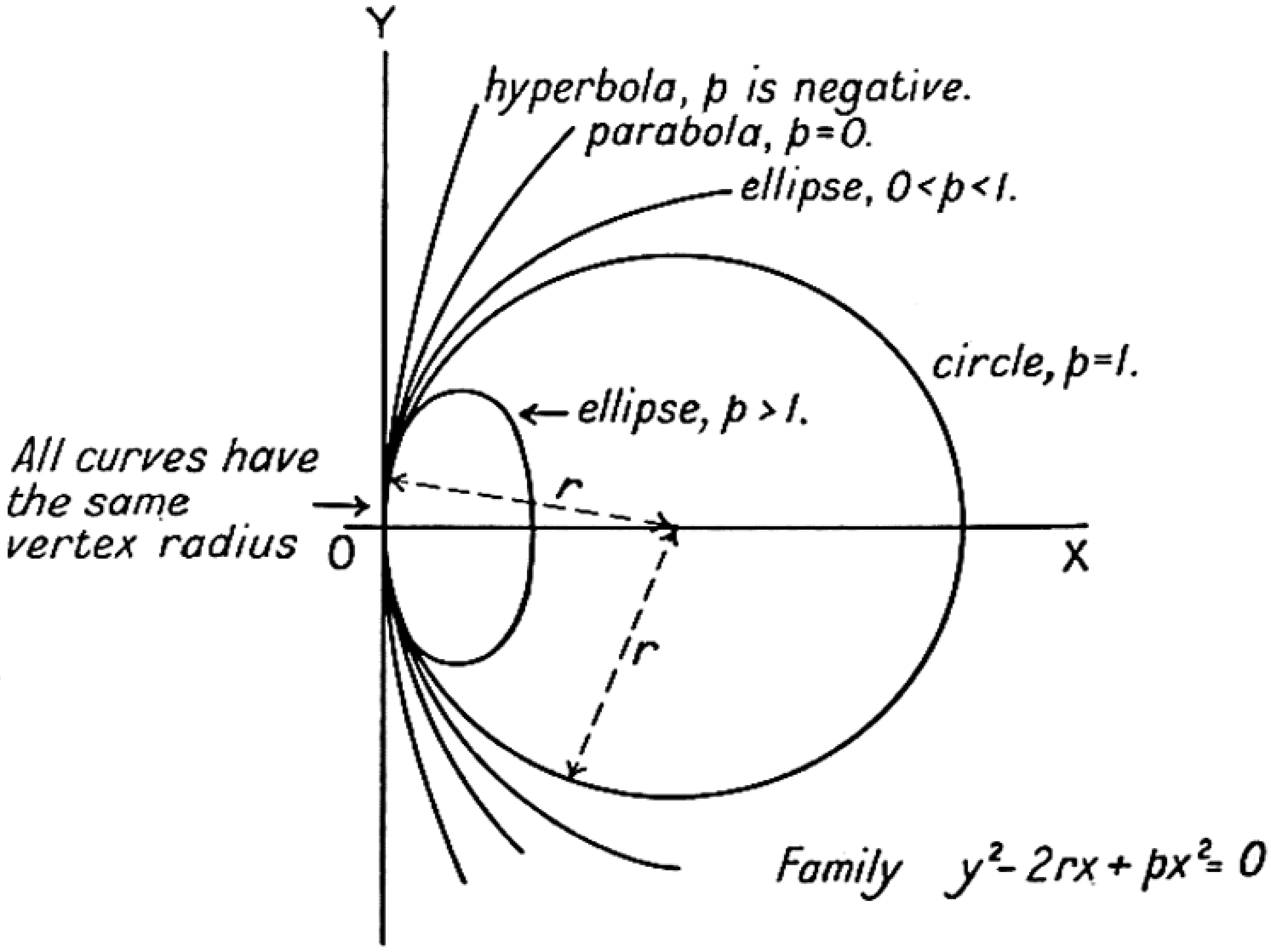


Fig. 264. Illustrating the Effect of the Parameter p

Hence the horizontal gap g between a conic of this family and the circle of the same vertex radius can be written

$$g = \frac{(p-1)\,y^4}{8r^3} + \frac{(p^2-1)\,y^6}{16r^5} + \text{etc.} \quad . \quad . \quad (8.03)$$

Curves (symmetrical about the axis) other than conics would have other coefficients for these terms in y^4, etc., but if they have the same curvature at the vertex the series for x will always begin with $y^2/2r$. A conic can always be found for which the term in y^4 will be identical with that for any symmetrical curve of degree higher than the second. For similar reasons the primary spherical aberration which entails optical path variations depending on y^4 can always be eliminated by changing from a spherical refracting surface into a conicoid. Or with two aspheric surfaces two parameters can be chosen so as, for example, to eliminate both primary spherical aberration and coma; and so on.

Taking the curve, $y^2 = 2rx - px^2$

the angle ϕ between the normal and the x axis is given by

$$\phi = \tan^{-1}\{-y/(r - px)\}$$

Hence $$\sin \phi = -y/\{y^2 + (r - px)^2\}^{\frac{1}{2}}$$

The sagittal radius of curvature r_s is

$$r_s = y \operatorname{cosec} \phi = \{y^2 + (r - px)^2\}^{\frac{1}{2}}$$

By the use of the usual formula for the ("tangential") radius of curvature r_t of the curve at the point x, y, we obtain the convenient result

$$r_t = r_s^3/r^2 \quad . \quad . \quad . \quad . \quad (8.04)$$

It will be seen that since $p = 1$ for a circle, any value of p less than unity (or a negative value of p) will increase the value of r_s for a given y, and (still more also) the value of r_t. The dioptric power will therefore diminish, and more for tangential than for sagittal fans of rays; this assumes of course that r is constant.

The expressions for all the primary aberrations of a system can be worked out in terms of the radii and aspheric parameters p for the various surfaces; it is then possible to assign solutions which will remove some or all of the errors. Once the value of p is known it is easy to calculate the depth of glass which should theoretically be removed from or added to the various zones of a spherical surface in order to produce the "figure" required.

It may be remarked that in very many cases only small deformations of spherical surfaces are required.

The question of the completion of an aspheric surface design by ray-tracing, etc., is dealt with in a discussion and symposium introduced by T. Y. Baker.[23]

Generation of Aspheric Surfaces. Convex and concave surfaces will in general fit each other in any relative position only if they are spheres of the same radius; consequently the methods of mutual grinding which produce the essential tendency to mutual correction for spheres is debarred from use for aspheric surfaces. However, if *conicoidal* surfaces are to be generated, mutual correction between a surface and a scraping edge is possible, as has been shown by J. H. Jeffree.[24]

Let AP, Fig. 265, represent the principal section of a conicoid. PT is the normal, and this length is the sagittal radius at P. However, if the conicoid be intersected by a new plane perpendicular to the principal section and passing through the focus S (PS is the trace of this plane) the radius at P of the curve of section of the conicoid by this plane is the resolved part PQ of the radius PT along the direction PS, moreover a section has a constant radius r equal to the radius of the principal section at the apex.

This can be demonstrated as follows. The polar equation of the conic AP referred to the focus S as origin is (Smith's *Conic Sections*, p. 167)

$$\rho = r/(1 + e \cos \theta),$$

where e is the eccentricity. Note that the height of the curve

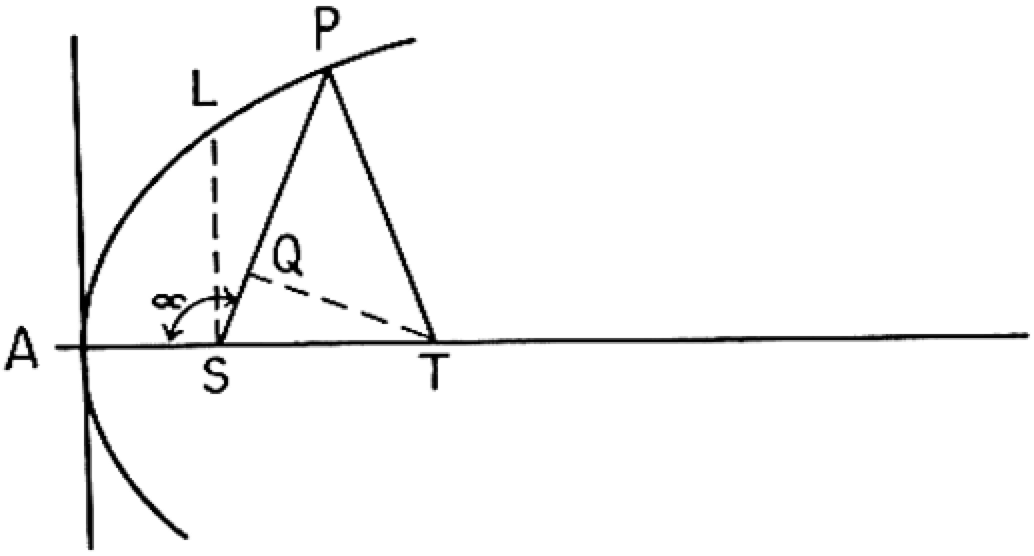


FIG. 265. THEORY OF THE JEFFREE MACHINE

above the focus (the semi latus rectum) is equal to r. Now at a point on the conic, defined by the angle $\widehat{ASP} = \alpha$ (say), the polar equation of the normal to the conic will be (*ibid.*, p. 175)

$$\frac{re \sin \alpha}{1 + e \cos \alpha} \cdot \frac{1}{\rho} = e \sin \theta + \sin (\theta - \alpha)$$

Thus when $\theta = 0$, $$\rho = -\frac{re}{1 + e \cos \alpha} = ST$$

The projected part of this on the original vector SP is therefore

$$-\frac{re \cos \alpha}{1 + e \cos \alpha} = SQ$$

Hence $$PQ = PS - SQ = \frac{r}{1 + e \cos \alpha} - \left(-\frac{re \cos \alpha}{1 + e \cos \alpha}\right)$$

i.e. $$PQ = r$$

The self-correcting contact is arranged as follows: if a template T is cut so as to fit the curve at the apex when held in an axial plane, it will also fit the conicoid closely if held in any plane perpendicular to an axial section and passing through the focus. Fig. 266 suggests the basic arrangement of Jeffree's machine adapted to generate a concave conicoid. The brass or copper plate, which cannot rotate in its own plane, passes through a slot in a cylindrical member which can rotate about a fixed axis passing through the focus of the conicoid and perpendicular to the axis A of symmetry of the latter; this plate is pressed by a suitable spring or weight against the concave surface which is continuously rotated while the plate is given an oscillation through a suitable angle large enough to make the lower curved end of the plate, which is charged with a suitable abrasive, grind the whole surface to be generated. The motion around the axis A is controlled by an "inching" mechanism in which a small amplitude movement, of frequency (say) 60 per minute, is combined with a much slower movement over the whole angular range.

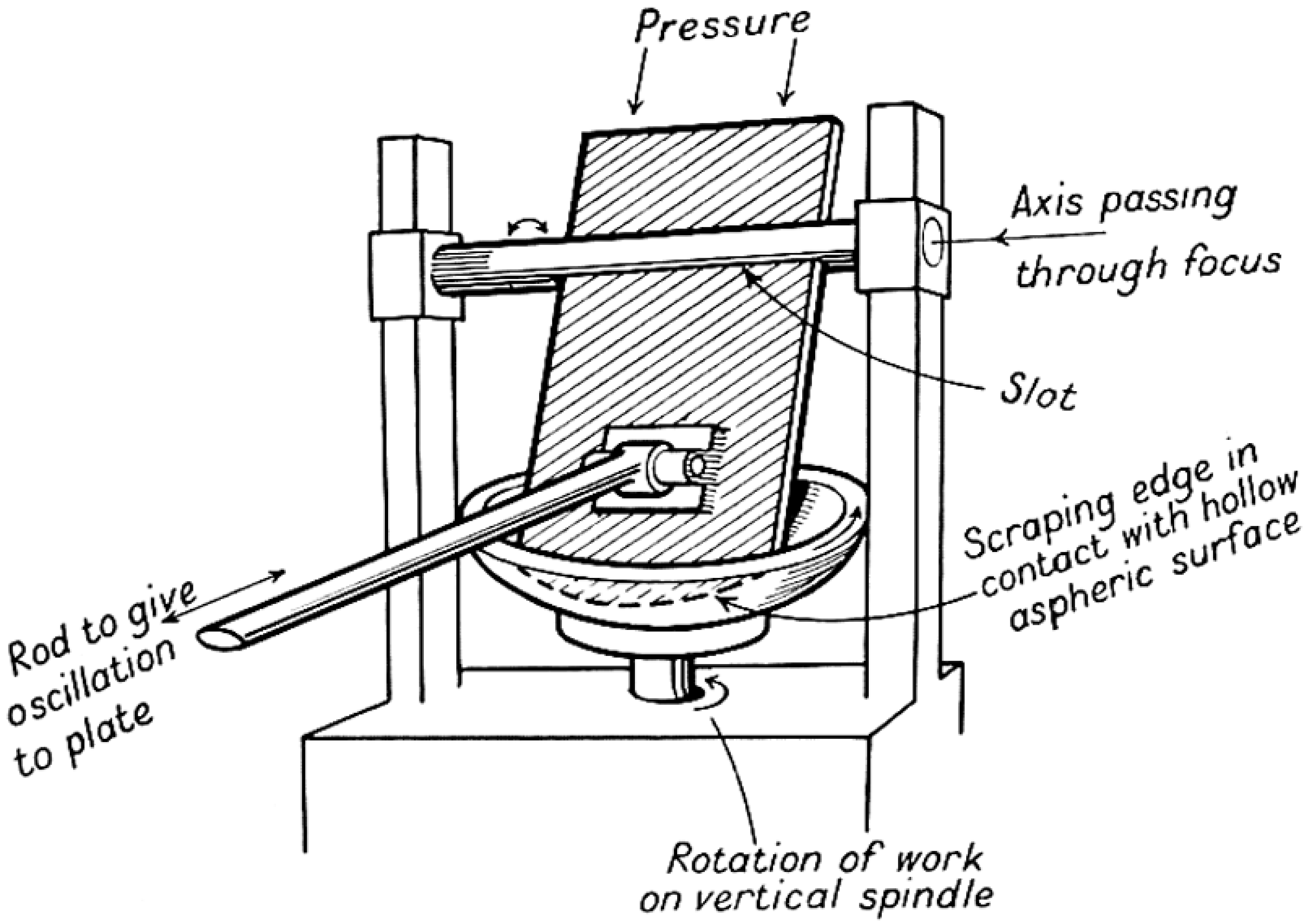

Pressure
Axis passing through focus
Slot
Scraping edge in contact with hollow aspheric surface
Rod to give oscillation to plate
Rotation of work on vertical spindle

FIG. 266. GENERAL PLAN FOR SELF-CORRECTING GENERATION OF CONICOID

Such a machine has been successfully employed to generate conicoidal moulds in stainless steel for the production of plastic lenses. While the system is self-correcting for the reasons given above, the probable sources of error are the inevitable shake and eccentricity on the rotary movements and the possible shake and angular shifting of the plate in its slot.

If a paraboloid is to be generated by this means the grinding plate has to "pass through an axis at infinity." It has, therefore, to be mounted with a linkage which keeps its plane always parallel to the axis of the surface.

Methods Based on Evolute and Other Cams. The locus of the centres of curvature of successive elements of a given plane curve is called the *evolute*. The normals of the original curve are tangents to the evolute; thus if a straight edge touches an evolute cam and rolls on it without slip, a single point on the straight edge will trace the original curve.

The straight-edge may carry a cutting or lapping element which travels in the locus of the principal section of the surface required. The material to be lapped is made to rotate round the axis of symmetry. In certain other optical production methods, high-speed lapping by diamond-charged wheels can in fact be carried out so well as to produce a finely smoothed surface on which the polishing process can be immediately carried out.

C. R. Burch[25] has exhibited a machine in which a *small* grinding or polishing lap merely has its tangential direction controlled by an evolute cam, no attempt being made forcibly to grind the surface, though it is necessary to start with the "work" already fairly close to the required figure. Zonal irregularities are avoided by a suitable "inching" mechanism. The method, though slow, produces surfaces of high regularity, an achievement which proved hardly possible with the earlier methods. There are many other useful mechanical arrangements such as are used in profiling machines, which can be employed to move a cutting or lapping tool in a non-circular locus, and they may possess the valuable feature that the locus can be obtained from a master cam made on a large scale, the actual movement being mechanically diminished. Such master cams can be made flexible at will so that their form can be adjusted by screws, etc., as in the Zeiss Machine (see below).

Linkages. Other early suggestions for the generation of aspheric surfaces turned upon the control of the motion of a small lap by attaching it to the end of a radial arm turning upon a symmetrical but aspherical pivot (Fraunhofer). Numerous mechanisms have been suggested to generate conic curves, and these can be used to guide cutting tools or laps. Thus if a carriage supporting one bearing is made to move along the directrix of a conic while another bearing remains fixed, it is simple to arrange a linkage of equal arms so that a third point remains equidistant from the two bearings, thus generating a parabola—or so that the third point has distances from the other two which maintain a fixed ratio (ellipse or hyperbola). Another class of mechanisms employs "trammel" arrangements, in which a simple case is that of a rod with two projecting pins which slide in mutually perpendicular slots; points on the rod then (in general) describe conic curves. A more elaborate system of linkages, calling for sliding carriages and cams, can be used to generate "polar reciprocal" curves; if the cam is circular, the polar reciprocal will be a conic.

The slider-crank mechanism typical of the ordinary piston engine is a case in which one end of the connecting rod executes a linear vibration, the other end a circular movement; while intermediate points execute movements represented by curves of the fourth degree. In their parts near the axis such curves approximate to ellipses of varying degrees of eccentricity according to the distance of the point from the crank bearing.

Gullstrand[26] made an elaborate analysis of the possibilities of generating convex aspheric surfaces by the action of a plane grinding surface which is given such a motion that the envelope of its successive positions is the curve required. The surface abrades the "work," which rotates around the axis of symmetry. A simple case to illustrate the method is represented in Fig. 267. It is well known that if a perpendicular FP is dropped from the focus F of a conic to the tangent, the locus of P is a circle. Thus if CP is a link of fixed length pivoted at a centre C, while PF is a rod which passes freely through a sleeve pivoted at F, and the line PQ is maintained perpendicular to PF, the envelope of the successive positions of PQ will be a conic. The work rotates around the axis CF; it can be arranged if desired that the grinding plane shown by the trace PQ rotates around PF.

If the point P be constrained to move in a symmetrical locus other than a circle, as is easily possible by some of the various mechanisms mentioned above, the envelope of the successive positions of the abrading surface can become a curve of degree higher than the second, and the constants of the various curves can thus be brought under control. However, while it is relatively simple to design the form of a surface of high degree to eliminate some specific aberration, the *general* analytical designing of systems with surfaces of degrees higher than the second is very complex, and not much progress, if any, has been made in this direction.

Limitations of Mechanical Systems. The elasticity of mechanical members and the various imperfections of pivots, slides, etc., prevent most of the above mechanisms from being used with any success in the direct lapping of high precision optical surfaces, which *should* conform to a prescribed locus within a small fraction of a wave-length; accordingly much attention has been given to means whereby accidental irregularities can be smoothed out. An example of this is found in the Zeiss Aspheric Lapping Machine, which follows the original suggestion of Descartes.

Zeiss Aspheric Grinding Machine. In this machine, Fig. 268, mostly used for aspherizing eyepiece lenses, and components for illumination systems, the non-circular profile is not imparted *directly* to the principal section of the work by the cutting tool, but is imparted through the agency of an aluminium grinding wheel, the rim of which is given the required aspheric section with the aid of the cutting tool of which the motion is controlled by an adjustable cam.

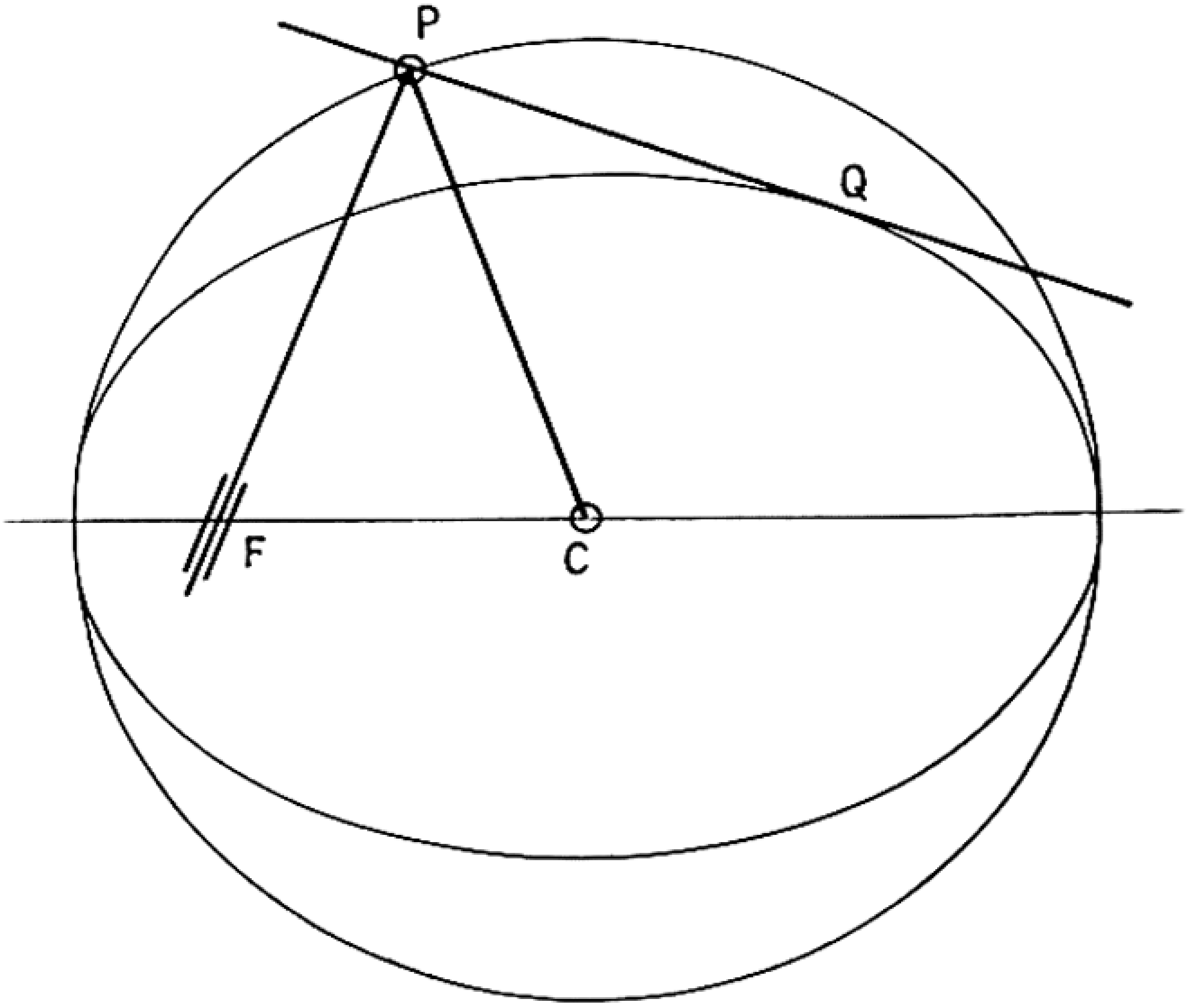


FIG. 267. THEORY OF GULLSTRAND MECHANISM

The wheel is given 600 revolutions per minute for "truing." Its circumference is channelled by a groove 20 to 30 mm. deep and filled with wood pitch. Emery 15 mμ → 8 mμ is used during grinding, which is done at 180 r.p.m. and is completed in two stages of about 5 minutes.

The lens to be ground is mounted on a spindle perpendicular to that of the grinding wheel, and symmetrical with a radial line in its median plane; it revolves at 240–450 r.p.m. according to the lens diameter. The result of these movements is that purely accidental local and zonal irregularities are suppressed in the figure of the lens in a manner not easily attained if the original lapping or cutting tool has to work directly on the glass.

In polishing, the lens remains cemented on the adapter which held it during grinding, and a flexible polisher of elastic felt or rubber is employed.

Sundry other Methods for Producing Aspheric Surfaces. A few other methods should be mentioned. They include—

(*a*) Vacuum distillation[27] of some transparent substance on to a spherical refracting surface to be aspherized; symmetry is secured by the rotation of the surface to be treated; the thickness of the deposit is controlled by the relative distance of the "source," and sometimes by the use of an appropriately shaped sector or diaphragm. This general idea might perhaps be styled "addition rather than attrition," a method which has hitherto found little scope owing to present difficulties of technique, but one which might possibly find a great application if suitable methods could be developed.

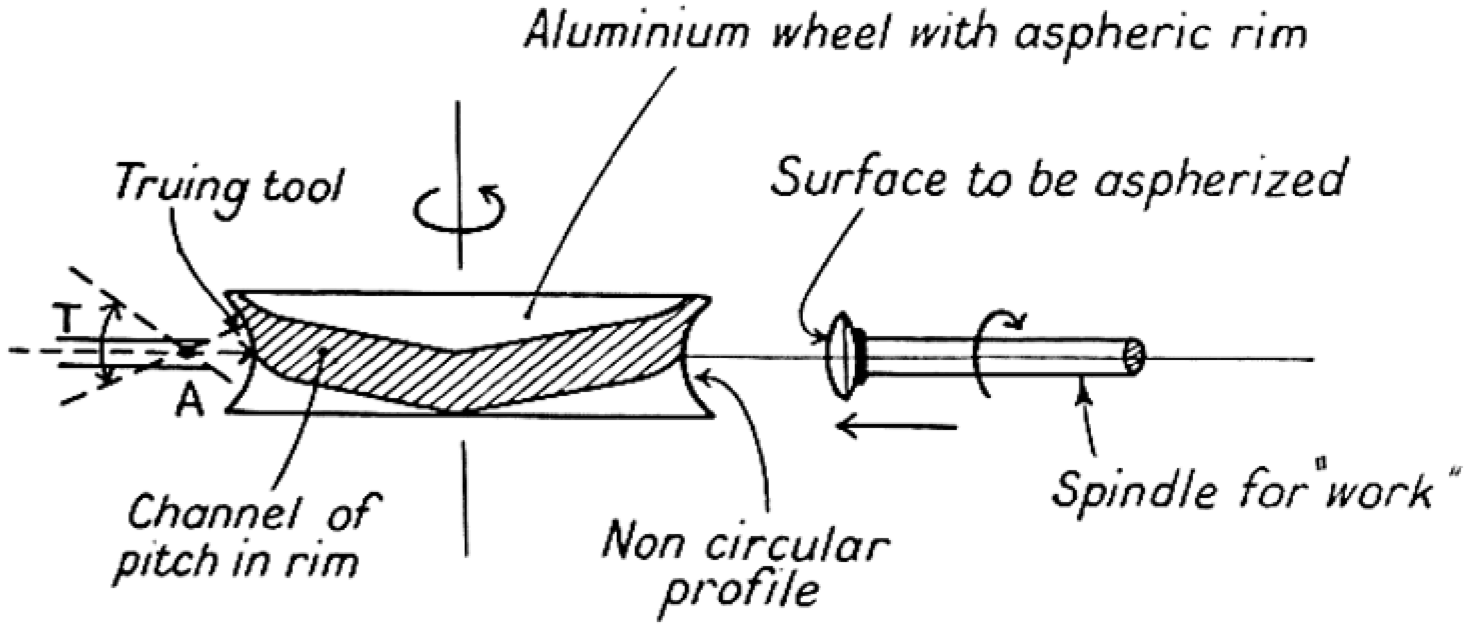


FIG. 268. PRINCIPLE OF ZEISS ASPHERIC GRINDING MACHINE

(*b*) A lens to be aspherized is given spherical surfaces and then placed under symmetrical stress so that it is deformed while one surface is given a new "spherical" grinding and polishing; when the stress is released the new surface takes an aspheric shape.

(*c*) "Dropping." The lens is given spherical surfaces, and is rested symmetrically on the rim of a mould with a concave surface. Sufficient heat is applied in an oven to soften the glass and drop it into the mould; the upper surface takes a calculable aspheric form without losing its polish.

(*d*) Flexible laps. These are specially used in the production of

Schmidt plates,[28] see above. The "work" has a plane surface which is slowly rotated about the vertical axis of an optical grinding machine. The lap consists of a thick sheet of rubber on the lower side of which is cemented a mosaic of lapping elements of uniform thickness; in some cases these are of porcelain. The lap can be given a slow lateral oscillatory movement of small amplitude in a normal way, but the relative grinding of the various zones is controlled by loading the upper surface of the rubber sheet by a ring of leaden weights so as to control the local pressure and hence the zonal abrasion. Normal abrasives are employed. The profile of the plate has to be checked fairly frequently by some suitable method or measuring instrument, and the zonal load is adjusted accordingly.

Polishing of Aspheric Surfaces. As is well known, the aspherizing of astronomical mirrors is nearly always done entirely during the "polishing" stage of production, and where slight departures from the spherical form are all that is necessary, the polished surface which permits of optical rather than mechanical control will be the starting point. But where deeply aspheric surfaces are in question the amount of material to be removed from a sphere is too great to allow of such methods; the aspherization has to be done in the stages of grinding or fine grinding and the figure in polishing has to be left to take care of itself, flexible polishing surfaces (perhaps of sponge rubber suitably covered or charged) being used to produce a uniform polish over the prepared surface so that uneven wear is avoided.

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APPENDIX I

SYMBOLS AND SIGN CONVENTION

It may be a convenience for some readers to have a concise statement of the sign conventions used in both volumes of the present book.

Symbols. Symbols relating to the image space are distinguished from those of the object space by the addition of a dash or accent, thus: l'.

Refractive indices (object and image spaces)	n, n'
Conjugate distances of object and image (measured from the principal points)	l, l'
Focal lengths	f, f'
Conjugate distances of object and image (measured from focal points)	x, x'
Perpendicular heights of object and image	h, h'
Angles between a ray and the axis (object and image spaces)	α, α'
The reciprocals of distances are denoted by **L**, thus $1/l$ and $1/l'$	**L** and **L'**
The power of an optical system ($n'/f' = -n/f$)	**F**

Sign Convention. Distances measured to the right along the axis are counted positive; those to the left are negative. Similarly, those measured upwards, perpendicular to the axis, are positive, and those measured downwards are negative.

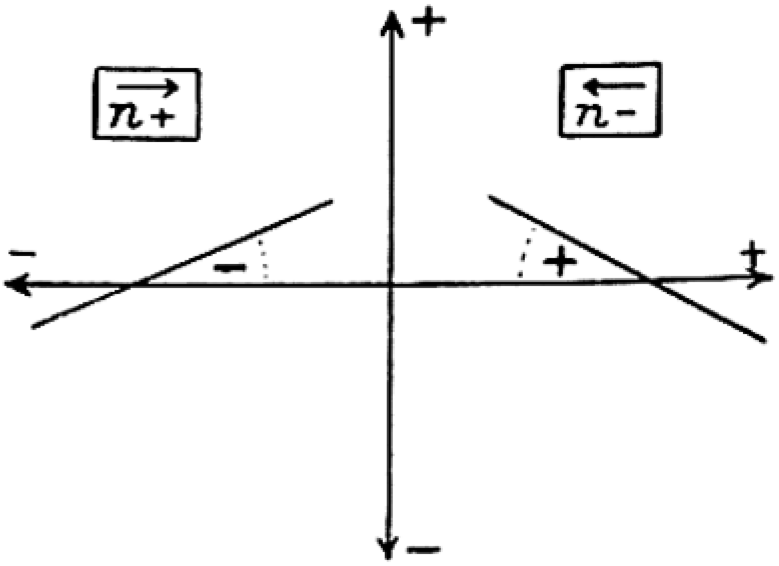

Fig. A1

The angles at which the ray directions meet the axis are counted positive if a clockwise turn will bring a line from the axis direction to the ray direction by the lesser angular movement.

The refractive indices are numerically positive when the direction of the light is from left to right, and numerically negative when the direction of the light is from right to left.

APPENDIX II

THEORY OF THE DIFFRACTION GRATING

THE simplest approach to the general theory is to consider the grating to consist of a series of parallel rectangular apertures. The investigation first deals then with the diffraction of a plane wave by a single rectangular aperture; the discussion is then extended to the case of a series of apertures.

1. Diffraction of a Plane Wave by a Rectangular Aperture. We may imagine a plane screen of indefinite extent with a rectangular aperture ABCD (Fig. A2). A wave-train with its wave-fronts parallel to the screen produces vibrations in the plane of the aperture which are all in the same phase. It is now desired to find the relative illumination, due to the aperture, at various points in a second screen at an indefinitely great distance, or a distance so great that if N is a point on the normal through the centre O of the aperture, the difference between the distances AN and ON, or CN and ON, is not more than a negligible fraction of the wave-length of light. In this way the disturbances spreading from all the parts of the aperture and reaching the point N will arrive in sensibly the same phase, and their amplitudes will simply be added together to find the resultant.

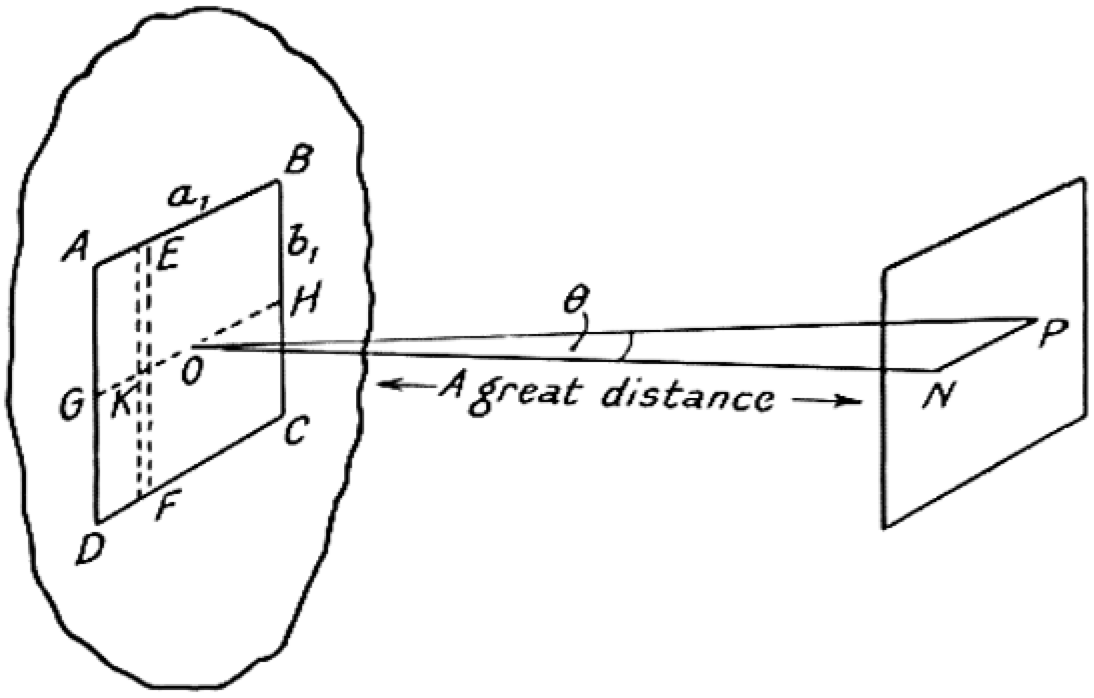

Fig. A2

Effects of this class were discussed by Fraunhofer, and are usually known as Fraunhofer diffraction phenomena. As will be explained below, the effects on a screen at an "infinite distance" are similar to those found in the focal surface of a lens placed behind the aperture and focusing the light; hence the discussion has more than a theoretical interest.

The second plane is also normal to the line ON; let us consider a point P in this plane such that $\widehat{NOP} = \theta$, say, and NP is parallel to GH, a diameter of the rectangle ABCD. It will be clear that the distances from P of all points on a line, such as AD or BC, perpendicular to GH, will still be sensibly equal, but the distances from P of various points on the line GH will differ considerably if θ is finite. The calculation of the effect due to the whole aperture may therefore be effected by dividing this aperture into a series of strips, such as EF, and imagining the whole effect of the strip as equivalent to a proportionately intense source of disturbance situated at the central point K of the strip.

The problem can now be discussed with a two-dimensional diagram (Fig. A3), as if the rectangular aperture were a line source of light, giving rise to disturbances starting with the same phase.

Let the sides of the rectangle be $m_1 = \text{AB}$, and $n_1 = \text{AD}$ respectively; the area is therefore $m_1 n_1$; let the amplitude at N be written $k m_1 n_1$ since all disturbances arrive at N in the same phase. In seeking the effects at a point P_1 the relative phases of the elementary disturbances arriving there must be considered. The dotted line MON in Fig. A3, is perpendicular to OP. All optical paths from points on MON to the point P are considered equal; hence it will be clear that the disturbance derived from the point R at a distance m from the point O will have an extra path to travel as compared with the disturbance derived from O. This extra path is $m \sin \theta$ (where $\text{OR} = m$) and the corresponding lag of phase is $(2\pi m \sin \theta)/\lambda$.

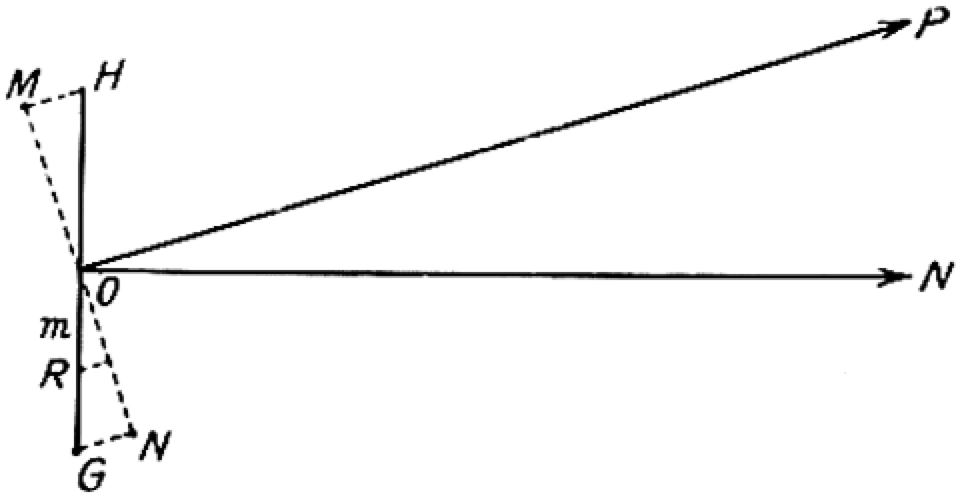

Fig. A3

In Vol. I, page 84, it was shown that the amplitude A resultant from a number of successive contributions with various phases is given by

$$\text{A}^2 = \{\Sigma a \sin \delta\}^2 + \{\Sigma a \cos \delta\}^2$$

where a is the amplitude and δ the phase angle of a single contribution. In our case the area will be divided up into equal strips of width dm so that the contributions of successive elements will have equal amplitudes but differing phases. The area of a single strip is $n_1 dm$, so that the amplitude at P produced by it will be $k n_1 dm$ where k has the same meaning as above.

The summation will therefore be

$$\text{A}^2 = \left[\sum_{m = -\frac{m_1}{2}}^{m = \frac{m_1}{2}} (k n_1 dm) \sin \{(2\pi m \sin \theta)/\lambda\} \right]^2$$

$$+ \left[\sum_{m = -\frac{m_1}{2}}^{m = \frac{m_1}{2}} (k n_1 dm) \cos \{(2\pi m \sin \theta)/\lambda\} \right]^2$$

Consider, however, the summation $\Sigma a \sin \delta$ characteristic of the first main term above. For one strip the component will be

$$kn_1dm \sin \{(2\pi m \sin \theta)/\lambda\},$$

but there will be another strip on the other side of O, at a distance $-m$ from O which will have an equal phase angle, but *negative*, so that the two terms will cancel each other. On the other hand, such terms will *add* numerically in the cosine summation, since the cosine of a negative angle is equal to that of the equal positive angle. We thus see that the first bracket above reduces to zero, and we get

$$\mathrm{A} = \sum_{m=-\frac{m_1}{2}}^{m=\frac{m_1}{2}} (kn_1dm) \cos \{(2\pi m \sin \theta)/\lambda\}$$

$$= kn_1\int_{m=-\frac{m_1}{2}}^{m=\frac{m_1}{2}} dm \cos \{2\pi m \sin \theta)/\lambda\}$$

$$= kn_1\left[\frac{\sin \{(2\pi m \sin \theta)/\lambda\}}{(2\pi \sin \theta)/\lambda}\right]_{m=-\frac{m_1}{2}}^{m=\frac{m_1}{2}}$$

$$= \frac{kn_1\lambda}{2\pi \sin \theta}\left[\sin\left(\frac{2\pi}{\lambda} \cdot \frac{m_1}{2} \sin \theta\right) + \sin\left(\frac{2\pi}{\lambda} \cdot \frac{m_1}{2} \sin \theta\right)\right]$$

$$\mathrm{A} = \frac{kn_1\lambda}{\pi \sin \theta} \sin\left(\frac{\pi m_1 \sin \theta}{\lambda}\right)$$

$$= km_1n_1 \frac{\sin\left(\frac{\pi m_1 \sin \theta}{\lambda}\right)}{\left(\frac{\pi m_1 \sin \theta}{\lambda}\right)}$$

$$= \mathrm{K}\frac{\sin \mathrm{U}}{\mathrm{U}},$$

where K is the central intensity, and $\mathrm{U} = (\pi m_1 \sin \theta)/\lambda$. A similar law must also hold for a direction from N taken parallel to the other side of the rectangle; in the case of $\mathrm{U} = (\pi n_1 \sin \theta)/\lambda$.

As regards the phase of the resultant vibration, we can see by the ordinary graphical construction, Vol. I, page 83, that if we have two equal components with phase angles of opposite sign but numerically equal, the resultant phase angle must be zero. In the above case, the resultant disturbance from the whole aperture has the phase of the component derived from the central point O.

Effect of Two Parallel Rectangular Apertures. The result of the investigation for a single aperture must now be extended to the case of a number of apertures. Let us first of all consider the case of two equal parallel rectangular apertures in a screen on which a plane wave-train is incident. The width of each aperture is m_1 as before, and its height n_1. In a very similar manner, we may reduce the effect of each aperture to that of a line source in the plane of the diagram.

We found above that the resultant of a single aperture has the phase of a disturbance starting from the mid-point.

Let the distance between the central points of each aperture be x and let θ be the angle of diffraction considered. In Fig. A4, the two apertures are A and B and the mid-point is C. Drawing a dotted line ECD through C perpendicular to the direction of diffraction, it will be clear that the disturbance from B would arrive with a lag of phase (δ say), while that from A would arrive with a numerically equal lead of phase, as compared with an (imaginary) distance derived from C. If A_1 is the amplitude due to a single aperture in the direction θ, then the resultant amplitude due to each of them will be

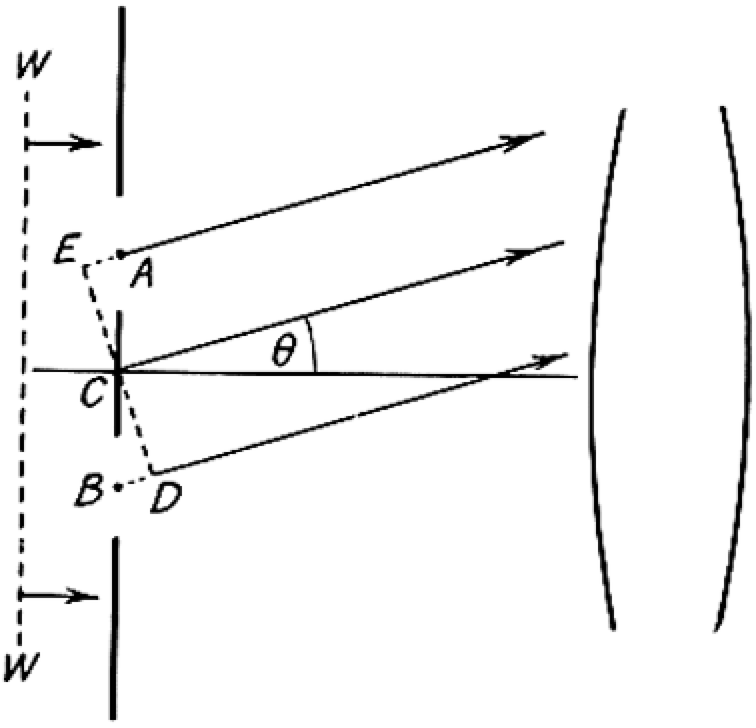

Fig. A4

$$A^2 = \{A_1 \sin \delta + A_1 \sin(-\delta)\}^2 + \{A_1 \cos \delta + A_1 \cos(-\delta)\}^2$$
$$= 4A^2_1 \cos^2 \delta$$

Now δ is evidently given by

$$\frac{2\pi}{\lambda}(\mathrm{BD}) = \frac{2\pi}{\lambda}\left(\frac{x}{2}\sin\theta\right) = \frac{\pi x \sin\theta}{\lambda},$$

so that the resultant amplitude is given by

$$A = 2A_1 \cos\left(\frac{\pi x \sin\theta}{\lambda}\right)$$

But from above we had that

$$A_1 = K(\sin U)/U$$

where $U = (\pi m_1 \sin\theta)/\lambda$, so that the complete expression for the amplitude is

$$A = \frac{2K \sin U}{U} \cos\left(\frac{\pi x \sin\theta}{\lambda}\right).$$

Hence the distribution due to a pair of very thin slits, which would be a set of interference fringes in which the amplitude follows the law $2K \cos(\pi x \sin\theta/\lambda)$, and the intensity follows the corresponding "$\cos^2$" law, will, in practice, when using slits of finite aperture, be modified by the $(\sin U)/U$ term. Students who are interested in the study of diffraction should make the experiment and plot the curves for particular cases, as they are most instructive. The subject is treated more fully in textbooks of Physical Optics.

Effect of a Series of Parallel Apertures. Let the width of each aperture be m_1 and the length n_1 as above. Exactly as before, we can reduce the effect of each aperture to that of a line source in its small diameter shown in the plane of the diagram (Fig. A5). Let the common distance between the centres of the apertures be x, then also, as before, the disturbances derived from each aperture will have the phase of one derived from the mid-point.

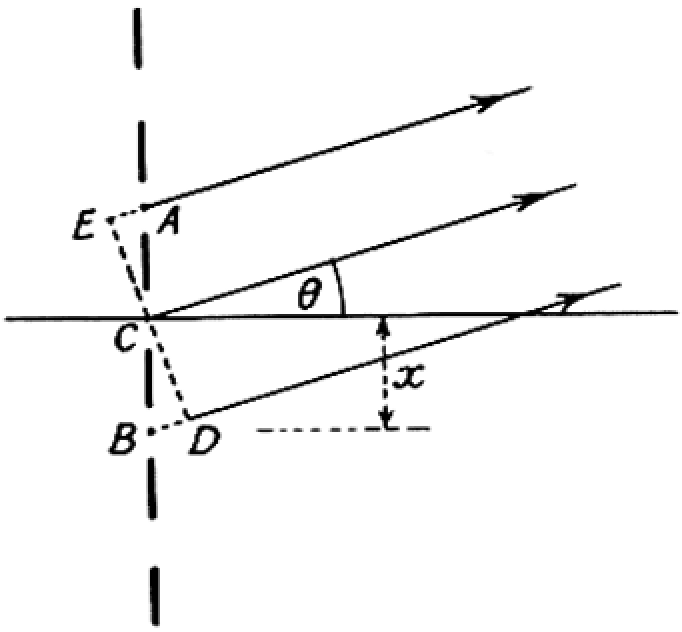

FIG. A5

Let C be the central aperture and A and B the first apertures on each side. It will be clear that the disturbance from B will have a lag, and that from A an equal lead in phase (δ say) considering the diffraction angle θ. For the pth aperture away from the centre the phase difference will be $p\delta$. Let A_1 be the amplitude due to a single aperture in the resultant taken for the angle θ, then the effect of all the apertures will be given by

$$A^2 = \left[\sum_{p=1}^{p=(N-1)/2} \{A_1 \sin p\delta + A_1 \sin(-p\delta)\}\right]^2 + \left[A_1 + \sum_{p=1}^{p=(N-1)/2} \{A_1 \cos p\delta + A_1 \cos(-p\delta)\}\right]^2$$

for a total number of apertures N including the central one. It will be clear that the sine terms will mutually cancel each other when N is odd, leaving

$$A = A_1 + 2\sum_{p=1}^{p=(N-1)/2} A_1 \cos p\delta$$

or putting in the value of δ, i.e. $(2\pi \,.\, x \sin\theta)/\lambda$

$$A = A_1 + 2\sum_{p=1}^{p=(N-1)/2} A_1 \cos\{(2\pi p x \sin\theta)/\lambda\}$$

The sum of the series may be found as in Vol. I, page 86; it proves to be given by

$$A = A_1 \frac{\sin\{(\pi N x \sin\theta)/\lambda\}}{\sin\{\pi x \sin\theta)/\lambda\}}$$

We can simplify the expression for the case when θ is small in the neighbourhood of the central maximum. It then becomes

$$\begin{aligned} A &= A_1 \frac{\sin\{(\pi N x \sin\theta)/\lambda\}}{(\pi x \sin\theta)/\lambda} \\ &= NA_1 \frac{\sin\{(\pi N x \sin\theta)/\lambda\}}{(\pi N x \sin\theta)/\lambda} \\ &= \text{(effect due to whole of apertures)}\ \{(\sin W)/W\} \end{aligned}$$

where $W = (\pi N x \sin\theta)/\lambda.$

This last expression will, however, become inaccurate, as θ becomes greater. It is not so easy to see the way in which the amplitude varies from this expression as from the one above, i.e.

$$A = A_1 + 2\sum_{p=1}^{p=(N-1)/2} A_1 \cos p\delta$$

where $\delta = (2\pi x \sin \theta/\lambda)$.

If we plot a succession of curves $y_1 = \cos \delta$, $y_2 = \cos 2\delta$, $y_3 = \cos 3\delta$, etc., and add the ordinates, we shall find that the sum of the ordinates keeps on increasing at $\delta = 0$, $\delta = 2\pi$, $\delta = 4\pi$, etc., but that at intermediate points the contributions of successive terms vary in sign and tend to cancel each other. With two terms

$$A_1 + 2A_1 \cos \delta$$

(which represents three grating apertures) there will be a result shown in Fig. 82 of this volume; the main maxima are separated by one intermediate maximum. With three terms (representing five grating apertures) there will be three intermediate maxima.

The student should draw the amplitude curve, and then the curve given by squaring the amplitudes, thus obtaining an intensity curve with all positive ordinates.

We find that when the number of terms is infinite (i.e. as the number of apertures increases greatly), the sum of the amplitudes is appreciable only when δ in the above expression is zero or some positive or negative multiple of 2π, i.e. when

$$x \sin \theta = \pm r\lambda,$$

where r is an integer or zero.

We may note that in Fig. A4, the amount of $x \sin \theta$ would be EA + BD, or simply the path difference between the disturbances derived from corresponding points of two adjacent apertures. If this is zero, or corresponds to one or more whole wave-lengths, we get a diffraction maximum. The more exact expression for the sum at the foot of page 367 gives

$$A = A_1 \{\sin (N\delta/2)/\sin \delta/2\}$$

We found that A_1, the effect of a single aperture, was given by $(K \sin U)/U$ so that the complete expression for the amplitude is

$$A = \text{constant} \times \frac{\sin\left(\frac{\pi m_1 \sin \theta}{\lambda}\right)}{\frac{\pi m_1 \sin \theta}{\lambda}} \cdot \frac{\sin\left(\frac{\pi N x \sin \theta}{\lambda}\right)}{\sin\left(\frac{\pi x \sin \theta}{\lambda}\right)}.$$

Wave Incident at Another Angle. We can easily see how the above expressions will be modified to deal with the case in which the plane wave-fronts incident at the apertures are not parallel to the screen, but are inclined with the normal in the plane of the diagram (which is supposed to be perpendicular to the screen) and when there is a change of refractive index at the plane of the screen.

We can show, as before, that the effect of the whole aperture will be equivalent to that of a line source at the middle of the aperture. This centre line can be shown in the diagram (Fig. A6). Let C be its central point, and let F be a point at a distance m from the centre.

The plane waves pass up to the aperture at an angle θ_o with the screen; we desire to find the diffraction effect in a direction making an angle θ_1 with the normal beyond the aperture. Draw the perpendicular CE and FD to the "incident" and "diffracted" paths.

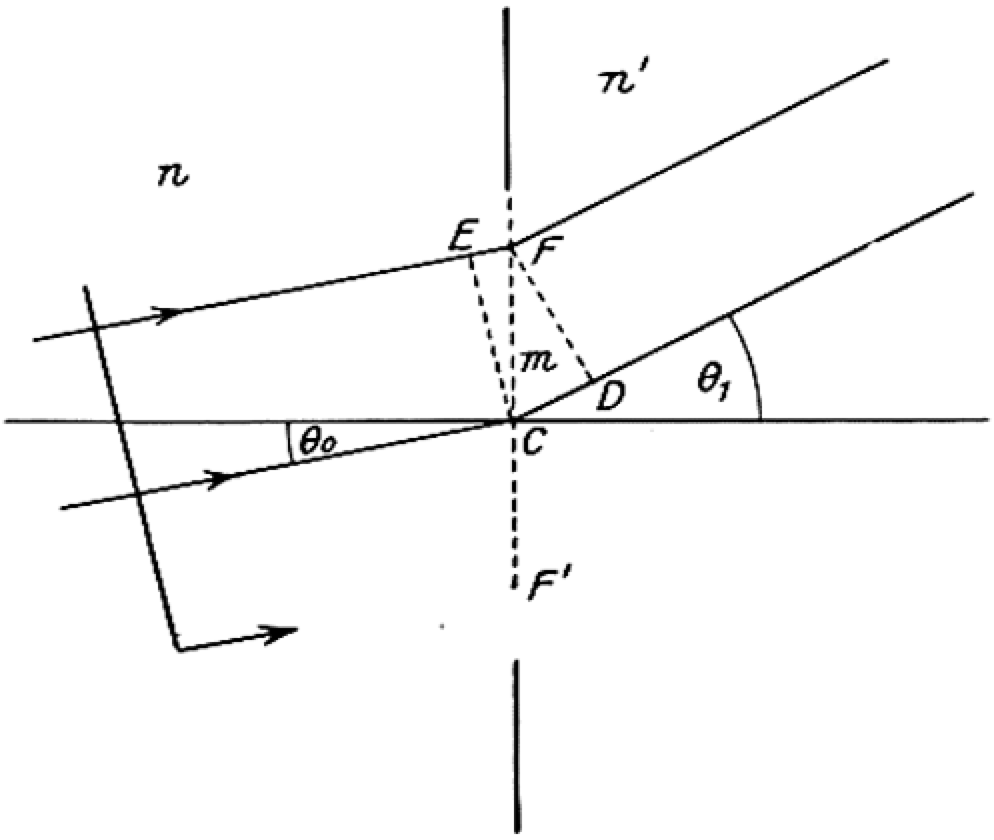

FIG. A6

The disturbances passing through F have a shorter optical path than those through C, the difference being

$$n' \,.\, \mathrm{CD} - n \,.\, \mathrm{EF} = n'm \sin \theta_1 - nm \sin \theta_o,$$

but for those passing through F′, the symmetrical point on the other side of C, there will be an equal and opposite path difference. Hence the term sin θ in the above expressions will be replaced by $n' \sin \theta_1 - n \sin \theta_o$. In the case of the diffraction grating, for example, the condition for a bright maximum will be

$$x(n' \sin \theta_1 - n \sin \theta_o) = \pm p\lambda$$

where p is zero or an integer.

APPENDIX III

THEORY OF CONTRAST TRANSMISSION

The Object Function. It was seen in the chapter on the microscope that if the coherently illuminated object is a grating of bars and spaces, the image can be regarded as composed of sinusoidal contributions of frequencies ϕ, 2ϕ, 3ϕ, etc. In this case the transmission of the object plane can be regarded as a periodic function of a "square-topped" character, the spaces having complete transmission (unity) and the bars no transmission (zero). The representation of such functions by trigonometric series was discussed by Fourier. Thus the Fourier series—

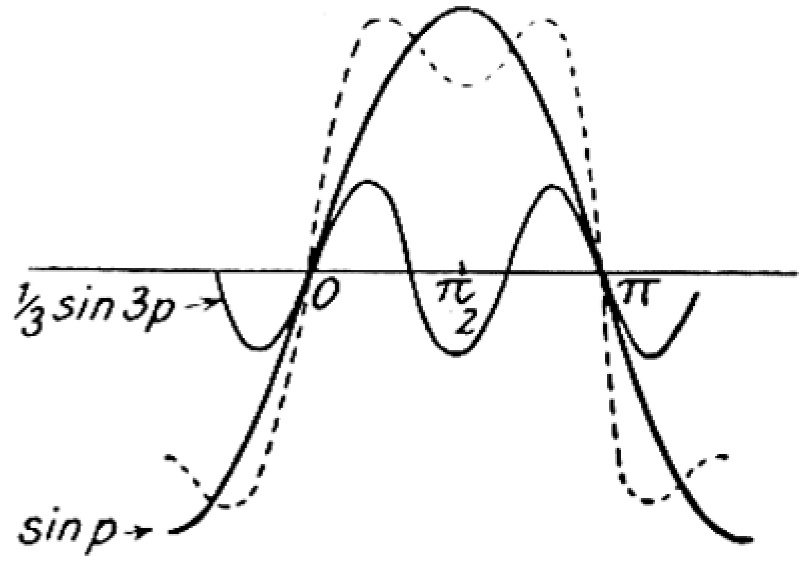

Fig. A7. Shows Superposition of First Two Terms in the Series $\sin p + \frac{1}{3} \sin 3p + \frac{1}{5} \sin 5p +$

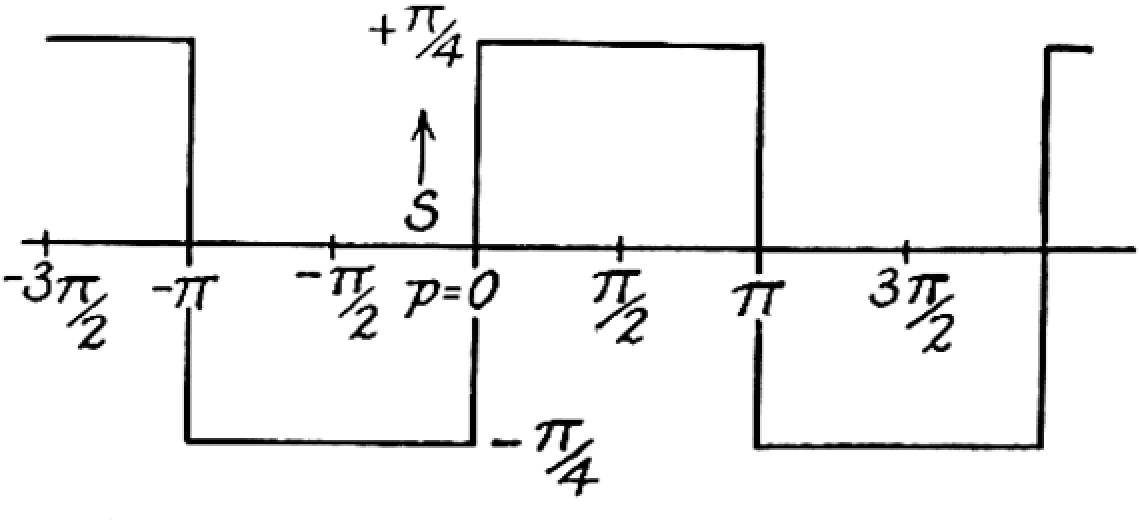

Fig. A8. Fourier Sine Series Resultant

$$\sin p + \tfrac{1}{3} \sin 3p + \tfrac{1}{5} \sin 5p +, \text{etc.}$$

shows a closer and closer approximation to a "square topped" sum as more and more terms are taken, the sum approaching alternately

$\pi/4$ and $-\pi/4$. See Figs. A7 and A8. Hence the series

$$\frac{2}{\pi}\left(\frac{\pi}{4} + \sin p + \tfrac{1}{3}\sin 3p + \tfrac{1}{5}\sin 5p +, \text{ etc. . . . to infinity}\right)$$

represents a transmission with maxima of unity, minima of zero, with equal bars and spaces. It can be shown that any real periodic function can be represented by either a sine series or a cosine series, or by the sum of both if the function is complex. Complex series occur if the function has to take account of variations in phase as well as of amplitude transmission; for example if

$$f(x) = \sum_{n=-\infty}^{n=+\infty} \{b_n \cos nx + a_n \sin nx\}$$

some of the coefficients may contain the "imaginary" factor i, i.e. $\sqrt{-1}$.

If the function is not periodic, a Fourier *series* can only be found to represent it *over some limited interval* of the variable. The sum of the series is periodic outside this interval, though the function itself may not have any finite value except within it. However, a Fourier *integral* may truly represent a limited function. Thus, for example, suppose that the function $f(x)$ has a constant value L between $x = -a/2$ and $x = +a/2$, and is zero elsewhere; the function can then be shown to be represented by

$$f(x) = 2\mathrm{L}a \int_{\nu=0}^{\nu=\infty} \frac{\sin \pi\nu a}{\pi\nu a} \cos 2\pi\nu x \, d\nu$$

Taking this as the sum of infinitesimal terms it will be noted that it is still a sum of cosine terms with various factors; the frequencies needed for the expression are no longer integral multiples of some fundamental frequency, but they have, in principle, an infinite range.

The quantity $(\sin \pi\nu a)/\pi\nu a$ is called a "Fourier transform" of $f(x)$, i.e. of the function limited as explained. In principle, *any* distribution of amplitude which can be described in terms of one variable can be expressed by such an integral. If the integral is complex, if, for example, it has such a form as

$$f(x) = \int_{-\infty}^{+\infty} g(\nu) \exp(2\pi i\nu x) \, d\nu$$

it can express a distribution of both amplitude and phase.

The same ideas can be extended to variations in two dimensions in the object plane; where the phase and amplitude may vary in both the x and y directions, the object function is again in general expressible as a double Fourier integral

$$f(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(u, v) \exp \{2\pi i\ (ux + vy)\}\, du\, dv$$

Again $g(u, v)$ is the Fourier transform of $f(x, y)$.

Remembering de Moivre's theorem it will be realized that all these formulae unite in expressing the object function, whether periodic or not, in terms of sinusoidal components of various frequencies.

If the image is to resemble the object perfectly in amplitude and phase, the corresponding sinusoidal components must all be present in the image; accordingly, the performance of the instrument can be assessed in terms of its capability of transmitting such components in their necessary proportions and phases. If the object itself happens to be simply periodic, like the "square-topped" series above, we need only know the transmission coefficients for a few frequencies in order to be able to make a tolerably accurate account of the image formation; cases involving limited functions are less easily dealt with.

The Cosine Grating. Consider a grating which has the transmission

$$f(x) = \tfrac{1}{2} \left(1 + \cos \frac{2\pi x}{p}\right)$$

It has maxima of unity and minima of zero. The spacing of the maxima is obviously equal to p.

Let such a grating be mounted in a microscope and illuminated coherently by a parallel beam of which any ray lies in a plane perpendicular to the lines of the grating;* if the incidence is normal the relative amount of amplitude A_θ diffracted into the direction θ will be (as on p. 367) proportional to

$$A_\theta = \int_0^X \left(1 + \cos \frac{2\pi x}{p}\right) \cos \left(\frac{2\pi}{\lambda}\, x \sin \theta\right) dx$$

where the range of the grating employed is 2X. The diffracted amplitude is thus (remembering the limited range of integration) a part of the Fourier transform of the object function.

$$A_\theta = \int_0^X \cos \frac{2\pi x \sin \theta}{\lambda}\, dx + \int_0^X \cos \frac{2\pi x}{p} \cos \frac{2\pi x \sin \theta}{\lambda}\, dx$$

As the first integral sum is taken over successive apertures of the grating the cosine is constantly changing sign, and no considerable building up of the amplitude can occur. The second integral* likeways remains negligible unless

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* We are in effect limiting the discussion to the two-dimensional case familiar in interference theory.

$$\frac{2\pi}{p} = \frac{2\pi \sin \theta}{\lambda}, \text{ i.e. } \sin \theta = \lambda/p,$$

when the approximate value of the whole expression is X/2 if X is great in comparison with p. As the grating interval is p, this value of θ expresses the direction of the first-order diffraction maximum. Consequently the diffraction given by a grating of the above character is confined to the first order on each side. As suggested in Figs. 90 and 91, the areas of the source-images capable of furnishing sets of homologous maxima depend on the areas of these transmitted by the objective aperture. As the grating interval diminishes, the first-order maxima move outwards from the central maximum and sooner or later become occluded; the contrast in the diffraction pattern must therefore depend upon the number of sets of homologous maxima left in operation. (The contrast does not change till occlusion begins.) Provided that the phase relations of the homologous points are identical, it may be expected that the contrast in the interference effect will be proportional to the area of either first-order maximum which is exposed; the relative amplitude A_0 of the zero-order maximum ($\theta = 0$) will, when X is comparatively great in comparison with p, be close to

$$A_0 \approx \int_0^X \left(1 + \cos \frac{2\pi x}{p}\right) dx \approx X$$

as against X/2 for the first order. The interference of the central maximum with *both* first-order maxima when they are not occluded can therefore produce complete extinction in the minima, and therefore the full possible contrast in the image.

The complete modern diffraction theory of optical images is beyond the scope of this book, but its main results should now be intelligible subject to a few definitions.

The Transmission Factor and other Definitions. It will be useful in the present connection (and in discussing the testing of optical systems) to explain some customary notations and conventions.

---

* *N.B.* $\int_0^X \cos \alpha x \cos \beta x dx = \left[\frac{\sin (\alpha - \beta)x}{2(\alpha - \beta)} + \frac{\sin (\alpha + \beta)x}{2(\alpha + \beta)}\right]_0^X$

Both the terms grow negligible as x increases, unless $\alpha = \beta$. When α is nearly equal to β, $\sin (\alpha - \beta)x \rightarrow (\alpha - \beta)x$, and the value of the integral approximates towards X/2.

The convention of the use of "optical co-ordinates" is based on the sine relation (Vol. I, equation (401))

$$nh \sin \alpha = n'h' \sin \alpha'$$

If we put

$$u = \frac{2\pi}{\lambda_0} nh \sin \alpha = \frac{2\pi}{\lambda_0} n'h' \sin \alpha' = u'$$

it will be clear that u and u' are, respectively, the (numerically equal) phase differences between central and marginal disturbances occasioned by small conjugate shifts h and h', respectively, in the object and image planes (see Vol. I, p. 78) and α and α' are the limiting angles of inclination of the marginal to the central rays allowed by the pupils. Then u and u' are called the "optical" distances corresponding to h and h'.

The frequency variable s, is defined as follows for an object containing a periodic "line" structure having R "lines" per mm. (and the conjugate image with R′ lines)

$$s = \frac{\lambda}{n \sin \alpha} \mathrm{R} = \frac{\lambda}{n' \sin \alpha'} \mathrm{R}'$$

Now if u is the "optical" displacement variable, the complete object intensity function $\mathrm{B}(u)$ for a linear object (one expressible in terms of only one variable) can be written as a Fourier integral in the usual conventional way

$$\mathrm{B}(u) = \sqrt{\frac{2}{\pi}} \int_0^\infty \beta(s) \cos \{us + \phi(s)\}\, ds$$

and in general it can be shown that the image can be described by a similar distribution

$$\mathrm{B}'(u') = \sqrt{\frac{2}{\pi}} \int_0^\infty \mathrm{T}(s)\, e^{i\theta(s)} \beta(s) \cos \{u's + \phi(s)\}\, ds$$

Note that the component of frequency s (the object and image frequencies being identical owing to the use of optical co-ordinates) has its "amplitude" modulated by a real factor $\mathrm{T}(s)$, and it may possibly be "phase-shifted" in its position relative to the true geometrical image by an amount $\theta(s)$. The complete "transmission factor" is $\mathrm{D}(s)$

$$\mathrm{D}(s) = \mathrm{T}(s)\, e^{i\theta(s)}$$

If the pupil is symmetrical about the axis and the system is free from aberration there will be no phase shift, $\theta(s)$ being zero, and the complete transmission factor is real.

The Convention for the "Fractional" Source and Pupil Co-ordinates. If we use u and v for optical co-ordinates in the object and image planes, it will be convenient to use x and y for co-ordinates in the pupil. The general scheme is shown conventionally in Fig. A9. The plane effective source is shown in some arbitrary position E_0, *not* in general coincident with the object unless the latter is self-luminous. The planes of the object and image are at O and O′, respectively, while E and E′ are the axial points of the entrance and exit pupils, respectively, of the system. The optical co-ordinates u and v defined above relate to a marginal ray which passes through the rim of the (usually circular) diaphragm at a radial distance ρ in the entrance pupil. If this ray is produced backward to intersect the effective source it passes through that plane at a distance ρ_0.

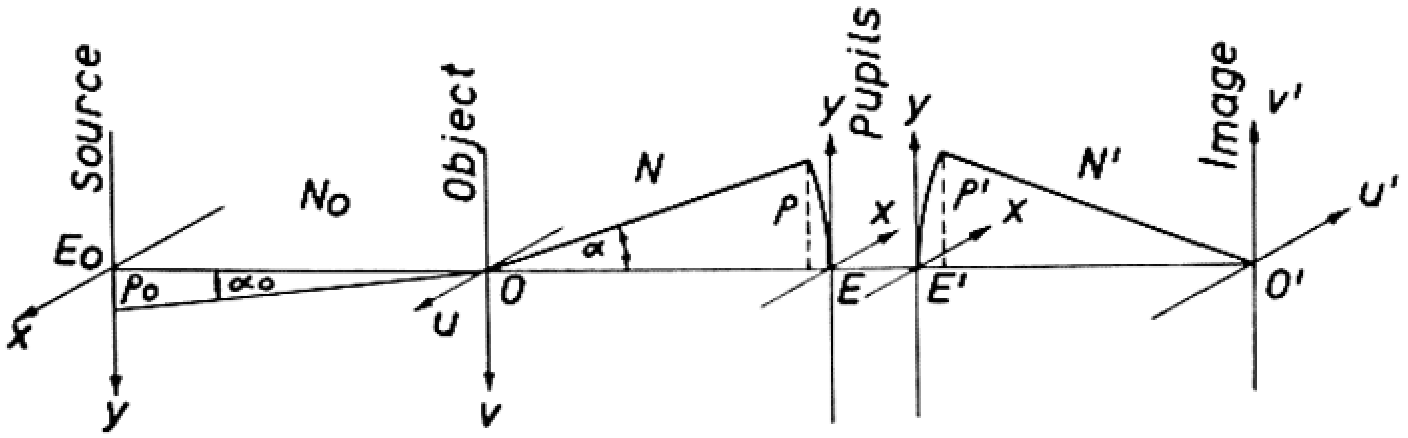

FIG. A9

Then if real co-ordinates in the effective source are a_0, b_0, the conventional co-ordinates are

$$x_0 = \frac{a_0}{\rho_0}, \text{ and } y_0 = \frac{b_0}{\rho_0}$$

In a similar way we may define points of the reference spheres intersecting the axial points E and E′ entrance and exit pupils and centred in the object and image O and O′, respectively. If the actual co-ordinates of conjugate points in the reference spheres are a, b; a', b', the fractional co-ordinates are

$$x = \frac{a}{\rho} \qquad y = \frac{b}{\rho}$$

$$x = \frac{a'}{\rho'} \qquad y = \frac{b'}{\rho'}$$

and it will be seen that in terms of (x, y), both pupils are of unit radius. In this way, the points x_0, y_0 of the source, and x, y in both entrance and exit reference spheres all lie on the same ray.

The source intensity function* is denoted by $\gamma(x_0, y_0)$. In simple applications of the theory the intensity will be unity within the limits of the source and zero outside them.

The complex *amplitude* transmission of the optical system is $f(x, y)$. This means that if the wave displacement in the entrance pupil is given by (Vol. I, p. 84)

$$A(x, y) = a(x, y) \exp \{i\delta(x, y)\}$$

where the amplitude a and the epoch angle δ are both shown as functions of the pupil co-ordinates; and further if the displacement in the exit pupil reference sphere is given by

$$A'(x, y) = c(x, y)\, a(x, y) \exp i\{\delta(x, y) + \varepsilon(x, y)\}$$

then the complex transmission is

$$f(x, y) = c(x, y) \exp \{i\varepsilon(x, y)\}$$

Note that the conjugate complement of this is

$$f^*(x, y) = c(x, y) \exp \{-i\varepsilon(x, y)\}$$

and the intensity transmission is

$$f(x, y)\, f^*(x, y) = c^2(x, y)$$

which is often written

$$| f(x, y) |^2$$

Expression for Contrast Transmission. The general diffraction theory of images developed by Duffieux[1] and by Hopkins,[2,3] naturally uses exponential expressions for object frequency functions. Thus, for example, in the function of one dimension

$$E = 1 + \cos su' = \tfrac{1}{2}(1 + e^{isu'})\,(1 + e^{-isu'})$$

we remember that $1 = e^{\circ}$. Hence we may say that we have three exponential amplitude frequencies represented in our cosine intensity function E, i.e. s, $-s$ and 0. In general one intensity frequency term results from the interaction of *two* amplitude terms, as in the second factor of the last expression; the third factor is the conjugate factor for finding the intensity.

Hopkins's general expression for a transmission factor in a two-dimensional object thus involves two pairs of spatial frequencies; but in a case where the object function is only one-dimensional, and the pair of spatial frequencies involved are called m, p, the expression for the general transmission factor is (putting zero signs for the unwanted ones)—

---

* Do not confuse with the coherence factor.

$$C(m, o\,;\ p, o) = \frac{1}{A}\iint_{-\infty}^{+\infty} \gamma(x, y)\, f(x + m, y)\, f^*(x + p, y)\, dx dy$$

where
$$A = C(o, o\,;\ o, o)$$

and the meanings of the γ and f functions are taken as defined above; this corresponds to an object in which there is no variation

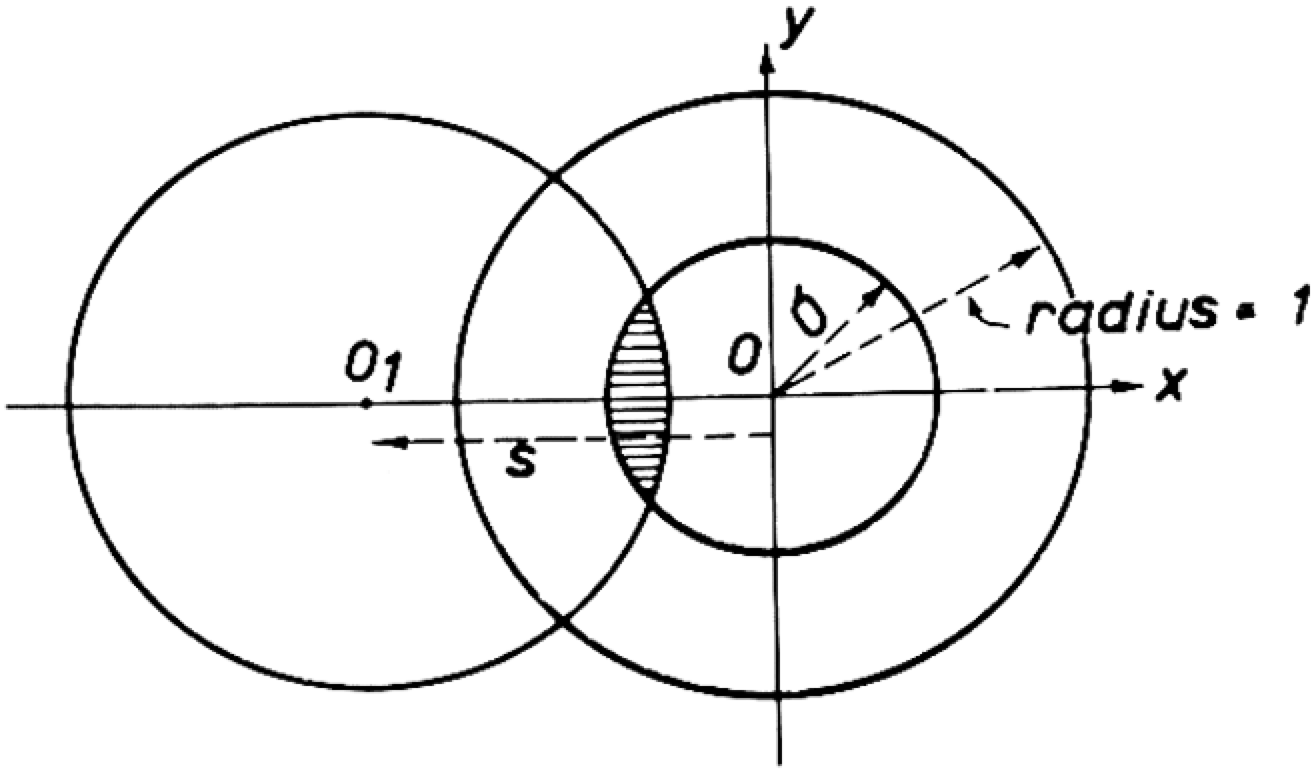

FIG. A10

(zero frequency) in the y direction; the integrations are to be taken over the entire range of x, y within which the products of the functions are appreciable.

In order to interpret this result suppose that the effective source is circular, of radius σ. Let us examine the transmission for the frequency pair "s and o," i.e.

$$\frac{1}{A}\iint_{-\infty}^{+\infty} \gamma(x, y)\, f(x + s, y)\, f^*(x, y)\, dx\, dy$$

Let us draw two circles with centre at the origin O, Fig. A10, one of radius σ which represents the limits of the effective source, and another concentric one of unit radius which represents the limit over which the pupil function (or its conjugate complex) has a value other than zero. If we now draw another circle with unit radius centred at a distance s to the left at O_1, it will be evident that any point x, y in the diagram will have co-ordinates $x + s$, o with respect to O_1. Hence it will be clear that the only points for which *all the three functions in the integral have values other than zero* lie within all three circles. The frequency pair "$-s$ and o" can be dealt with by adding another circle at a distance s to the right, and we should then get a symmetrical diagram. The formal condition for the occurrence of contrast in the image is identical with that inferred from the Abbe principle.

If the aperture of the effective source is too small there is no overlap, and the integral is zero. The condition that resolution may just begin is

$$\sigma = s - 1$$

But $\sigma = NA_C/NA_O$ where NA_C is the numerical aperture of the effective source and NA_O that of the pupil; also, s is the ratio of the numerical aperture NA_D of the angle of diffraction of the first order maximum to the NA of the pupil, i.e. if this angle of diffraction is θ and the spacing of the lines is $1/R$

$$\frac{n \sin \theta}{R} = \lambda$$

Hence
$$s = \frac{n \sin \theta}{n \sin \alpha} = \frac{NA_D}{NA_O}$$

The condition for the limit of resolution is thus

$$\frac{NA_C}{NA_O} = \frac{NA_D}{NA_O} - 1, \text{ i.e. } \frac{1}{R} = \frac{\lambda}{NA_C + NA_O}$$

If the optical system is free from aberration both $f(x, y)$ and $f^*(x, y)$ will be constant over the area of the pupil. Taking their value as unity, and using the same convention for $\gamma(x, y)$ it will be seen that the value of the transmission-factor integral becomes equal to the area of the shaded sector in the Fig. A10. The area of such a sector is easily written down; if the radii of two intersecting circles are r_1 and r_2 (Fig. A11), while the base angles* are θ and ϕ,

$$\text{area} = r_1^2 (\theta - \cos \theta \sin \theta) + r_2^2 (\phi - \cos \phi \sin \phi)$$

In our case $r_1 = 1$ and $r_2 = \sigma$, while the area of the effective source is $\pi\sigma^2$. Thus

$$\frac{C(s, o;\ o, o)}{C(o, o;\ o, o)} = \frac{\text{area of overlap}}{\text{area of effective source}}$$
$$= \frac{(\theta - \cos \theta \sin \theta) + \sigma^2(\phi - \cos \phi \sin \phi)}{\pi\sigma^2}$$

Note that
$$\phi = \cos^{-1} \{(s - \cos \theta)/\sigma\}$$

---

* Do not confuse this angle θ with the angle of diffraction above.

Self-luminous Object (Incoherent Illumination). For a self-luminous object, $\gamma(x, y)$ may be put equal to unity for all parts of the pupil, and this term therefore disappears from the integration. In the absence of aberration the integration is therefore equivalent to finding the area of the overlapping of two circles of radius unity

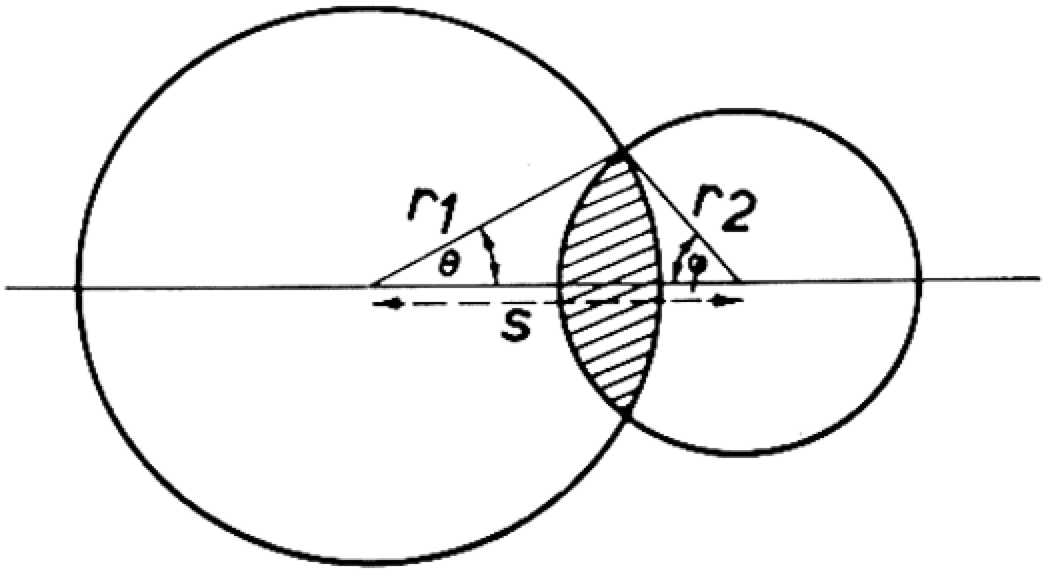

FIG. A11

whose centres are separated by a distance s. The area of overlap is now

$$r_1^2(2\theta - \sin 2\theta)$$

where $\theta = \cos^{-1}(s/2)$ and the ratio of this area to the whole pupil gives $D(s)$

$$D(s) = \frac{1}{\pi}(2\theta - \sin 2\theta)$$

The limit of resolution occurs when the circles touch, i.e. when $s = 2$; that is when

$$\frac{1}{R} = \frac{0{\cdot}5\lambda}{n \sin \alpha}$$

Effects of Change of Focus, Aberrations, etc. Any change of focus, or the presence of aberration will give new values in the pupil function, and these have to be taken account of in the integrations. However, the transmission[3] factors* for various sinusoidal frequencies can be calculated, numerically or otherwise, and thus the performance of the system can be assessed for any object characterized by an explicit "object function."

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2. HOPKINS, *Proc. Roy. Soc.* A., **217,** 408 (1953).
3. HOPKINS, *Proc. Roy. Soc.* A., **231,** 91 (1955).

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* Some recent authors prefer to use the expression "contrast transfer factor."

PROPERTIES OF A SYSTEM OF CO-AXIAL THIN LENSES

1. Focal Length. If a paraxial ray initially parallel to the axis at a height y_a traverses a series of thin lenses of powers $\mathbf{F}_a$, $\mathbf{F}_b$, etc., the angular deviation produced by the first is $\mathbf{F}_a y_a$; by the second it is $\mathbf{F}_b y_b$, and so on*. The whole series of k lenses causes the ray to converge at the angle

$$\sum_{j=1}^{j=k} \mathbf{F}_j y_j$$

towards the axis.

Now since the first principal Gaussian plane and the second are traversed at the same height (y_a in this case), the distance f' from the plane at which the ray has a height y_a to the final focus is

$$f' = \frac{y_a}{\sum \mathbf{F}_j y_j}$$

If the system is to have a fixed focal length (say, unity) the condition is $\sum \mathbf{F} \frac{y_j}{y_a} = 1$

2. Axial Chromatic Aberration. The conjugate distance relations for a thin lens, for the two coloured rays C and F, are

$$\frac{1}{l'_C} = \frac{1}{l_C} + (n_C - 1)\mathbf{R} \qquad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{1}{l'_F} = \frac{1}{l_F} + (n_F - 1)\mathbf{R} \qquad . \quad . \quad . \quad . \quad . \quad (2)$$

Subtracting (2) from (1) and writing $\delta l' = l'_C - l'_F$, and so on, we easily get

$$\delta l' = \delta l \left(\frac{l'_C \, l'_F}{l_C \, l_F}\right) + (n_F - n_C)\mathbf{R} \, l'_C \, l'_F.$$

To multiply the last term on the right by $(n_d - 1)/(n_d - 1)$, i.e. by unity, does not upset the equation. We get

$$\delta l' = \delta l \left(\frac{l'_C \, l'_F}{l_C \, l_F}\right) + \frac{\mathbf{F}}{V} \, l'_C \, l'_F$$

where $\mathbf{F}$ is the power of the lens for the "d" line and V is the constringence, $(n_d - 1)/(n_F - n_C)$.

---

* See pages 240–1, Vol. I.

Now $l'_C \, l'_F \approx (l')^2$, where l' is the conjugate distance for the mean ray, "d"; and similarly $l_C \, l_F \approx l^2$. Thus the equation is written

$$\delta l' \approx \delta l \left(\frac{l'}{l}\right)^2 + \frac{\mathbf{F}}{V}\,(l')^2$$

Put into words, this equation says that the chromatic aberration in the image space of the lens is found by *first* taking the chromatic aberration δl in the object space and multiplying this by the axial magnification coefficient $(l'/l)^2$ (*see* Vol. I, p. 46); we then add to this the new amount $(\mathbf{F}/V)\,(l')^2$ due to the action of the lens. Since $l\alpha = l'\alpha'$, (p. 24, Vol. I) we can re-write the equation

$$\delta l' \approx \delta l \left(\frac{\alpha}{\alpha'}\right)^2 + \frac{\mathbf{F}}{V}\left(\frac{y}{\alpha'}\right)^2.$$

where y is the incidence height of the ray in the lens. Now if the lens were one of a series we should have to multiply the amount arising in this lens by the axial magnification coefficient which transforms axial lengths in *its* image space to the corresponding axial lengths in the final image space. If this lens is "j" and the final lens is "k," the axial magnification coefficient is $(\alpha'_j/\alpha'_k)^2$. Thus the contribution of lens j to the final total is

$$\frac{\mathbf{F}_j}{V_j}\left(\frac{y_j}{\alpha'_j}\right)^2\left(\frac{\alpha'_j}{\alpha'_k}\right)^2 = \frac{\mathbf{F}_j}{V_j}\left(\frac{y_j}{\alpha'_k}\right)^2$$

and the grand total for all the lenses can be written

$$\begin{aligned} \delta l'_k &= \delta l_1 \left(\frac{\alpha_1}{\alpha'_k}\right)^2 + \sum_1^k \frac{\mathbf{F}_j}{V_j}\left(\frac{y_j}{\alpha'_k}\right)^2 \\ &= \delta l_1 \left(\frac{\alpha_1}{\alpha'_k}\right)^2 + l'^2_k \sum_1^k \frac{\mathbf{F}_j}{V_j}\left(\frac{y_j}{y_k}\right)^2 \end{aligned}$$

Thus for two lenses "a" and "b," with a real object point, so that $\delta l_1 = 0$, the condition that the final axial chromatic aberration may be zero is

$$\frac{\mathbf{F}_a}{V_a}\left(\frac{y_a}{y_b}\right)^2 + \frac{\mathbf{F}_b}{V_b} = 0$$

If the object distance is infinite we shall have

$$\frac{y_b}{y_a} = \frac{f'_a - d}{f'_a}$$

and these equations can be transformed to give

$$V_b f'_b + V_a f'_a \left(1 - \frac{d}{f'_a}\right)^2 = 0$$

So that if d is small in comparison with f'_a we have approximately

$$V_b f'_b + V_a (f'_a - 2d) = 0$$

For a series of lenses, the condition that the axial chromatic aberration due to the system may vanish is

$$\sum_1^k \frac{\boldsymbol{F}_j}{V_j} y_j^2 = 0$$

3. Transverse Chromatic Aberration. The expression above for the axial chromatic aberration for a single thin lens when $\delta l = 0$, is

$$\delta l' = \frac{\boldsymbol{F}}{\mathrm{V}} l'^2$$

This leads easily to the angular chromatic aberration a' after refraction of a white principal ray at a height $\bar{y}$ above the axis of the lens; it is independent of the (paraxial) angle of the ray with the axis, and is—

$$a' = \frac{\boldsymbol{F}}{\mathrm{V}} \bar{y}$$

This gives a transverse chromatic intercept t' in the image plane at a distance l' from the lens

$$t' = \frac{\boldsymbol{F}}{\mathrm{V}} \bar{y} l'$$

If the lens (j) is one of a series, as above, terminating with lens k this transverse aberration due to j will be represented in the final image plane by an amount found by multiplying t'_j by the factor m of transverse magnification. We have since $t'_j \alpha'_j = t'_k \alpha'_k$

$$m = \alpha'_j / \alpha'_k$$

Hence the contribution (C_j) of lens j to the final transverse chromatic aberration is

$$\mathrm{C}_j = \left\{ \left(\frac{\boldsymbol{F}}{\mathrm{V}} \right)_j \bar{y}_j \, l'_j \right\} \frac{\alpha'_j}{\alpha'_k}$$

$$= \left(\frac{\boldsymbol{F}}{\mathrm{V}} \right)_j \bar{y}_j \frac{y_j}{\alpha'_k}$$

where y_j is the height of passage, through the lens j, of the ray which finally traverses the axial image point at an angle α'_k. The total transverse aberration produced by the system is therefore

$$C = \frac{1}{\alpha'_k} \sum_{j=1}^{j=k} \left(\frac{\mathsf{F}}{V}\right)_j \bar{y}_j y_j$$

For a series of lenses the condition that the transverse chromatic effect may be zero can be written

$$\sum_1^k \left(\frac{\mathsf{F}}{V}\right)_j y^2 \left(\frac{\bar{y}}{y}\right) = 0$$

APPENDIX V

DATA FOR PHOTOGRAPHIC LENSES

(1) Typical "New Achromat" landscape lens. $f/16$, $f' = 100$ approx.

$r_1 = -15{\cdot}70$			
	$d = 0{\cdot}83$	$N_D = 1{\cdot}5472$	$V_{D-G'} = 35{\cdot}7$
$r_2 = 162{\cdot}50$			
	,, 2·08	,, 1·6118	,, 46·8
$r_3 = -14{\cdot}66$			

Diaphragm at a distance = 5·8 in front of 1st surface.

(2) Early Protar. $f/12{\cdot}5$, $f' = 100$.

$r_1 = 17{\cdot}5$			
	$d = 2{\cdot}9$	$N_D = 1{\cdot}6489$	$V_{D-G'} = 25{\cdot}4$
$r_2 = 5{\cdot}8$			
	,, 1·3	,, 1·6031	,, 29·0
$r_3 = 18{\cdot}6$			
	$\Delta = 3{\cdot}0$	Air (stop at mid-point).	
$r_4 = -12{\cdot}8$			
	$d = 1{\cdot}1$	,, 1·5154	,, 42·6
$r_5 = 18{\cdot}6$			
	,, 1·8	,, 1·6112	,, 41·3
$r_6 = -14{\cdot}3$			

(3) Dagor. $f/6{\cdot}8$, $f' = 100$.

$r_1 = 20{\cdot}8$			
	$d = 3{\cdot}0$	$N_D = 1{\cdot}6122$	$V_{D-G'} = 43{\cdot}73$
$r_2 = -34{\cdot}4$			
	,, 0·9	,, 1·5481	,, 41·52
$r_3 = 8{\cdot}5$			
	,, 0·5	,, 1·5120	,, 47·41
$r_4 = 21{\cdot}1$			

(Symmetrical) $\Delta/2 = 2{\cdot}7$. Air (stop at mid-point).

(4) Four-component air-spaced lens. Rodenstock. D.R.G.M. 262201 1905 (Gauss Type). $f/5$, $f' = 100$.

$r_1 = 28{\cdot}1$			
	$d = 3{\cdot}6$	$N_D = 1{\cdot}527$	$V = 59{\cdot}8$
$r_2 = 93{\cdot}5$			
	$\Delta = 1{\cdot}3$	air	
$r_3 = 30{\cdot}3$			
	$d = 3{\cdot}6$	,, 1·610	35·0
$r_4 = 20$			

(Symmetrical) $\Delta/2 = 3{\cdot}0$. (Stop at mid-point.)

(5) Four-component air-spaced lens. Aviar. B.P. 1136590/1918, Celor type. $f/4{\cdot}5$, $f' = 1$.

$r_1 =$ 0·2976			
	$d =$ 0·0455	$N_D =$ 1·6116	$V =$ 56·4
$r_2 = -$ 0·8152			
	$\Delta =$ 0·0203	air	
$r_3 = -$ 0·4285			
	$d =$ 0·0100	,, 1·5682	,, 43·4
$r_4 =$ 0·4464			
	$\Delta =$ 0·0362	air	
$r_5 = -$ 0·5041			
	$d =$ 0·0100	,, 1·5502	,, 45·8
$r_6 =$ 0·4330			
	$\Delta =$ 0·0193	air	
$r_7 =$ 0·8152			
	$d =$ 0·0455	,, 1·6116	,, 56·4
$r_8 = -$ 0·2976			

(Stop at mid-point of central air space.)

(6) Cooke Triplet Objective (Series IV). $f/5{\cdot}6$, $f' = 100$.

$r_1 =$ 19·4			
	$d =$ 4·29	$N_D =$ 1·6110	$V_{D-G'} =$ 40·94
$r_2 = -$ 128·3			
	$\Delta =$ 1·63	air	
$r_3 = -$ 57·8			
	$d =$ 0·93	,, 1·5744	,, 31·08
$r_4 =$ 18·9			
	$\Delta =$ 12·90	air	
$r_5 =$ 311·3			
	$d =$ 3·03	,, 1·6110	,, 40·94
$r_6 = -$ 66·4			

(Diaphragm in 2nd air space, against 2nd element.)

(7) Pentac. B.P. 151506. $f/3$, $f' = 6{\cdot}25$.

$r_1 =$ 2·635			
	$d =$ 0·49	$N_D =$ 1·6109	$V =$ 46·27
$r_2 = -$ 2·953			
	$d =$ 0·1	,, 1·5485	,, 34·93
$r_3 = -$ 94·1			
	$\Delta =$ 0·52	air	
$r_4 = -$ 2·74			
	$d =$ 0·1	,, 1·5485	,, 34·93
$r_5 =$ 2·375			
	$\Delta =$ 0·53	air	
$r_6 =$ 52·4			
	$d =$ 0·1	,, 1·5485	,, 34·93
$r_7 =$ 2·451			
	$d =$ 0·49	,, 1·6109	,, 46·27
$r_8 = -$ 2·451			

(Diaphragm in 2nd air space, against 2nd element.)

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